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A COMPLEX SURFACE OF GENERAL TYPE WITH $p_g = 0,\ K^2 = 2$ AND $H_1 = \mathbb{Z}/2\mathbb{Z}$

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Abstract. As the sequel to [4], we construct a minimal complex surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$ using a rational blow-down surgery and $\mathbb{Q}$-Gorenstein smoothing theory. We also present an example of $p_g = 0, K^2 = 2$ and $H_1 = \mathbb{Z}/3\mathbb{Z}$.

One of the fundamental problems in the classification of complex surfaces is to find a new family of surfaces of general type with $p_g = 0$ and $K^2 > 0$. In this paper we construct new complex surfaces of general type with $p_g = 0$ and $K^2 = 2$. The first example of a minimal complex surface of general type with $p_g = 0$ and $K^2 = 2$ was constructed by Campedelli [3] in the 1930's as a ramified double cover of $\mathbb{P}^2$; more precisely as the double cover of $\mathbb{P}^2$ branched along a reducible curve of degree 10 with 6 $[3, 3]$ points not lying on a conic. Nowadays minimal surfaces of general type with $p_g = 0$ and $K^2 = 2$ are called (numerical) Campedelli surfaces. For Campedelli surface X, the number of elements in the torsion subgroup of $H^2(X;\mathbb{Z})$ is bounded by 9 [8]. Although many families of non-simply connected complex surfaces of general type with $p_g = 0$ and $K^2 = 2$ have been constructed (refer to Chapter VII, [1]) and the classifications are completed for some torsion groups [6], until now it is not known whether there is a minimal complex surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$.

Recently we constructed a simply connected surface of general type with $p_g = 0$ and $K^2 = 2$ using a rational blow-down surgery and $\mathbb{Q}$-Gorenstein smoothing theory [4]. In this paper we continue to construct a minimal complex surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$ using the same technique. The first key ingredient of this paper is to find a right rational surface Z which makes it possible to get such a complex surface. Once we have a right rational surface Z, then we can obtain a minimal complex surface of general type with $p_g = 0$ and $K^2 = 2$ by applying a rational blow-down surgery and $\mathbb{Q}$-Gorenstein smoothing theory developed in [4] to Z. And then we show that the surface has $H_1 = \mathbb{Z}/2\mathbb{Z}$, which is the second key ingredient of this article. Since almost all the proofs except the computation of homology groups are basically the same as the proofs of the main construction in [4], we only explain how to construct a such surface and how to compute the first homology group in Section 2. The main result of this paper is the following

Theorem 1. *There exists a minimal complex surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$.*

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Furthermore, using the same method as above, we are also able to construct a Campedelli surface with $H_1 = \mathbb{Z}/3\mathbb{Z}$ which will be addressed in Section 3.

Corollary 1. *There exists a minimal complex surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/3\mathbb{Z}$.*

Remark. Recently I. Bauer, F. Catanese, F. Grunewald and R. Pignatelli also constructed two Campedelli surfaces with fundamental group $\mathbb{Z}/3\mathbb{Z}$ using completely different methods [2].

1. The main construction of a surface with $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$

We begin with a rational elliptic surface $Y = \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2$ which has one $\tilde{E}_6$-singular fiber, one I_2-singular fiber, and two nodal singular fibers used in Section 3 in [4] (Figure 1).

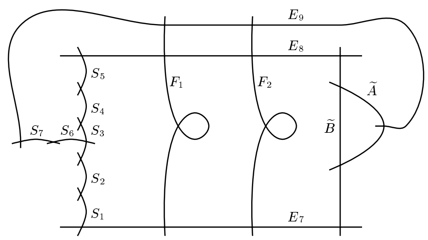

FIGURE 1. A rational elliptic surface $Y = \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2$

Notations. We denote a line in $\mathbb{P}^2$ by H and exceptional curves in $Y = \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2$ by $E_1, E_2, \ldots, E_9$. Equivalently we use the same notation $H, E_1, E_2, \ldots, E_9$ for the standard generators of $H_2(Y;\mathbb{Z})$ which represent a line and exceptional curves in Y respectively. We also denote the rational curves lying in the $\tilde{E}_6$-singular fiber by $S_1, S_2, \ldots, S_7$ and two nodal fibers by F_1, F_2 and two rational curves lying in the I_2-singular fiber by $\tilde{A}, \tilde{B}$ respectively. In fact $\tilde{A}$ and $\tilde{B}$ are proper transforms of a line A and a conic B lying in $\mathbb{P}^2$ respectively.

Main Construction. We first blow up at two singular points in the nodal fibers F_1, F_2 on Y. Then the proper transforms $\tilde{F}_1, \tilde{F}_2$ of F_1, F_2 will be rational (-4)-curves whose homology classes are $[\tilde{F}_1] = [F_1] - 2e_1$ and $[\tilde{F}_2] = [F_2] - 2e_2$, where e_1, e_2 are new exceptional curves in $Y \sharp 2\overline{\mathbb{P}}^2$ coming from two blowing ups. Next, we blow up six times at the intersection points between two sections E_7, E_8 and $\tilde{F}_1, \tilde{F}_2, \tilde{B}$. It makes the self-intersection number of the proper transforms E_7, E_8 and $\tilde{B}$ to be -4 respectively. Let us denote six new exceptional curves arising from six times blowing ups by $e_3, e_4, \ldots, e_8$ respectively. Now we blow up twice successively at the intersection point between the proper transform of E_7 and the exceptional curve e_3 in the total transform of F_1. It makes a chain of $\mathbb{P}^1$'s, $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, lying in the total transform of F_1. Let us denote two new exceptional curves arising from twice blowing ups by e_9, e_{10} respectively. We blow up again four times successively at the intersection point between the proper transform of E_7 and the exceptional curve e_4 in the total transform of F_2, so that a chain of $\mathbb{P}^1$'s, $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, lies in the total transform of F_2. Let us denote four new exceptional curves arising from four times blowing ups at this step by $e_{11}, e_{12}, e_{13}, e_{14}$ respectively. We note that it makes the self-intersection number of the proper transform of E_7 to be -10. Then we blow up twice successively at the intersection point between rational (-2)-curve in the end of linear chain $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$ and the exceptional curve e_{14}. Let us denote two new exceptional curves arising from twice blowing ups at this step by e_{15}, e_{16} respectively. It changes $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$ to $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ}$, and it produces a chain of $\mathbb{P}^1$'s, $\overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-10}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ}$, which contains the proper transform of two sections E_7, E_8 and a linear chain of $\mathbb{P}^1$'s in the $\tilde{E}_6$-singular fiber. We also blow up twice successively at one of the two intersection points between rational (-6)-curve in $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ}$ and the exceptional curve e_2 appeared by the blowing up at the singular point of one nodal fiber F_2. Let us denote two new exceptional curves arising from twice blowing ups at this step by e_{17}, e_{18} respectively. Then the chain $\overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ}$ changes to $\overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-8}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ}$ by adding two new rational (-2)-curves. Finally, we have a rational surface $Z := Y \sharp 18\overline{\mathbb{P}}^2$ which contains four disjoint configurations: $C_{22,15} = \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-10}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ}$, $C_{4,1} = \overset{-6}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$, $C_{16,11} = \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-8}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-4}{\circ}$, and $C_{2,1} = \overset{-4}{\circ}$ (Figure 2).

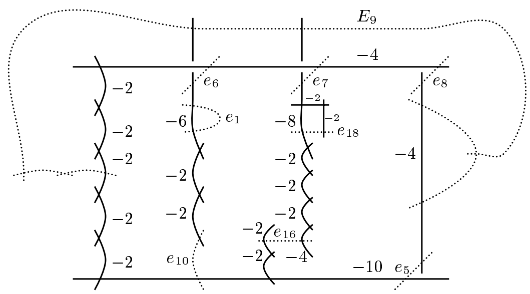

FIGURE 2. A rational surface $Z = Y \sharp 18\overline{\mathbb{P}}^2$

Notations. We use the same notation $C_{p,q}$ for both a smooth 4-manifold obtained by plumbing disk bundles over the 2-sphere instructed by $\underset{u_k}{\overset{-b_k}{\circ}} - \underset{u_{k-1}}{\overset{-b_{k-1}}{\circ}} - \cdots - \underset{u_2}{\overset{-b_2}{\circ}} - \underset{u_1}{\overset{-b_1}{\circ}}$ and a linear chain of 2-spheres, $\{u_k, u_{k-1}, \ldots, u_1\}$. Here $\frac{p^2}{pq-1} = [b_k, b_{k-1}, \ldots, b_1]$ is a continued fraction with all $b_i \geq 2$ uniquely determined by p, q, and u_i represents an embedded 2-sphere as well as a disk bundle over 2-sphere whose Euler number is $-b_i$. Note that the boundary of a configuration $C_{p,q}$ is a lens space $L(p^2, 1-pq)$, which also bounds a rational ball $B_{p,q}$. Similarly we use a notation $D_{2,3}$ for $\overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-4}{\circ}$. Note that the boundary of a configuration $D_{2,3}$ is a lens space $L(18, -11)$, which bounds a Milnor fiber $M_{2,3}$. It is well known that the Milnor fiber $M_{2,3}$ is not a rational ball but a negative definite 4-manifold with second Betti number 1 (cf. [5]).

Then we contract these four disjoint chains of $\mathbb{P}^1$'s from Z so that it produces a normal projective surface, denoted by X, with four permissible singularities. Using the same technique as in [4], we are able to prove that X has a $\mathbb{Q}$-Gorenstein smoothing and a general fiber X_t of the $\mathbb{Q}$-Gorenstein smoothing is a minimal complex surface of general type with $p_g = 0$ and $K^2 = 2$. Furthermore, the general fiber X_t is diffeomorphic to a rational blow-down 4-manifold $Z_{22,16,4,2}$ which is obtained from $Z = Y \sharp 18\overline{\mathbb{P}}^2$ by replacing four disjoint configurations $C_{22,15}$, $C_{16,11}$, $C_{4,1}$ and $C_{2,1}$ with corresponding rational balls $B_{22,15}$, $B_{16,11}$, $B_{4,1}$ and $B_{2,1}$ respectively. In the next section we will prove that the rational blow-down 4-manifold $Z_{22,16,4,2}$ has $H_1(Z_{22,16,4,2};\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Hence we obtain the main result of this paper - Theorem 1.

2. Proof of $H_1(Z_{22,16,4,2};\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

In this section we compute the first homology group of a rational blow-down 4-manifold $Z_{22,16,4,2}$ using geometric arguments and some elementary homology sequences.

First note that the rational surface $Z = Y \sharp 18\overline{\mathbb{P}}^2$ contains four disjoint configurations - $C_{22,15}$, $C_{16,11}$, $C_{4,1}$ and $C_{2,1}$. Let us decompose the rational surface Z into

$$Z = Z_0 \cup \{C_{22,15} \cup C_{16,11} \cup C_{4,1} \cup C_{2,1}\}.$$

Then the rational blow-down 4-manifold $Z_{22,16,4,2}$ can be decomposed into

$$Z_{22,16,4,2} = Z_0 \cup \{B_{22,15} \cup B_{16,11} \cup B_{4,1} \cup B_{2,1}\}.$$

Before we prove that $H_1(Z_{22,16,4,2};\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, we introduce several lemmas which are critical in the computation of $H_1(Z_{22,16,4,2};\mathbb{Z})$. Let us first consider the following two sets of homology classes lying in $H_2(Z;\mathbb{Z})$: $\mathcal{D} = \{\tilde{A},\, E_3 - E_6,\, E_6 - E_9\}$ and $\mathcal{E} = \{E_9,\, e_1,\, e_5,\, e_6,\, e_7,\, e_8,\, e_{10},\, e_{16},\, e_{18}\}$. Then one can easily get the following lemmas.

Lemma 2.1. *(i) The homology classes of* $\{u_i \,|\, 1 \le i \le 9\}$ *in* $C_{22,15}$ *can be represented by* $\{e_{15} - e_{16},\, e_{14} - e_{15},\, E_7 - e_3 - e_4 - e_5 - e_9 - \cdots - e_{14},\, E_4 - E_7,\, E_1 - E_4,\, H - E_1 - E_2 - E_3,\, E_2 - E_5,\, E_5 - E_8,\, E_8 - e_6 - e_7 - e_8\}$.
(ii) The homology classes of $\{u_i \,|\, 1 \le i \le 7\}$ *in* $C_{16,11}$ *can be represented by* $\{e_{17} - e_{18},\, e_2 - e_{17},\, F_2 - 2e_2 - e_4 - e_7 - e_{17} - e_{18},\, e_4 - e_{11},\, e_{11} - e_{12},\, e_{12} - e_{13},\, e_{13} - e_{14} - e_{15} - e_{16}\}$.
(iii) The homology classes of $\{u_i \,|\, 1 \le i \le 3\}$ *in* $C_{4,1}$ *can be represented by* $\{F_1 - 2e_1 - e_3 - e_6,\, e_3 - e_9,\, e_9 - e_{10}\}$.
(iv) The homology classes of u_1 *in* $C_{2,1}$ *can be represented by* $\{\tilde{B} - e_5 - e_8\}$.

Lemma 2.2. *The set of homology classes representing all generators in* $C_{22,15} \cup C_{16,11} \cup C_{4,1} \cup C_{2,1} \cup \mathcal{D} \cup \mathcal{E}$ *spans* $H_2(Z;\mathbb{Z})$.

Lemma 2.3. $H_2(Z_0, \partial B_{22,15} \cup \partial B_{16,11} \cup \partial B_{4,1} \cup \partial B_{2,1};\mathbb{Z})$ *is spanned by the images of homology classes representing all generators in* $C_{22,15} \cup C_{16,11} \cup C_{4,1} \cup C_{2,1} \cup \mathcal{D} \cup \mathcal{E}$ *under a composition of homomorphisms* $H_2(Z;\mathbb{Z}) \to H_2(Z, C_{22,15} \cup C_{16,11} \cup C_{4,1} \cup C_{2,1};\mathbb{Z}) \cong H_2(Z_0, \partial B_{22,15} \cup \partial B_{16,11} \cup \partial B_{4,1} \cup \partial B_{2,1};\mathbb{Z})$.

Proof. Since an induced homomorphism $H_2(Z;\mathbb{Z}) \to H_2(Z, C_{22,15} \cup C_{16,11} \cup C_{4,1} \cup C_{2,1};\mathbb{Z})$ by an inclusion is surjective, which follows from the long exact homology sequence of the pair $(Z, C_{22,15} \cup C_{16,11} \cup C_{4,1} \cup C_{2,1})$, and since $H_2(Z, C_{22,15} \cup C_{16,11} \cup$

$C_{4,1} \cup C_{2,1}; \mathbb{Z})$ is isomorphic to $H_2(Z_0, \partial C_{22,15} \cup \partial C_{16,11} \cup \partial C_{4,1} \cup \partial C_{2,1}; \mathbb{Z}) = H_2(Z_0, \partial B_{22,15} \cup \partial B_{16,11} \cup \partial B_{4,1} \cup \partial B_{2,1}; \mathbb{Z})$ by an excision principle, the statement follows from Lemma 2.2. □

Lemma 2.4. *Suppose that* $\partial_* : H_2(Z_0, \partial B_{22,15} \cup \partial B_{16,11} \cup \partial B_{4,1} \cup \partial B_{2,1}; \mathbb{Z}) \to H_1(\partial B_{22,15} \cup \partial B_{16,11} \cup \partial B_{4,1} \cup \partial B_{2,1}; \mathbb{Z}) = \mathbb{Z}_{22^2} \oplus \mathbb{Z}_{16^2} \oplus \mathbb{Z}_{4^2} \oplus \mathbb{Z}_{2^2}$ *is a homomorphism induced by a boundary map* $\partial : (Z_0, \partial Z_0) \to \partial Z_0$. *And let* $i_* : H_1(\partial B_{22,15} \cup \partial B_{16,11} \cup \partial B_{4,1} \cup \partial B_{2,1}; \mathbb{Z}) = \mathbb{Z}_{22^2} \oplus \mathbb{Z}_{16^2} \oplus \mathbb{Z}_{4^2} \oplus \mathbb{Z}_{2^2} \to H_1(B_{22,15} \cup B_{16,11} \cup B_{4,1} \cup B_{2,1}; \mathbb{Z}) = \mathbb{Z}_{22} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$ *be a homomorphism induced by an inclusion* i. *Then we have*

(0) $\partial_*(u_i) = (0,0,0,0) \xrightarrow{i_*} (0,0,0,0)$ *for any class* $u_i \in C_{22,15} \cup C_{16,11} \cup C_{4,1} \cup C_{2,1}$

(i) $\partial_*(\tilde{A}) = (0,0,0,2) \xrightarrow{i_*} (0,0,0,0)$

(ii) $\partial_*(E_3 - E_6) = (10,0,0,0) \xrightarrow{i_*} (10,0,0,0)$

(iii) $\partial_*(E_6 - E_9) = (0,0,0,0) \xrightarrow{i_*} (0,0,0,0)$

(iv) $\partial_*(E_9) = (0,13,3,0) \xrightarrow{i_*} (0,13,3,0)$

(v) $\partial_*(e_1) = (0,0,6,0) \xrightarrow{i_*} (0,0,2,0)$

(vi) $\partial_*(e_5) = (19,0,0,1) \xrightarrow{i_*} (19,0,0,1)$

(vii) $\partial_*(e_6) = (1,0,3,0) \xrightarrow{i_*} (1,0,3,0)$

(viii) $\partial_*(e_7) = (1,13,0,0) \xrightarrow{i_*} (1,13,0,0)$

(ix) $\partial_*(e_8) = (1,0,0,1) \xrightarrow{i_*} (1,0,0,1)$

(x) $\partial_*(e_{10}) = (19,0,1,0) \xrightarrow{i_*} (19,0,1,0)$

(xi) $\partial_*(e_{16}) = (329,1,0,0) \xrightarrow{i_*} (21,1,0,0)$

(xii) $\partial_*(e_{18}) = (0,188,0,0) \xrightarrow{i_*} (0,12,0,0)$.

Proof. (0) and *(iii)* follow from the fact that they do not intersect with ∂Z_0. For the rest, we choose generators $\{\alpha = (1,0,0,0), \beta = (0,1,0,0), \gamma = (0,0,1,0), \delta = (0,0,0,1)\}$ of $H_1(B_{22,15}; \mathbb{Z}) \oplus H_1(B_{16,11}; \mathbb{Z}) \oplus H_1(B_{4,1}; \mathbb{Z}) \oplus H_1(B_{2,1}; \mathbb{Z})$ so that α, β, γ and δ are represented by circles $\partial C_{22,15} \cap e_6$ (equivalently $\partial C_{22,15} \cap e_7$ or $\partial C_{22,15} \cap e_8$), $\partial C_{16,11} \cap e_{16}$, $\partial C_{4,1} \cap e_{10}$ and $\partial C_{2,1} \cap e_5$ (equivalently $\partial C_{2,1} \cap e_8$), respectively. Then one can easily see that the rest of computation follows from Figure 2. For example, we can compute *(iv)* as follows: Note that an exceptional curve E_9 intersects with (-8)-curve in $C_{16,11}$ and it also intersects with (-6)-curve in $C_{4,1}$. Since $\partial_*(E_9)$(= a normal circle of (-8)-curve) is $13\beta \in H_1(\partial B_{16,11})$ and $\partial_*(E_9)$(= a normal circle of (-6)-curve) is $3\gamma \in H_1(\partial B_{4,1})$, we have $\partial_*(E_9) = (0,13,3,0)$. The images of i_* follow from the fact that $i_* : H_1(\partial B_{p,q}) = \mathbb{Z}_{p^2} \to H_1(B_{p,q}) = \mathbb{Z}_p$ sends a generator to a generator, i.e. $i_*(1) = 1$. □

Finally, we are now in a position to compute $H_1(Z_{22,16,4,2}; \mathbb{Z})$.

Proposition 2.1. $H_1(Z_{22,16,4,2}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Proof. First let us consider the following commutative diagram between two exact homology sequences with $\mathbb{Z}$-coefficients for pairs $(Z_0, \partial Z_0)$ and $(Z_{22,16,4,2}, B_{22,15} \cup B_{16,11} \cup B_{4,1} \cup B_{2,1})$:

$$\begin{array}{ccccc} H_2(Z_0, \partial B_{22,15} \cup \partial B_{16,11} \cup \partial B_{4,1} \cup \partial B_{2,1}) & \xrightarrow{\partial_*} & H_1(\partial B_{22,15} \cup \partial B_{16,11} \cup \partial B_{4,1} \cup \partial B_{2,1}) & \xrightarrow{j_*} & H_1(Z_0) \to 0 \\ i_* \downarrow \cong & & \downarrow i_* & & \\ H_2(Z_{22,16,4,2}, B_{22,15} \cup B_{16,11} \cup B_{4,1} \cup B_{2,1}) & \xrightarrow{\partial_*} & H_1(B_{22,15} \cup B_{16,11} \cup B_{4,1} \cup B_{2,1}) & \xrightarrow{j_*} & H_1(Z_{22,16,4,2}) \to 0 \end{array}$$

where i_* and j_* are induced homomorphisms by inclusions i and j respectively, and ∂_* is an induced homomorphism by a boundary map ∂. Note that the first i_* is an isomorphism by an excision principle and the second i_* is surjective. Hence it follows from the diagram above that

$$\begin{aligned} H_1(Z_{22,16,4,2};\mathbb{Z}) &\cong H_1(B_{22,15}\cup B_{16,11}\cup B_{4,1}\cup B_{2,1};\mathbb{Z})/\operatorname{Im}(i_*\circ\partial_*) \\ &= \mathbb{Z}_{22}\oplus\mathbb{Z}_{16}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2/\operatorname{Im}(i_*\circ\partial_*). \end{aligned}$$

Note that, by Lemma 2.3 and Lemma 2.4 above, $\operatorname{Im}(i_*\circ\partial_*)$ is spanned by a set of columns of the following matrix

$$\begin{pmatrix} 10 & 0 & 0 & 19 & 1 & 1 & 1 & 19 & 21 & 0 \\ 0 & 13 & 0 & 0 & 0 & 13 & 0 & 0 & 1 & 12 \\ 0 & 3 & 2 & 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Applying elementary column operations to the matrix above, we obtain a reduced form

$$\begin{pmatrix} 10 & 0 & 6 & 18 & 10 & 14 & 1 & 19 & 21 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore it is easy to see that $\operatorname{Im}(i_*\circ\partial_*)$ is spanned by

$$\{(10,0,0,0),\ (6,0,0,0),\ (1,0,0,1),\ (19,0,1,0),\ (21,1,0,0)\}$$

and $\{(10,0,0,0),(6,0,0,0),(1,0,0,1),(19,0,1,0),(21,1,0,0)\}\cup\{(0,0,0,1)\}$ spans the space $H_1(B_{22,15}\cup B_{16,11}\cup B_{4,1}\cup B_{2,1};\mathbb{Z})=\mathbb{Z}_{22}\oplus\mathbb{Z}_{16}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2$. Note that $(0,0,0,1)\notin \operatorname{Im}(i_*\circ\partial_*)$ but $2(0,0,0,1)=2(10,0,0,0)+2(1,0,0,1)\in\operatorname{Im}(i_*\circ\partial_*)$. Hence we get

$$H_1(Z_{22,16,4,2};\mathbb{Z})\cong\mathbb{Z}_{22}\oplus\mathbb{Z}_{16}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2/\operatorname{Im}(i_*\circ\partial_*)=\langle\overline{(0,0,0,1)}\rangle=\mathbb{Z}/2\mathbb{Z}. \qquad \square$$

3. A surface with $K^2=2$ and $H_1=\mathbb{Z}/3\mathbb{Z}$

In this section we construct a surface of general type with $p_g=0, K^2=2$ and $H_1=\mathbb{Z}/3\mathbb{Z}$ using the same technique. As we mentioned in the Introduction, the key ingredient in the construction of such a surface is to find a right rational elliptic surface Z'. Once we have a right rational elliptic surface Z' for $K^2=2$ and $H_1=\mathbb{Z}/3\mathbb{Z}$, the remaining argument is the same as before. Hence we describe here only how to get such a rational elliptic surface Z' which is following.

Main Construction. We begin with a rational elliptic surface $Y'=\mathbb{P}^2\sharp 9\overline{\mathbb{P}}^2$ which has one I_8-singular fiber, one I_2-singular fiber, and two nodal singular fibers used in Section 3 in [7] (Figure 3). From this rational elliptic surface Y', we first blow up at two singular points in the nodal fibers F_1, F_2 on Y'. Then the proper transforms $\tilde{F}_1,\tilde{F}_2$ of F_1,F_2 will be rational (-4)-curves whose homology classes are $[\tilde{F}_1]=[F_1]-2e_1$ and $[\tilde{F}_2]=[F_2]-2e_2$, where e_1,e_2 are new exceptional curves in $Y'\sharp 2\overline{\mathbb{P}}^2$ coming from two blowing ups. Next, we blow up at two black circled points lying in an I_8-singular fiber and we also blow up at four black circled points which are intersection points between sections S_i and nodal fibers F_i in Figure 3 above. We blow up again twice at one of the two intersection points between the proper transform of F_i and the exceptional

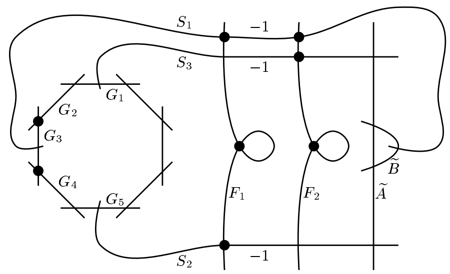

FIGURE 3. A rational elliptic surface $Y' = \mathbb{P}^2 \sharp 9\overline{\mathbb{P}}^2$

curve e_i in the total transform of F_i for $i = 1, 2$. Then we get a rational surface $Z' := Y' \sharp 10\overline{\mathbb{P}}^2$ which contains three disjoint configurations: $C_{9,5} = \overset{-2}{\circ} - \overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$ (which consists of $\tilde{e}_1, \tilde{F}_1, \tilde{S}_3, G_1, \tilde{G}_2$), $C_{9,5} = \overset{-2}{\circ} - \overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$ (which consists of $\tilde{e}_2, \tilde{F}_2, \tilde{S}_2, G_5, \tilde{G}_4$) and $D_{2,3} = \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-4}{\circ}$ (which consists of $\tilde{B}, \tilde{S}_1, \tilde{G}_3$) (Figure 4). Note that the boundary of a configuration $C_{9,5}$ is a lens space $L(81, -44)$ which also bounds a rational ball $B_{9,5}$, and the boundary of a configuration $D_{2,3}$ is a lens space $L(18, -11)$ which bounds a Milnor fiber $M_{2,3}$.

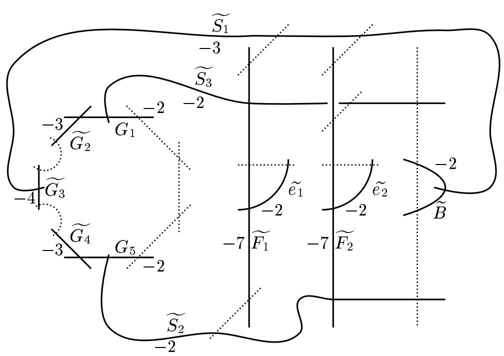

FIGURE 4. A rational surface $Z' = Y' \sharp 10\overline{\mathbb{P}}^2$

Then we contract these three disjoint chains $C_{9,5}$, $C_{9,5}$, $D_{2,3}$ of $\mathbb{P}^1$'s from Z' so that it produces a normal projective surface, denoted by X, with three permissible singularities. Using the same technique as in [4], we are able to prove that X has a $\mathbb{Q}$-Gorenstein smoothing and a general fiber X_t of the $\mathbb{Q}$-Gorenstein smoothing is a minimal complex surface of general type with $p_g = 0$ and $K^2 = 2$. Furthermore, the general fiber X_t is diffeomorphic to a rational blow-down 4-manifold $Z'_{9,9,2}$ which is obtained from $Z' = Y' \sharp 10\overline{\mathbb{P}}^2$ by replacing three disjoint configurations $C_{9,5}$, $C_{9,5}$ and $D_{2,3}$ with corresponding Milnor fibers $B_{9,5}$, $B_{9,5}$ and $M_{2,3}$ respectively. Finally, using a similar technique in Section 2, it is easy to prove that the rational blow-down 4-manifold $Z'_{9,9,2}$ has $H_1(Z'_{9,9,2}; \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$, which is Corollary 1.

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