COMPOSITION WITH A TWO VARIABLE FUNCTION

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1. Introduction

In [11] and [12], A. Némethi studied the Milnor fiber and monodromy zeta function of composed functions of the form $f(g_1, g_2)$ with f a two variable polynomial and g_1 and g_2 polynomials with distinct sets of variables. The present paper addresses the question of proving similar results for the motivic Milnor fiber introduced by Denef and Loeser, cf. [1],[3],[10],[4]. In fact, Némethi later considered in [13] the more general situation of a composition $f \circ \mathbf{g} \colon (X, x) \to (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, where $\mathbf{g}$ has a reasonable discriminant. Still later, Némethi and Steenbrink [14] proved similar results at the level of the Hodge spectrum [16],[17],[18], using the theory of mixed Hodge modules. In particular, they were able to compute, under mild assumptions, the Hodge spectrum of composed functions of the form $f(g_1, g_2)$ without assuming the variables in g_1 and g_2 are distinct. Their result involves the discriminant of the morphism $\mathbf{g} = (g_1, g_2)$. In a previous paper [7], we computed the motivic Milnor fiber for functions of the form $g_1 + g_2^\ell$ when ℓ is large without assuming the variables in g_1 and g_2 are distinct. The corresponding result for the Hodge spectrum goes back to M. Saito [15] and is a special case of the results of Némethi and Steenbrink [14]. So, it seems very natural to search for a full motivic analogue of the results of [14]. At the present time, we are unable to realize this program and we have to limit ourself, as we already mentioned, to the case when g_1 and g_2 have no variable in common. Already extending our result to the case when one only assumes the discriminant of the morphism $\mathbf{g}$ is contained in the coordinate axes seems to require new ideas.

In this paper we consider a polynomial f in $k[x, y]$ and we assume that $f(0, y)$ is non zero of degree m. We denote by i_p the closed embedding into $\mathbb{A}_k^2$ of a point p in $F_0 = f^{-1}(0) \cap x^{-1}(0)$. We consider the motivic Milnor fiber $\mathcal{S}_f$ of the function $f \colon \mathbb{A}_k^2 \longrightarrow \mathbb{A}_k^1$ whose restriction $i_p^* \mathcal{S}_f$ above p is an element of the Grothendieck ring $\mathcal{M}_{\mathbb{G}_m}^{\mathbb{G}_m}$. We then reformulate Guibert's computation of the motivic Milnor fiber of germs of plane curve singularities [6] using generalized convolution operators of [8]. More precisely, we express it in terms of the tree $\tau(f, p)$ associated to f, depending on the given coordinate system (x, y) on the affine plane $\mathbb{A}_k^2$. Let us recall the tree $\tau(f, p)$ is obtained by considering the Puiseux expansions of the roots of f at p, cf. [9],[5]. To any so-called rupture vertex v of this graph, we attach a weighted homogeneous polynomial $Q_{v,f}$ in $k[c, d]$. We have defined in [8] a generalized convolution by such

a polynomial. It is a morphism from $\mathcal{M}^{\mathbb{G}_m}_{\mathbb{G}_m \times \mathbb{G}_m}$ to $\mathcal{M}^{\mathbb{G}_m}_{\mathbb{G}_m}$, but can be extended to a morphism from $\mathcal{M}^{\mathbb{G}_m}_{\mathbb{A}^1_k \times \mathbb{G}_m}$ to $\mathcal{M}^{\mathbb{G}_m}_{\mathbb{G}_m}$. We denote by ϖ_j the morphism $x \longmapsto x^j$ from $\mathbb{G}_m$ to $\mathbb{G}_m$ and by m_p the order of p as a root of $f(0, y)$. One can then reformulate Guibert's theorem as

$$(*) \qquad i_p^* \mathcal{S}_f = [\varpi_{m_p} : \mathbb{G}_m \longrightarrow \mathbb{G}_m] - \sum_v \Psi_{Q_{v,f}}([\mathrm{Id} : \mathbb{A}^1_k \times \mathbb{G}_m \longrightarrow \mathbb{A}^1_k \times \mathbb{G}_m])$$

where the sum runs over the set of rupture vertices of $\tau(f, p)$.

For $1 \leq j \leq 2$, let $g_j : X_j \longrightarrow \mathbb{A}^1_k$ be a function on a smooth k-variety X_j. By composition with the projection, g_j becomes a function on the product $X = X_1 \times X_2$ and we write $\mathbf{g}$ for the map $g_1 \times g_2 : X \to \mathbb{A}^2_k$. The main result of this paper, Theorem 4.2, gives a formula for $i^* \mathcal{S}_{f \circ \mathbf{g}}$, where i denote the inclusion of $g_1^{-1}(0) \cap g_2^{-1}(0)$, similar to $(*)$, with $[\mathrm{Id} : \mathbb{A}^1_k \times \mathbb{G}_m \longrightarrow \mathbb{A}^1_k \times \mathbb{G}_m]$ replaced by a virtual object A_v. The virtual object A_v is defined inductively in terms of the tree associated to f at the origin of $\mathbb{A}^2_k$ and of A_{v_0}, where v_0 is the first (extended) rupture vertex of $\tau(f, p)$, and A_{v_0} depends only on $\mathbf{g}$.

2. Preliminaries and combinatorial set up

2.1. We fix an algebraically closed field k of characteristic 0. For a variety X over k, we denote by $\mathcal{L}(X)$ and $\mathcal{L}_n(X)$ the spaces of arcs, resp. arcs mod t^{n+1} as defined in [2]. As in [7], we denote by $\mathcal{M}_X$ the localisation of the Grothendieck ring of varieties over X with respect to the class of the relative line. We shall also use the $\mathbb{G}_m$-equivariant variant $\mathcal{M}^{\mathbb{G}_m}_{X \times \mathbb{G}^p_m}$ defined in [8], which is generated by classes of objects $Y \to X \times \mathbb{G}^p_m$ endowed with a monomial $\mathbb{G}_m$-action.

Also, if p is a closed point of X we denote by i_p the inclusion $i_p : p \to X$ and by i_p^* the corresponding pullback morphism at the level of rings $\mathcal{M}$.

2.2. Let us start by recalling some basic constructions introduced by Denef and Loeser in [1], [4] and [3].

Let X be a smooth variety over k of pure dimension d and $g : X \to \mathbb{A}^1_k$. We set $X_0(g)$ for the zero locus of g, and consider, for $n \geq 1$, the variety

$$(2.2.1) \qquad \mathcal{X}_n(g) := \left\{ \varphi \in \mathcal{L}_n(X) \ \middle| \ \mathrm{ord}_t g(\varphi) = n \right\}.$$

Note that $\mathcal{X}_n(g)$ is invariant by the $\mathbb{G}_m$-action on $\mathcal{L}_n(X)$. Furthermore g induces a morphism $g_n : \mathcal{X}_n(g) \to \mathbb{G}_m$, assigning to a point φ in $\mathcal{L}_n(X)$ the coefficient $\mathrm{ac}(g(\varphi))$ of t^n in $g(\varphi)$, which we shall also denote by $\mathrm{ac}(g)(\varphi)$. This morphism is homogeneous of weight n with respect to the $\mathbb{G}_m$-action on $\mathcal{X}_n(g)$ since $g_n(a \cdot \varphi) = a^n g_n(\varphi)$, so we can consider the class $[\mathcal{X}_n(g)]$ of $\mathcal{X}_n(g)$ in $\mathcal{M}^{\mathbb{G}_m}_{X_0(g) \times \mathbb{G}_m}$.

We now consider the motivic zeta function

$$(2.2.2) \qquad Z_g(T) := \sum_{n \geq 1} [\mathcal{X}_n(g)] \, \mathbb{L}^{-nd} \, T^n$$

in $\mathcal{M}^{\mathbb{G}_m}_{X_0(g) \times \mathbb{G}_m}[[T]]$. Note that $Z_g = 0$ if $g = 0$ on X.

Denef and Loeser showed in [1] and [3] (see also [4]) that $Z_g(T)$ is a rational series by giving a formula for $Z_g(T)$ in terms of a resolution of f. They also showed that one can consider $\lim_{T \mapsto \infty} Z_g(T)$ in $\mathcal{M}^{\mathbb{G}_m}_{X_0(g) \times \mathbb{G}_m}$ and they define the motivic Milnor fiber of g as

$$(2.2.3) \qquad\qquad \mathcal{S}_g := - \lim_{T \mapsto \infty} Z_g(T).$$

2.3. In this subsection we do not assume X to be smooth. For technical reasons we shall use in the present paper the following innocuous variant of $\mathcal{M}^{\mathbb{G}_m}_{X \times \mathbb{G}^p_m}$: replacing everywhere in the definition the first $\mathbb{G}_m$-factor endowed with the $\mathbb{G}_m$-action by multiplicative translation $\lambda \cdot x = \lambda x$ by $\mathbb{A}^1_k$ with "the same" $\mathbb{G}_m$-action one gets a ring $\mathcal{M}^{\mathbb{G}_m}_{X \times \mathbb{A}^1_k \times \mathbb{G}^{p-1}_m}$ generated by classes of objects $Y \to X \times \mathbb{A}^1_k \times \mathbb{G}^{p-1}_m$ endowed with a monomial $\mathbb{G}_m$-action.

If Q is a quasihomogeneous polynomial in p variables, we defined in [8] a convolution operator

$$\Psi_Q : \mathcal{M}^{\mathbb{G}_m}_{X \times \mathbb{G}^p_m} \longrightarrow \mathcal{M}^{\mathbb{G}_m}_{X \times \mathbb{G}_m}.$$

In this paper we shall use the slight variant, still denoted by Ψ_Q, which is obtained with the same definition, replacing $\mathbb{G}^p_m$ by $\mathbb{A}^1_k \times \mathbb{G}^{p-1}_m$,

$$\Psi_Q : \mathcal{M}^{\mathbb{G}_m}_{X \times \mathbb{A}^1_k \times \mathbb{G}^{p-1}_m} \longrightarrow \mathcal{M}^{\mathbb{G}_m}_{X \times \mathbb{G}_m}.$$

In fact, such constructions carry over for any toric variety with torus $\mathbb{G}^p_m$, not only for $\mathbb{A}^1_k \times \mathbb{G}^{p-1}_m$.

2.4. Fix a positive integer N and consider the ring of fractional power series $k[[x^{\frac{1}{N}}]]$. Given a positive rational number r we denote by $I_{\geq r}$ the ideal of power series of order at least r in $k[[x^{\frac{1}{N}}]]$. We call the quotient $k[[x^{\frac{1}{N}}]]/I_{\geq r}$ the ring of r-truncated fractional power series.

To a r-truncated fractional power series y one assigns a labelled rooted real metric tree $\tau_r(y)$ in the following way. The total space $\tau_r(y)$ is the half-open interval $[0, r)$ and its vertices are the positive exponents with non zero coefficients of the expansion of y in powers of x together with the origin which is the root. We define the *height* of a vertex to be its distance to the root. We label each vertex by the coefficient of the corresponding term (this coefficient is non-zero for all vertices except maybe for the root). The vertices are ordered by the height. Starting above a vertex there is only one edge. This edge ends with the next vertex if there is one and remains open above the last vertex. We label each edge by 0 and we say $\tau_r(y)$ is of height r. We denote by $|\tau_r(y)|$ the underlying unlabelled tree. Notice that we can see the labels as degree 1 polynomials of $k[X]$ (or cycles in $\mathbb{A}^1_k$), that is, X for an edge and $X - a$ for a vertex labelled by a.

If now y is a power series in $k[[x^{\frac{1}{N}}]]$, we denote by $\tau_r(y)$ the height r tree associated to its truncation at order r. Thus, for $r < r'$, $\tau_r(y)$ is obtained from $\tau_{r'}(y)$ by truncating up to height r. We denote by $\tau(y)$ the inductive limit of the system $(\tau_r(y))_{r \in \mathbb{Q}}$ and call it the tree associated to the power series y.

2.5. We consider a two variable polynomial f in $k[x, y]$. We assume that $f(0, y)$ is non zero of degree m and we consider the m Newton-Puiseux expansions y_i, $1 \leq i \leq m$, associated to f at the points of $f^{-1}(0) \cap x^{-1}(0)$. There exists an integer N such that these roots are elements of the ring of fractional power series $k[[x^{\frac{1}{N}}]]$, namely, they are the roots of the polynomial f in $k[[x^{\frac{1}{N}}]]$.

Fix a positive rational number r. We denote by $\bigcup_{i=1}^m \tau_r(y_i)$ the labelled rooted real metric tree which is obtained as follows. In the disjoint union of the trees $\coprod_{i=1}^m |\tau_r(y_i)|$ we identify two vertices (resp. two edges) if they have same height and same label. If v is a vertex (resp. an edge) shared by trees $\tau_r(y_i)$ for i in J, then its label on the union is the $|J|$-th power of its label on any of the $\tau_r(y_i)$, i in J.

Since the group of N-roots of unity acts on the r-truncated expansions $(y_1, \ldots, y_m)$, it also acts on $|\bigcup_{i=1}^m \tau_r(y_i)|$. We denote by $|\tau_r(f)|$ the separated quotient and by $\pi : |\bigcup_{i=1}^m \tau_r(y_i)| \to |\tau_r(f)|$ the quotient morphism. Note that the connected components of $|\tau_r(f)|$ are in natural bijection with points of $f^{-1}(0) \cap x^{-1}(0)$. For any such point p, we denote by $|\tau(f, p)|$ the corresponding connected component which is naturally endowed with the structure of a rooted real metric tree. We attach labels to the vertices and edges of $|\tau_r(f)|$ in the following way:

- If e is an edge of $|\tau_r(f)|$, the label attached to e is the label on any element of $\pi^{-1}(e)$. It is a power of X in $k[X]$ and we denote it by $P_{e,f}$. We will call *degree of the edge* e the degree of $P_{e,f}$.
- If v is a vertex of $|\tau_r(f)|$, the label on v is the product of the labels on $\pi^{-1}(v)$. We denote it by $P_{v,f}$. We will call *degree of the vertex* v the degree of $P_{v,f}$. Notice that the degree of a vertex v is equal to the degree of the edge e which ends in v.

For $r < r'$, the graph $\tau_r(f)$ is the truncation of $\tau_{r'}(f)$ at height r. The *graph of contacts* $\tau(f)$ defined by f along $f^{-1}(0) \cap x^{-1}(0)$ is the inductive limit of the graphs $\tau_r(f)$, $r \in \mathbb{Q}$, cf. [9], [5]. We say that a vertex v of $\tau(f)$ is a *rupture vertex* if the set of zeroes of $P_{v,f}$ contains at least two points in $\mathbb{A}_k^1$. We define the augmented set of rupture vertices of the tree $\tau(f, p)$ as the set of rupture vertices of $\tau(f, p)$ together with the vertex of minimal non zero height on $\tau(f, p)$.

We fix from now on a point p which will be assumed for simplicity to be the origin in $\mathbb{A}_k^2$. For any arc φ in $\mathcal{L}(\mathbb{A}_k^2)$ such that $\varphi(0) = p$ and $x(\varphi) \neq 0$, there exist power series ω in $k[[t]]$ and $\sum_j b_j \omega^j$ in $k[[\omega]]$, and an integer M such that $\gcd(M, \{j \mid b_j \neq 0\}) = 1$ and

$$
\begin{aligned}
x(\varphi(t)) &= \omega(t)^M \\
y(\varphi(t)) &= \sum_j b_j \, \omega(t)^j.
\end{aligned}
$$

Hence

$$
y(\varphi(t)) = \sum_j b_j (x(\varphi(t))^{\frac{j}{M}}
$$

is a fractional power series in $x(\varphi(t))$. We consider the tree $\tau(y)$ with

$$
y(x) := \sum_j b_j x^{\frac{j}{M}}
$$

in $k[[x^{\frac{1}{M}}]]$.

The power series ω is defined up to an M-rooth of unity so that the b_j's are defined up to a factor ζ^j with ζ an M-root of unity. Two different choices lead to trees in

the same μ_N-orbit. This orbit is denoted by $\tau(\varphi)$. Notice that $\tau(\varphi)$, as well as $\tau(f)$, depends on the system of coordinates (x, y).

2.6. Definition. Consider φ in $\mathcal{L}(\mathbb{A}_k^2)$ and f in $k[x, y]$ as before. The *order of contact* of φ with f is the maximum number s in $\mathbb{Q} \cup \{\infty\}$ such that $\tau_s(\varphi)$ is included in $\tau_s(f)$ (it is infinite if and only if $f(\varphi) = 0$). The *contact* of φ with f is the tree $\tau_r(\varphi)$ where r is the order of contact of φ with f.

2.7. From now on the polynomial f is fixed in $k[x, y]$ and we denote by m the degree of $f(0, y)$. For a positive rational number r, by a contact τ of order r, we mean a subtree of $\tau(f, p)$ which is isomorphic to $[0, r)$. In particular τ is rooted at p and its closure in $\tau(f, p)$ contains a unique point of height r, not necessarily a vertex of $\tau(f, p)$, which completely determines τ. To such a contact τ we assign a polynomial $P_{\tau, f}$ in the following way. The last and (semi)open edge of τ is contained in a unique edge e of $\tau(f, p)$.

- If e ends at a vertex v at height r of $\tau(f, p)$ (in this case we say that τ ends at the vertex v), we will set $P_{\tau, f} = P_{\tau, v}$.
- Otherwise, (in that case we say that τ ends at the edge e) we set $P_{\tau, f} = P_{\tau, e}$.

By definition of contact, there is an integer M and a polynomial y_τ in $k[\omega]$, of degree strictly smaller than rM, both depending only on τ, such that for any arc φ of contact τ with f, there exists a series ω in $k[[t]]$, $\mathrm{ord}_t(\omega) = \ell$, such that

$$
\begin{aligned}
x(\varphi(t)) &= \omega(t)^M \\
y(\varphi(t)) &= y_\tau(\omega(t)) \quad [\mathrm{mod}(t^{\lceil rM\ell \rceil})].
\end{aligned}
$$

For an arc φ of contact τ with f, the quotient $\mathrm{ord}_t(f(\varphi))/\ell$ is an integer and depends only on τ. We denote it by $\nu(\tau)$. One always has the inequality $\nu(\tau) \geq Mr$.

The tree $\tau(f, p)$ is built from the Puiseux expansions of the m roots of $f(x, y)$ in the ring of fractional power series $\bigcup_N k[[x^{1/N}]]$. Conversely, to any finite subtree ς of $\tau(f, p)$, we can associate a polynomial f_ς in $k[x, y]$ which is the minimal polynomial of the m Puiseux expansions restricted to ς. Considering the tree associated to the polynomial f_ς, we get a tree $\tau(f_\varsigma, p)$ which is an infinite tree with a finite number of vertices. The intersection of $\tau(f_\varsigma, p)$ with $\tau(f, p)$ contains ς. As an example, we can consider the tree τ_r obtained from $\tau(f, p)$ by truncation at height r. We will denote by $\overline{\tau_r}$ the tree $\tau(f_{\tau_r}, p)$.

3. Guibert's theorem revisited

3.1. We consider the following set of arcs:

$$
\mathcal{X}_{\tau, \ell} := \left\{ \varphi \in \mathcal{L}_{\bar{\nu}(\tau)\ell}(\mathbb{A}_k^2) \;\middle|\; \varphi \text{ has contact } \tau \text{ with } f, \; \mathrm{ord}_t x(\varphi) = M\ell \right\}.
$$

where $\bar{\nu}(\tau)$ is the maximum of the integers $\nu(\tau)$ and M.

We denote by $Q_{\tau, f}$ the function $(\omega, c) \mapsto \omega^{\nu(\tau)} P_{\tau, f}(\omega^{-Mr} c)$. One should note that $Q_{\tau, f}$ is a polynomial in $k[c, \omega]$, even if Mr may not be an integer.

3.2. Lemma. *Consider a contact τ and an integer ℓ and denote by $N(\tau, \ell)$ the integer $2\bar{\nu}(\tau)\ell - M\ell - \lfloor Mr\ell \rfloor$. For any arc φ in $\mathcal{X}_{\tau, \ell}$, there exist two series ω and ε in $k[t]/t^{\bar{\nu}(\tau)\ell + 1}$ such that*

(1) $\mathrm{ord}_t(\omega) = \ell$, $x(\varphi) = \omega^M$

(2) $\mathrm{ord}_t(\varepsilon) \geq Mr\ell$ *(resp. $= Mr\ell$ if τ ends in an edge),* $y(\varphi) = y_\tau(\omega) + \varepsilon$.
The mapping $(\varepsilon, \omega) \longmapsto (\omega^M, y_\tau(\omega) + \varepsilon)$ *induces an isomorphism*

$$\Phi : ((\mathbb{A}^1_k \times \mathbb{G}_m) \setminus Q^{-1}_{\tau,f}(0)) \times \mathbb{A}^{N(\tau,\ell)}_k \longrightarrow \mathcal{X}_{\tau,\ell}$$

sending (c, ω_ℓ, a) *to*

$$\Big(t^{\ell M}(\omega_\ell + \sum_{k=1}^{\ell(\bar{\nu}(\tau)-M)} a_k t^k)^M [t^{\ell\bar{\nu}(\tau)+1}], \, y_\tau(\omega) + ct^{Mr\ell} + \sum_{\ell(\bar{\nu}(\tau)-M)<k\leq\ell\bar{\nu}(\tau)} a_k t^k [t^{\ell\bar{\nu}(\tau)+1}] \Big).$$

Via the isomorphism Φ, the angular coefficient $\mathrm{ac}(f(\varphi))$ is given, up to a non-zero constant, by the following formula:

$$\mathrm{ac}(f(\varphi)) \sim \omega_\ell^{\nu(\tau)} P_{\tau,f}(\omega_\ell^{-Mr} c) = Q_{\tau,f}(c, \omega_\ell).$$

For the $\mathbb{G}_m$-action σ on $(\mathbb{A}^1_k \times \mathbb{G}_m)$ given by $\sigma(\lambda) \cdot (c, \omega_\ell) = (\lambda^{Mr\ell} c, \lambda^\ell \omega_\ell)$, the polynomial $Q_{\tau,f}$ is homogeneous of degree $\nu(\tau)\ell$.

Proof. We did already notice that the map Φ is surjective. Conversely ω is determined by $x(\varphi)$ up to a M-th root of unity, and uniquely determined by $x(\varphi)$ and $y(\varphi)$ for the *gcd* of M and exponents of non zero terms in $y_\tau(\omega)$ is equal to 1. $\square$

3.3. On the constructible set $\mathcal{X}_{\tau,\ell}$, via the isomorphism Φ, we have a morphism to $\mathbb{A}^1_k \times \mathbb{G}_m$ induced by the first projection from $(\mathbb{A}^1_k \times \mathbb{G}_m) \times \mathbb{A}^{N(\tau,\ell)}_k$. The constructible set $\mathcal{X}_{\tau,\ell}$ defines a class in $\mathcal{M}^{\mathbb{G}_m}_{\mathbb{A}^1_k \times \mathbb{G}_m}$ we denote by $[\mathcal{X}_{\tau,\ell}]$. On the other hand, the function $\mathrm{ac}(f)$ induces a $\mathbb{G}_m$-equivariant morphism from $\mathcal{X}_{\tau,\ell}$ to $\mathbb{G}_m$, hence defines a class in $\mathcal{M}^{\mathbb{G}_m}_{\mathbb{G}_m}$ we denote by $[\mathcal{X}_{\tau,\ell}(f)]$. By Lemma 3.2, the morphism $\mathcal{X}_{\tau,\ell} \to \mathbb{G}_m$ is equal to the composition of the morphism $\mathcal{X}_{\tau,\ell} \to \mathbb{A}^1_k \times \mathbb{G}_m$ with $Q_{\tau,f}$.

3.4. For any rupture vertex v, there is only one contact ending in v that we denote by τ_v. We set $Q_{v,f} := Q_{\tau_v,f}$.

We are now in position to restate Guibert's theorem [6] in the following form:

3.5. Theorem (Guibert)**.** *With the above notation, the following holds:*

$$i_p^* \mathcal{S}_f = [\varpi_{m_p} : \mathbb{G}_m \longrightarrow \mathbb{G}_m] - \sum_v \Psi_{Q_{v,f}}([\mathrm{Id} : \mathbb{A}^1_k \times \mathbb{G}_m \longrightarrow \mathbb{A}^1_k \times \mathbb{G}_m])$$

where the second sum runs over the rupture vertices of $\tau(f)$ above p.

Proof. Note that for a two variable quasihomogeneous polynomial Q

$$(3.5.1) \quad \Psi_Q([\mathrm{Id} : \mathbb{A}^1_k \times \mathbb{G}_m \longrightarrow \mathbb{A}^1_k \times \mathbb{G}_m]) =$$
$$- [Q : (\mathbb{A}^1_k \times \mathbb{G}_m) \setminus Q^{-1}(0) \longrightarrow \mathbb{G}_m] + \mathcal{S}_Q([\mathbb{A}^1_k \times \mathbb{G}_m])$$

where $\mathcal{S}_Q$ is defined as in [7], [8]. We denote by π_E the morphism $(a, b) \longmapsto a^E$ from $\mathbb{G}_m \times \mathbb{G}_m$ to $\mathbb{G}_m$. When the zeroes of Q are a disjoint union of one dimensional $\mathbb{G}_m$-orbits, $\mathcal{S}_Q(\mathbb{A}^1_k \times \mathbb{G}_m)$ decomposes into a sum

$$\mathcal{S}_Q(\mathbb{A}^1_k \times \mathbb{G}_m) = - \sum_i [\pi_{E_i} : \mathbb{G}_m \times \mathbb{G}_m \longrightarrow \mathbb{G}_m]$$

where E_i is the multiplicity of Q along the i-th component of $Q^{-1}(0)$. If v is a rupture vertex we consider the following zeta function:

$$Z_f^v(T) := \sum_{\ell \geq 1} \sum_\tau [\mathcal{X}_{\tau,\ell}(f)] \, \mathbb{L}^{-2\nu(\tau)\ell} \, T^{\nu(\tau)\ell}$$

where the second sum is extended to the set of contacts τ which contain τ_v and do not contain or end in any successor of v in the set of rupture vertices. From [6] (3.3) and (5.2), we deduce that $Z_f^v(T)$ has a limit $-\mathcal{S}_f^v$ in the Grothendieck ring $\mathcal{M}_{\mathbb{G}_m}^{\mathbb{G}_m}$ when T goes to infinity, which is given by the formula

$$\mathcal{S}_f^v = -\Psi_{Q_{v,f}}([\mathrm{Id} : \mathbb{A}_k^1 \times \mathbb{G}_m \longrightarrow \mathbb{A}_k^1 \times \mathbb{G}_m]).$$

We consider now the series

$$Z_f^p(T) := \sum_{\ell \geq 1} \sum_\tau [\mathcal{X}_{\tau,\ell}(f)] \, \mathbb{L}^{-2\nu(\tau)\ell} \, T^{\nu(\tau)\ell}$$

where the second sum is extended to the set of contacts τ starting from the root corresponding to p, which do not contain or end in any rupture vertex. Again by [6], loc. cit., $Z_f^p(T)$ has limit

$$-[\varpi_{m_p} : \mathbb{G}_m \longrightarrow \mathbb{G}_m]$$

when T goes to infinity. The restriction $i_p^* \mathcal{S}_f$ is the limit as $T \mapsto \infty$ of $-i_p^* Z_f(T)$ which decomposes into

$$-i_p^* Z_f(T) = -Z_f^p(T) - \sum_v Z_f^v(T)$$

where the sum extends to all the rupture vertices of $\tau(f)$ above p. The result follows.

□

4. Composition with a morphism

4.1. For $1 \leq j \leq 2$, let $g_j : X_j \longrightarrow \mathbb{A}_k^1$ be a function on a smooth k-variety X_j. By composition with the projection, g_j becomes a function on the product $X = X_1 \times X_2$. We write d_j for the dimension of X_j, j from 1 to 2, and d for $d_1 + d_2$. Define $\mathbf{g}$ as the map $g_1 \times g_2$ on X and G as the product $G = g_1 g_2$. For any subvariety Z of the set $X_0(G)$ containing $X_0(\mathbf{g}) := g_1^{-1}(0) \cap g_2^{-1}(0)$, we denote by i the closed immersion of $X_0(\mathbf{g})$ in Z.

As in section 2, we denote by f a two variable polynomial, we assume that $f(0, y)$ is a nonzero polynomial, we denote by p the origin and will denote by m_p the order of 0 as a root of $f(0, y)$. We consider the augmented set of rupture vertices of the tree $\tau(f, p)$, namely the set of rupture vertices together with the vertex of minimal non zero height on $\tau(f, p)$. Denote that vertex by v_0 and consider its associated polynomial Q_{v_0}. We denote by $\mathcal{S}'_{g_2}$ the element in $\mathcal{M}_{X_0(g_2) \times \mathbb{A}_k^1}^{\mathbb{G}_m}$ which is the "disjoint sum" of $\mathcal{S}_{g_2}$ in $\mathcal{M}_{X_0(g_2) \times \mathbb{G}_m}^{\mathbb{G}_m}$ and $X_0(g_2)$ in $\mathcal{M}_{X_0(g_2)}$. We set $A_{v_0} := \mathcal{S}'_{g_2} \boxtimes \mathcal{S}_{g_1}$, considered as an element in $\mathcal{M}_{X_0(\mathbf{g}) \times (\mathbb{A}_k^1 \times \mathbb{G}_m)}^{\mathbb{G}_m}$. For any rupture vertex v of the tree $\tau(f, p)$, we denote by $a(v)$ the predecessor of v in the augmented set of rupture vertices and we define by induction a virtual variety A_v in $\mathcal{M}_{X_0(\mathbf{g}) \times \mathbb{A}_k^1 \times \mathbb{G}_m}^{\mathbb{G}_m}$. We assume we are given a virtual variety $A_{a(v)}$ in $\mathcal{M}_{X_0(\mathbf{g}) \times \mathbb{A}_k^1 \times \mathbb{G}_m}^{\mathbb{G}_m}$ whose restriction over $X_0(\mathbf{g}) \times \mathbb{G}_m \times \mathbb{G}_m$ is diagonally monomial in the sense of [7] (2.3), or more precisely whose $\mathbb{G}_m$-action is

diagonally induced from a diagonally monomial $\mathbb{G}_m^2$-action in the sense of [7]. To any successor of $a(v)$ corresponds a factor of the polynomial $Q_{a(v)}$. Denote by $Q_{a(v)}^v$ the factor associated to v. Notice that $(Q_{a(v)}^v)^{-1}(0)$ is a smooth subvariety in $\mathbb{G}_m \times \mathbb{G}_m$ equivariant under a diagonal $\mathbb{G}_m$-action and that the second projection pr_2 of the product $\mathbb{A}_k^1 \times \mathbb{G}_m$ induces an homogeneous fibration from $(Q_{a(v)}^v)^{-1}(0)$ to $\mathbb{G}_m$. We denote by B_v the restriction of $A_{a(v)}$ above $(Q_{a(v)}^v)^{-1}(0)$. The external product of the identity of the affine line $\mathbb{A}_k^1$ by the induced map $\mathrm{pr}_2 : B_v \longrightarrow \mathbb{G}_m$ defines an element A_v in $\mathcal{M}_{X_0(\mathbf{g}) \times (\mathbb{A}_k^1 \times \mathbb{G}_m)}^{\mathbb{G}_m}$ which is diagonally monomial when restricted to $X_0(\mathbf{g}) \times \mathbb{G}_m \times \mathbb{G}_m$.

4.2. Theorem. *With the previous notations and hypotheses, the following formula holds*

$$(4.2.1) \qquad i^* \mathcal{S}_{f \circ \mathbf{g}} = \mathcal{S}_{(g_2)^{m_p}}(X_0(g_1)) - \sum_v \Psi_{Q_v}(A_v),$$

where the sum is runs over the augmented set of rupture vertices of the tree $\tau(f,p)$.

Proof. We first reduce to the case where $\tau(f,p)$ has only a finite number of vertices.

4.3. Lemma. *There exists a rational number γ such that, for any r greater than γ,*

$$(4.3.1) \qquad i^* \mathcal{S}_{f \circ \mathbf{g}} = i^* \mathcal{S}_{f_{\tau_r} \circ \mathbf{g}}.$$

Proof. Consider a rupture vertex v. The quotient $\mathrm{ord}_t(f \circ \mathbf{g}(\varphi))/\mathrm{ord}_t(g_1(\varphi))$ is an affine function of r whenever τ contains τ_v and does not contain or end in any rupture vertex greater than v. Hence the quotient $\mathrm{ord}_t(f \circ \mathbf{g}(\varphi))/\mathrm{ord}_t(g_1(\varphi))$ is a function on the tree $\tau(f)$, the restriction of which on each semi-open branch joining two consecutive rupture vertices (resp. on each infinite branch above a rupture vertex) is an increasing affine function of the height.

We consider the following set of arcs

$$\mathcal{X}_{n_1,n_2}(x \circ \mathbf{g}, f \circ \mathbf{g}) := \left\{ \varphi \in \mathcal{L}_{n_1+n_2}(X) \;\middle|\; \mathrm{ord}_t x \circ \mathbf{g} = n_1, \; \mathrm{ord}_t f \circ \mathbf{g} = n_2 \right\}.$$

For γ large enough, the zeta function

$$Z_{x \circ \mathbf{g}, f \circ \mathbf{g}}^{\gamma}(T) = \sum_{n_2 \geq \gamma n_1} [\mathcal{X}_{n_1,n_2}(x \circ \mathbf{g}, f \circ \mathbf{g})] \, \mathbb{L}^{-(n_1+n_2)d} \, T^{n_2}$$

goes to zero as T goes to infinity, cf. [7]. The lemma follows.   □

To prove the theorem, it is enough to consider the case when the tree $\tau(f,p)$ has a finite number of vertices. The proof goes by induction on the number of vertices of the tree $\tau(f,p)$. Certainly the result holds if there is no vertex. Assume first the tree has only one vertex v_0. The formula is then a particular case of the main formula in [8]. Assume now we have at least two vertices. Choose a maximal vertex v for the height function on $\tau(f,p)$ and consider the subtree τ^- obtained from $\tau(f,p)$ by deleting the vertex v. Denote by $a(v)$ the predecessor of v on $\tau(f,p)$ and by f^- the polynomial associated to τ^-, cf. 2.7.

Consider an arc φ in $\mathbb{A}_k^2$ with origin p. Then one of the following two statements holds:

- The contact of φ with f does not contain τ_v. Then $\mathrm{ord}_t(f(\varphi)) = \mathrm{ord}_t(f^-(\varphi))$ and $\mathrm{ac}(f(\varphi)) = \mathrm{ac}(f^-(\varphi))$.
- The contact of φ with f contains τ_v.

According to these two different cases, we can split the zeta function $Z_{f \circ \mathbf{g}}$ in two pieces, namely

$$(4.3.2) \qquad Z_{f \circ \mathbf{g}} = Z_{<v} + Z_{\geq v},$$

Similarly, the zeta function $Z_{f^- \circ \mathbf{g}}$ decomposes in

$$(4.3.3) \qquad Z_{f^- \circ \mathbf{g}} = Z_{<v}^- + Z_{\geq v}^-.$$

We noticed that $Z_{<v}^- = Z_{<v}$, hence we get

$$(4.3.4) \qquad Z_{f \circ \mathbf{g}} - Z_{f^- \circ \mathbf{g}} = Z_{\geq v} - Z_{\geq v}^-.$$

In section 3.3, for any contact τ and integer ℓ, we have considered a set $\mathcal{X}_{\tau,\ell}$ associated to a polynomial f. Similarly we have a set $\mathcal{X}_{\tau,\ell}^-$ associated to f^-. These two sets map to $\mathbb{A}_k^1 \times \mathbb{G}_m$. Consider now the inverse image by $\mathbf{g}$ of $\mathcal{X}_{\tau,\ell}$ (resp. $\mathcal{X}_{\tau,\ell}^-$) and denote it by $\mathcal{X}_{\tau,\ell}(\mathbf{g})$ (resp. $\mathcal{X}_{\tau,\ell}^-(\mathbf{g})$). We assume, by induction, that the motivic nearby cycles of f^- have the given form and that for any contact τ greater than $\tau_{a(v)}$ the set $\mathcal{X}_{\tau,\ell}^-(\mathbf{g})$ is a piecewise affine bundle on $X_0(\mathbf{g}) \times \mathbb{G}_m \times B_{a(v)}$.

An arc $\mathbf{g} \circ \varphi$ in $\mathcal{L}_{\nu(\tau)\ell}(\mathbb{A}_k^2)$ having contact τ with f^- has contact τ_v with f if and only if $\tau = \tau_v$ and $Q_{a(v)}^v$ does not vanish on φ or if τ contains strictly τ_v. In that case φ maps to $\{0\} \times B_{a(v)}$. Hence the set $\mathcal{X}_{\tau_v,\ell}(\mathbf{g})$ is a disjoint union of piecewise affine bundles on $X_0(\mathbf{g}) \times ((\mathbb{A}_k^1 \times B_{a(v)}) \setminus B_v)$ and the function $\mathrm{ac}(f \circ \mathbf{g})$ is given by the composition of the canonical map with Q_v.

An arc $\mathbf{g} \circ \varphi$ in $\mathcal{L}_{\nu(\tau)\ell}(\mathbb{A}_k^2)$ has contact greater than τ_v with f if and only if it has contact τ_v with f^- and $Q_{a(v)}^v$ vanish on φ. Hence, for any contact τ greater than τ_v, the set $\mathcal{X}_{\tau,\ell}(\mathbf{g})$ is a disjoint union of piecewise affine bundles on $X_0(\mathbf{g}) \times \mathbb{G}_m \times B_v$ and the function $\mathrm{ac}(f \circ \mathbf{g})$ is given by the composition of the canonical map with the projection $X_0(\mathbf{g}) \times \mathbb{G}_m \times B_v \longrightarrow \mathbb{G}_m$.

We can compute the difference $Z_{\geq v} - Z_{\geq v}^-$ and check that it has limit $\Psi_{Q_v}(A_v)$ as T goes to infinity.

It is a consequence of the following lemma, which follows from direct computation, that only extended rupture vertices have a non zero contribution. $\square$

4.4. Lemma. *Consider a vertex v of $\tau(f,p)$ and assume it is not an extended rupture vertex. Then the polynomial Q_v is of the form: $Q_v(c,\omega) = ((c - \alpha\omega^R)\omega^N)^E$ where R, N and E are integers and α a non-zero constant. It defines a map from $\mathbb{A}_k^1 \times \mathbb{G}_m$ to $\mathbb{A}_k^1$ the zero set of which is isomorphic to $\mathbb{G}_m$. Then:*

- $\Psi_{Q_v}(A_v) = 0$ *in* $\mathcal{M}_{X_0(\mathbf{g}) \times \mathbb{G}_m}^{\mathbb{G}_m}$.
- *For the unique successor $s(v)$ of v, the equality*

$$A_{s(v)} = A_v$$

holds in $\mathcal{M}_{X_0(\mathbf{g}) \times \mathbb{A}_k^1 \times \mathbb{G}_m}^{\mathbb{G}_m}$.

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