

MONODROMY AND ISOTOPY OF MONOTONE LAGRANGIAN TORI.

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ABSTRACT. We define new Hamiltonian isotopy invariants for a 2-dimensional monotone Lagrangian torus embedded in a symplectic 4-manifold. We show that, in the standard symplectic \mathbb{R}^4 , these invariants distinguish a monotone Clifford torus from a Chekanov torus.

1. Introduction

This article concerns the Hamiltonian isotopy problem of *monotone* Lagrangian tori, which is a special case of the Lagrangian knot problem as formulated by Eliashberg and Polterovich in [5]. Two Lagrangian tori L_0, L_1 embedded in a symplectic 4-manifold (M, ω) are said to be *Hamiltonian isotopic* if there exists a smooth isotopy of Hamiltonian diffeomorphisms $\phi_t \in \text{Ham}(M)$ with compact support, $t \in [0, 1]$, $\phi_0 = \text{id}$, such that $\phi_1(L_0) = L_1$.

We study the monodromy group \mathcal{H}_L of the Hamiltonian self-isotopies of a monotone Lagrangian torus L . We define two new Hamiltonian isotopy invariants for L : the *twist number* $t(L) \in \mathbb{N} \cup \{0\}$ and the *spectrum* $s(L) \in \mathbb{N} \cup \{0\}$ of L . The twist number $t(L)$ is related to Dehn twists along a embedded curve with 0 Maslov number, while the spectrum $s(L)$ involves the Maslov number of the primitive integral 1-eigenvectors of involutions of \mathcal{H}_L (see Definition 2.9 and Proposition 2.10).

We then apply our construction to the cases when $L \subset \mathbb{R}^4$ is either a monotone Clifford torus $T_{b,b}$ or a Chekanov torus $T'_{0,b}$ (called *special torus* in [1]) in the standard symplectic 4-space \mathbb{R}^4 . We obtain the following:

Theorem 1.1. *Let $b > 0$. Let \mathcal{H}_b denote the Hamiltonian monodromy group of $T_{b,b}$, \mathcal{H}'_b the Hamiltonian monodromy group of $T'_{0,b}$. Then $\mathcal{H}_b \cong \mathbb{Z}_2 \cong \mathcal{H}'_b$ as abstract groups and hence $t(T_{b,b}) = 0 = t(T'_{0,b})$. However,*

$$s(T_{b,b}) = 2, \quad s(T'_{0,b}) = 1.$$

Hence $T_{b,b}$ and $T'_{0,b}$ are not Hamiltonian isotopic in \mathbb{R}^4 .

Thus our approach provides a new way to distinguish $T_{b,b}$ from $T'_{0,b}$ up to Hamiltonian isotopy.

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It should be pointed out that, the Hamiltonian non-isotopy between $T_{b,b}$ and $T'_{0,b}$ has been proved by Chekanov [1, 2]. Chekanov gave two proofs on this result. The first proof in [1] utilized the symplectic capacities introduced by Ekeland and Hofer [3, 4], whilst the second proof [2] employed pseudoholomorphic curves with boundaries [6]. Indeed, Chekanov dealt with Clifford tori and Chekanov tori in symplectic \mathbb{R}^{2n} with $n \geq 2$, and completely classified such tori in all \mathbb{R}^{2n} .

Comparing with Chekanov's proofs, our approach is more algebraic in nature, and seemingly simpler and more elementary. For technical simplicity, we do not venture into higher dimensional cases here. However, we expect that, given suitable generalization, invariants similar to $t(L)$ and $s(L)$ can be defined for monotone Lagrangian tori of general dimensions. We hope to come back to this topic later.

This paper is organized as follows: In Section 2 we start with some conditions on symplectic 4-manifolds, in order for the monotonicity of a Lagrangian torus L to be well-defined. We then proceed to define the Hamiltonian monodromy group \mathcal{H} of a monotone torus L and study its properties. Then follows the definition of the invariants $t(L), s(L)$. In Section 3 we determine the Hamiltonian monodromy group as well as the values of $t(L), s(L)$ for $L = T_{b,b}$ (Lemma 3.2) and $L = T'_{0,b}$ (Lemma 3.3). We end up this note with several open questions.

2. Hamiltonian monodromy of monotone Lagrangian tori

Let $L \hookrightarrow M$ be an embedded Lagrangian torus in a symplectic 4-manifold (M, ω) . One can endow the tangent bundle of M with an almost complex structure compatible with ω . This turns TM into a complex vector bundle of which the Chern classes depend only on ω . From now on, unless otherwise mentioned, we assume that M satisfies the following two conditions that (i) the first Chern class $c_1(M) = c_1(TM) \in H^2(M, \mathbb{Z})$ vanishes and (ii) $H^1(M, \mathbb{R}) = 0$.

That $c_1(M) = 0$ ensures that the Maslov class $\mu \in H^1(L, \mathbb{Z})$ is well-defined. There is a unique integer $m_L \geq 0$ such that $\mu(H_1(L, \mathbb{Z})) = m_L \mathbb{Z}$. We call m_L the *divisibility* of μ on L .

Near L the symplectic form ω is exact, i.e., there exists a 1-form λ defined on a tubular neighborhood U_L of L such that $\omega = d\lambda$. The pull-back 1-form $\iota^*\lambda \in \Omega^1(L)$ is closed, we denote its cohomology class in $H^1(L, \mathbb{R})$ as α . The class α is independent of the choice of λ due to the assumption that $H^1(M, \mathbb{R}) = 0$.

Definition 2.1. Assume that $\mu \neq 0$ and $\alpha \neq 0$. Then L is *monotone* if $\alpha = c\mu$ for some $c \in \mathbb{R} \setminus \{0\}$.

Let $\text{Ham}(M, L)$ denote the group of all symplectomorphisms $\phi : (M, \omega) \rightarrow (M, \omega)$ such that $\phi(L) = L$ and ϕ is the time one map of some time dependent Hamiltonian vector field on M , and the vector field has compact support.

A map $\phi \in \text{Ham}(M, L)$ induces an isomorphism on $H^1(L, \mathbb{Z}) \cong \mathbb{Z}^2$ and hence on $H^1(L, \mathbb{R})$, preserving both μ and α . If μ and α are \mathbb{R} -linearly independent then $\phi^* = \text{id}$ on both $H^1(L, \mathbb{Z})$ and $H^1(L, \mathbb{R})$. Below we consider the monotone case only. For $\phi \in \text{Ham}(M, L)$ we call the induced isomorphism $\phi_* : H_1(L, \mathbb{Z}) \rightarrow H_1(L, \mathbb{Z})$ the *monodromy* of ϕ .

Definition 2.2. We define the *Hamiltonian monodromy group* of $L \subset M$ (or $\text{Ham}(M, L)$) to be

$$\mathcal{H}_L = \{\phi_* \in \text{Isom}(H_1(L, \mathbb{Z})) \mid \phi \in \text{Ham}(M, L)\}.$$

Consider the subgroup of $\text{Isom}(H_1(L, \mathbb{Z}))$:

$$G_\mu := \{g \in \text{Isom}(H_1(L, \mathbb{Z})) \mid \mu \circ g = \mu\}.$$

Clearly \mathcal{H}_L is a subgroup of G_μ .

If we fix a basis for $H_1(L, \mathbb{Z})$ then the group $\text{Isom}(H_1(L, \mathbb{Z}))$ is identified with the integral general linear group

$$GL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1 \right\}.$$

In the following we fix a basis γ, σ for $H_1(L, \mathbb{Z})$ so that $\mu(\gamma) = \min\{\mu(\eta) \mid \eta \in H_1(L, \mathbb{Z}), \mu(\eta) > 0\}$, and $\sigma \in \ker \mu$. We identify an element $p\gamma + q\sigma$ of $H_1(L, \mathbb{Z})$ with the column vector $(p, q)^t \in \mathbb{Z}^2$. Then μ is identified with the row vector $(m_L, 0) \in \text{Hom}(H_1(L, \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}^2$ where $m_L = \mu(\gamma)$, and ϕ_* is identified with an element of $GL(2, \mathbb{Z})$ such that $\mu \circ \phi_* = \mu$. We have the following:

Proposition 2.3. *The group $G_\mu := \{g \in GL(2, \mathbb{Z}) \mid \mu \circ g = \mu\}$ is isomorphic to the infinite dihedral group $D_\infty := \langle f_0, f_1 \mid f_0^2 = e = f_1^2 \rangle = \mathbb{Z}_2 * \mathbb{Z}_2$ (see [7]), where e denotes the identity element.*

Proof. A direct computation (identifying μ with $(m_L, 0)$) shows that, with respect to the basis $\{\gamma, \sigma\}$ for $H_1(L, \mathbb{Z})$ as described above,

$$G_\mu = \left\{ \begin{pmatrix} 1 & 0 \\ k & \pm 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\}.$$

Let $f_k := \begin{pmatrix} 1 & 0 \\ k & -1 \end{pmatrix}$ and $g_k := \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$. We have

- (i). $f_k^2 = e$ and $g_k = g_1^k$ for $k \in \mathbb{Z}$, g_k has infinite order for $k \neq 0$,
- (ii). $g_1 = f_1 f_0$ and hence $g_{-1} = g_1^{-1} = f_0 f_1$,
- (iii). $f_{k+1} = g_k f_1 = (f_1 f_0)^k f_1 = f_1 g_{-k}$ and $f_{-k} = g_{-k} f_0 = (f_0 f_1)^k f_0 = f_0 g_k$ for $k \geq 0$.

Readers can check that G_μ is indeed freely generated by the two elements f_0 and f_1 of order 2, hence G_μ is isomorphic to the infinite dihedral group. \square

Remark 2.4. Geometrically g_k is a $(\pm k)$ -Dehn twist along the curve representing the class σ . The \pm -sign ambiguity is to the non-uniqueness of σ which is unique only up to signs.

Remark 2.5. Sometimes it is more convenient to consider a different basis $\{\gamma, \gamma' := \gamma + \sigma\}$ for $H_1(L, \mathbb{Z})$. The transformation matrix from basis $\{\gamma, \gamma'\}$ to basis $\{\gamma, \sigma\}$ is $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We use \sim to denote the matrix representation with respect to the basis $\{\gamma, \gamma'\}$. Then for $k \in \mathbb{Z}$,

$$\tilde{f}_k := T^{-1} f_k T = \begin{pmatrix} 1-k & 2-k \\ k & k-1 \end{pmatrix}, \quad \tilde{g}_k := T^{-1} g_k T = \begin{pmatrix} 1-k & -k \\ k & 1+k \end{pmatrix}.$$

Lemma 2.6. *Let f_k, g_k be as in the proof of Proposition 2.3. Then $f_k f_l = g_{k-l}$ and hence $f_k = g_{k-l} f_l$ for $k, l \in \mathbb{Z}$.*

Proof. By applying (iii) from the proof of Proposition 2.3, we have $f_k f_0 = g_k$ and $f_0 f_l = g_{-l}$ for $k, l \in \mathbb{Z}$. Hence $f_k f_l = (f_k f_0)(f_0 f_l) = g_k g_{-l} = g_{k-l}$. i.e., $f_k = g_{k-l} f_l$, for $k, l \in \mathbb{Z}$. \square

Lemma 2.6 in particular implies a different (but well-known) way of expressing $G_\mu \cong D_\infty$ by generators and relations:

$$(1) \quad G_\mu \cong D_\infty = \langle f, g \mid f^2 = e, f g f = g^{-1} \rangle.$$

Definition 2.7. An element $f \in G_\mu$ is called an *involution* if $f^2 = e$ and $f \neq e$, a *twist* if $\text{ord}(f) = \infty$.

Then, with the matrix representations adapted in the proof of Proposition 2.3, $f_k \in G_\mu$ is an involution for any $k \in \mathbb{Z}$, and g_k a twist for any $k \in \mathbb{Z} \setminus \{0\}$.

Lemma 2.8. *Let $\mathcal{H} \subset G_\mu$ be a subgroup of G_μ . We adapt the notations f_k, g_k from the proof of Proposition 2.3. Then we have the following classification of \mathcal{H} :*

- (i). $\mathcal{H} = \{e\}$.
- (ii). If $\mathcal{H} \neq \{e\}$ and \mathcal{H} has no involutions, then $\mathcal{H} = \langle g_d \rangle \cong \mathbb{Z}$ for some $d \in \mathbb{N}$.
- (iii). If \mathcal{H} contains exactly one involution say, f_k , then $\mathcal{H} = \langle f_k \rangle \cong \mathbb{Z}_2$.
- (iv). If \mathcal{H} contains two involutions, then there exist some $k, d \in \mathbb{N}$ such that $\mathcal{H} = \langle f_k, g_d \rangle \cong D_\infty$.

Proof. For computational simplicity, we adapt the matrix representations and notations from the proof of Proposition 2.3.

Assume that \mathcal{H} has no involutions. Then \mathcal{H} is a subgroup of the free abelian group $\langle g_1 \rangle$ generated by $g_1 \in G_\mu$. Hence $\mathcal{H} = \langle g_d \rangle$ for some $d \in \mathbb{N}$, $\mathcal{H} \cong \mathbb{Z}$.

Now suppose that \mathcal{H} contains exactly one involution say, f_k , for some $k \in \mathbb{Z}$. We claim that $\mathcal{H} = \{e, f_k\}$ and hence $\mathcal{H} \cong \mathbb{Z}_2$. Otherwise, we would have $g_l \in \mathcal{H}$ for some $l \in \mathbb{Z}$, $l \neq 0$. But then $f_{k+l} = g_l f_k \in \mathcal{H}$ by Lemma 2.6, which contradicts with our assumption on \mathcal{H} . Hence $\mathcal{H} = \{e, f_k\} \cong \mathbb{Z}_2$ for some $k \in \mathbb{Z}$.

Finally, assume that \mathcal{H} contains more than one involution. Let

$$d := \min\{|k - l| \mid f_k, f_l \in \mathcal{H}, f_k \neq f_l\} \in \mathbb{N}$$

Fix a number $k \in \mathbb{Z}$ such that $f_k, f_{k+d} \in \mathcal{H}$. We claim that \mathcal{H} is generated by f_k and f_{k+d} . More precisely we will show that

- (i). $g_m \in \mathcal{H}$ iff d divides m ,
- (ii). $f_n \in \mathcal{H}$ iff d divides $n - k$.

Assume that $m = ds$ for some $s \in \mathbb{Z}$. Then $g_m = g_{ds} = g_d^s = (f_{k+d} f_k)^s \in \mathcal{H}$. Similarly, assume that $n = k + dr$ for some $r \in \mathbb{Z}$. then $f_n = f_{k+dr} = g_{dr} f_k \in \mathcal{H}$. Conversely, assume that $g_m \in \mathcal{H}$ for some $m = ds + r \in \mathbb{Z}$ with $d, s, r \in \mathbb{Z}$, $0 < r < d$. Then $g_r = g_m g_{-ds} \in \mathcal{H}$ and $f_{k+r} = g_r f_k \in \mathcal{H}$. But then $|(k+r) - k| < d$, which contradicts with the minimality of d . Similarly, if $f_n \in \mathcal{H}$ for some $n \in \mathbb{Z}$, then $g_{n-k} = f_n f_k \in \mathcal{H}$, which implies that d divides $n - k$. This verifies the claim. So \mathcal{H} is freely generated by f_k, f_{k+d} , two elements of order 2. Hence \mathcal{H} is isomorphic to D_∞ . Since $g_d = f_{k+d} f_k$, \mathcal{H} is also generated by f_k and g_d . Finally, replacing f_k by $f_{k+sd} = g_{sd} f_k$ for some $s \in \mathbb{N}$ large enough if necessary, we may assume that $k > 0$. This completes the proof. \square

Definition 2.9. Let $f \in \mathcal{H}_L$ be an involution. Then f is diagonalizable with eigenvalues $1, -1$. Let $\text{Fix}(f) := \{\eta \in H_1(L, \mathbb{Z}) \mid f(\eta) = \eta\}$. There is a unique element $\eta_f \in \text{Fix}(f)$ such that $\text{Fix}(f)$ is generated by η_f and $\mu(\eta_f) > 0$. Define

$$m_f := \mu(\eta_f)/m_L \in \mathbb{N}.$$

Recall that $\sigma \in H_1(L, \mathbb{Z})$ denotes a fixed primitive class with $\mu(\sigma) = 0$.

Proposition 2.10. Let m_f be as defined above. Then $m_f = 1$ or 2 . Moreover, if we fix an arbitrary basis for $H_1(L, \mathbb{Z})$ and represent η_f, σ respectively as the first and second column vectors of an integral 2×2 matrix A , then $m_f = |\det(A)|$ the absolute value of the determinant of A .

Proof. Fix a basis for $H_1(L, \mathbb{Z}) \cong \mathbb{Z}^2$ and express $\eta_f = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\sigma = \begin{pmatrix} b \\ d \end{pmatrix}$ as column vectors with respect to the basis. Let $m := \det A = ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z} \setminus \{0\}$. Now in matrix form

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 + \frac{2bc}{m} & -\frac{2ab}{m} \\ \frac{2cd}{m} & 1 - \frac{2ad}{m} \end{pmatrix} \in GL(2, \mathbb{Z})$$

So $\frac{ab}{m}, \frac{ad}{m}, \frac{cb}{m}, \frac{cd}{m} \in \frac{1}{2}\mathbb{Z}$. Note that b, d are coprime, so there exist $r_1, r_2 \in \mathbb{Z}$ such that $r_1 b + r_2 d = 1$. Then $r_1 \cdot \frac{ba}{m} + r_2 \cdot \frac{da}{m} = \frac{a}{m} \in \frac{1}{2}\mathbb{Z}$ and $r_1 \cdot \frac{bc}{m} + r_2 \cdot \frac{dc}{m} = \frac{c}{m} \in \frac{1}{2}\mathbb{Z}$. Since a, c are coprime we must have $m \mid 2$.

Elements of $H_1(L, \mathbb{Z})$ are represented as column vectors with integral coefficients. Let $n_1, n_2 \in \mathbb{Z}$ so that $n_1 m_L = \mu\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ and $n_2 m_L = \mu\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$. By definition we have $\mu\left(\begin{pmatrix} a \\ c \end{pmatrix}\right) = m_f m_L$ and $\mu\left(\begin{pmatrix} b \\ d \end{pmatrix}\right) = 0$. So

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} m_f \\ 0 \end{pmatrix}, \quad \text{i.e.,} \quad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \frac{m_f}{m} \begin{pmatrix} d \\ -b \end{pmatrix}.$$

Note that n_1, n_2 are coprime since $\mu(H_1(L, \mathbb{Z})) = m_L \mathbb{Z}$. Also, $0 = \mu\left(\begin{pmatrix} b \\ d \end{pmatrix}\right) = b n_1 + d n_2$, so we have $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \pm \begin{pmatrix} d \\ -b \end{pmatrix}$, hence $m_f = |m| = |\det(A)|$ which equals 1 or 2.

Now if we choose another basis for $H_1(L, \mathbb{Z})$ and correspondingly represent the ordered pair (η_f, σ) by a matrix A' . Then $A' = BA$ for some $B \in GL(2, \mathbb{Z})$, hence $|\det(A')| = |\det(A)| = m_f$. This completes the proof. \square

Example 2.11. Let $f_k = \begin{pmatrix} 1 & 0 \\ k & -1 \end{pmatrix}$ be as defined in the proof of Proposition 2.3.

Then $m_{f_k} = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 2 & \text{if } k \text{ is odd.} \end{cases}$

This is a straightforward computation. Let $f = f_k$. Write $\eta_f = \begin{pmatrix} a \\ b \end{pmatrix}$ then $\eta_f = f(\eta_f) = \begin{pmatrix} a \\ ak - b \end{pmatrix}$. Since a, b are coprime we have that $a = 1$ (the positive sign follows from $\mu(\eta_f) > 0$) provided that $b = 0$. In this case we obtain that $\mu(\eta_f) = m_L$ and hence $m_f = 1$.

If $b \neq 0$ then, since $ak = 2b$ we have $k \neq 0$ and $\begin{pmatrix} a \\ b \end{pmatrix} = t \cdot \begin{pmatrix} 2 \\ k \end{pmatrix}$ for some $t \in \mathbb{Q}$. It follows that

- (i). $t = 1$ if k is odd, then $\mu(\eta_f) = 2m_L$, $m_f = 2$;
- (ii). $t = \frac{1}{2}$ if k is even and nonzero, then $\mu(\eta_f) = m_L$, $m_f = 1$.

This completes the computation.

From the perspective of the Maslov class μ , involutions of G_μ fall into two different types according to their values of m_f . Thus in addition to the group type of \mathcal{H}_L the Hamiltonian monodromy group of L as listed in Lemma 2.8, m_f can be used to construct further invariants for \mathcal{H}_L provided that some element of \mathcal{H} is an involution. Also, if \mathcal{H}_L contains some twist elements, the twist number can also be defined for \mathcal{H}_L . Below we define the new invariants for L .

Definition 2.12. Let L be monotone and \mathcal{H}_L its Hamiltonian monodromy group. We adapt the notations f_k, g_k from the proof of Proposition 2.3.

Let $\mathcal{T} \subset \mathcal{H}$ be the subset of all twists of \mathcal{H} . The *twist number* of \mathcal{H}_L is defined to be

$$t(L) := \begin{cases} d = \min\{k > 0 \mid g_k \in \mathcal{T}\} & \text{if } \mathcal{T} \neq \emptyset, \\ 0 & \text{if } \mathcal{T} = \emptyset. \end{cases}$$

Let $\mathcal{S} \subset \mathcal{H}$ denote the subset of all involutions of \mathcal{H} . The *spectrum* of \mathcal{H}_L is defined to be

$$s(L) := \begin{cases} \min\{m_f \mid f \in \mathcal{S}\} & \text{if } \mathcal{S} \neq \emptyset, \\ 0 & \text{if } \mathcal{S} = \emptyset. \end{cases}$$

In particular, if $\mathcal{H}_L \cong \mathbb{Z}_2$, then $t(L) = 0$, and $s(L) = 1$ or 2 .

Lemma 2.13. *The numbers $t(L), s(L)$ are invariants of monotone Lagrangian torus L up to Hamiltonian isotopies.*

Proof. Given a pair of Hamiltonian isotopic monotone Lagrangian tori L_0, L_1 and let $L_t := \phi_t(L_0)$, $t \in [0, 1]$, be a Hamiltonian isotopy between L_0 and L_1 . Here ϕ_t is the time t map of a time dependent Hamiltonian vector field. The L_t is monotonic for all $t \in [0, 1]$. Clearly $\phi_t^{-1} \circ \text{Ham}(M, L_t) \circ \phi_t = \text{Ham}(M, L_0)$ for all t and hence $\phi_t^* \mathcal{H}_t = \mathcal{H}_0$ where \mathcal{H}_t is the Hamiltonian monodromy group of L_t . By continuity we have $t(L_t) = t(L_0)$ and $s(L_t) = s(L_0)$ for all $t \in [0, 1]$. This completes the proof. \square

3. Examples in \mathbb{R}^4 : Clifford tori and Chekanov tori

Basic properties of Lagrangian tori in \mathbb{R}^4 . Let $M = \mathbb{R}^4$ with the standard symplectic structure $\omega = \sum_{j=1}^2 dx_j \wedge dy_j$. Readers can check that $c_1(\mathbb{R}^4) = 0$ and $H^1(\mathbb{R}^4, \mathbb{R}) = 0$. Let λ denote a primitive of ω , $d\lambda = \omega$. Let $L \hookrightarrow \mathbb{R}^4$ be an embedded torus. Using pseudoholomorphic curves, Gromov [6] showed L is *not exact*, i.e., the closed 1-form $\iota^*\lambda \in \Omega^1(L)$ is not exact. So the action class $\alpha := [\iota^*\lambda] \in H^1(L, \mathbb{R})$ is nontrivial. Polterovich [9] proved that the Maslov class $\mu \in H^1(L, \mathbb{Z})$ has divisibility 2, i.e., $2 = \min\{\mu(\gamma) \mid \gamma \in H_1(L, \mathbb{Z}), \mu(\gamma) > 0\}$.

Clifford tori. For $a, b > 0$ the *Clifford torus*

$$T_{a,b} := \{|z_1| = a, |z_2| = b\} \subset \mathbb{R}^4$$

is Lagrangian. It is monotone iff $a = b$.

Using symplectic capacities introduced by Ekeland and Hofer, Chekanov [1] proved the following:

Proposition 3.1 (Chekanov [1]). *Two Clifford tori $T_{a,b}, T_{a',b'}$ are Hamiltonian isotopic iff $T_{a',b'} = T_{a,b}$ or $T_{b,a}$.*

Lemma 3.2. *The Hamiltonian monodromy group \mathcal{H}_b of $T_{b,b}$ is a group of order 2, i.e., it is generated by a single involution, hence $t(T_{b,b}) = 0$. Moreover $s(T_{b,b}) = 2$.*

Proof. We take $\gamma \in H_1(T_{b,b}, \mathbb{R})$ to be the class represented by the curve $\{(be^{i\theta}, b) \in \mathbb{C} \times \mathbb{C} \mid \theta \in [0, 2\pi]\}$. We also take $\gamma' \in H_1(T_{b,b}, \mathbb{R})$ to be the class represented by the curve $\{(b, be^{i\theta}) \in \mathbb{C} \times \mathbb{C} \mid \theta \in [0, 2\pi]\}$. It can be checked that $\mu(\gamma) = 2 = \mu(\gamma')$. With this understood we adapt the notations \tilde{f}_k, \tilde{g}_k for monodromies from Remark 2.5 in Section 2.

We identify $T_{b,b}$ with $L = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with coordinates (t_1, t_2) so that, for $t \in \mathbb{R}/\mathbb{Z}$,

(2) $\{(t, 0)\}$ represents the class γ , and $\{(0, t)\}$ represents the class γ' .

Also let (s_1, s_2) be the dual coordinates for fibers of the cotangent bundle T^*L . The cotangent bundle $(T^*L, -d\lambda_{\text{can}})$ is symplectic, where λ_{can} is the canonical 1-form (see [8]), $\lambda_{\text{can}} = s_1 dt_1 + s_2 dt_2$. We also use the identification $\mathbb{R}^4 \cong \mathbb{C}^2 = \{(r_1 e^{\sqrt{-1}\theta_1}, r_2 e^{\sqrt{-1}\theta_2}) \mid r_i \geq 0, \theta_i \in \mathbb{R}/2\pi\mathbb{Z}\}$.

Now consider the map $\Phi : T^*L \rightarrow \mathbb{R}^4 \cong \mathbb{C}^2 = \{(r_1 e^{\sqrt{-1}\theta_1}, r_2 e^{\sqrt{-1}\theta_2})\}$,

$$\Phi(t_1, t_2, s_1, s_2) := \left(\sqrt{b^2 - \frac{s_1}{\pi}} e^{2\pi\sqrt{-1}t_1}, \sqrt{b^2 - \frac{s_2}{\pi}} e^{2\pi\sqrt{-1}t_2} \right).$$

The map Φ is defined on the domain $U_L := \{s_1 < \pi b^2, s_2 < \pi b^2\}$, and is a symplectic embedding from U_L into \mathbb{R}^4 , $T_{b,b} \subset \Phi(U_L)$.

Consider the primitive 1-form $\lambda = \frac{1}{2}(r_1^2 d\theta_1 + r_2^2 d\theta_2)$ of ω . We have $\Phi^*\lambda = \pi b^2(dt_1 + dt_2) - \lambda_{\text{can}}$. Also, Let $L_{c_1, c_2} := \{s_1 = c_1, s_2 = c_2\} \subset U_L$, then

$$(3) \quad \Phi(L_{c_1, c_2}) = T_{\sqrt{b^2 - \frac{c_1}{\pi}}, \sqrt{b^2 - \frac{c_2}{\pi}}}.$$

In the following we use Φ to identify a small neighborhood of $T_{b,b}$ with $U_\delta := \{|s_1| < \delta, |s_2| < \delta\}$ for $\delta > 0$ small.

We claim that $\tilde{g}_k \notin \mathcal{H}_b$ for any $k \neq 0$. Assume in the contrary that $\tilde{g}_k \in \mathcal{H}_b$ for some $k \neq 0$. Let $\phi \in \text{Ham}(\mathbb{R}^4, T_{b,b})$ be one with $\phi_* = g_k$. Modifying ϕ by a L -preserving Hamiltonian isotopy if necessary, we may assume that, on U_δ for some $\delta > 0$,

$$\phi(t_1, t_2, s_1, s_2) = ((1-k)t_1 - kt_2, kt_1 + (1+k)t_2, (1+k)s_1 - ks_2, ks_1 + (1-k)s_2).$$

Then, by taking $c_1 = 0$ and $c_2 = \epsilon > 0$ very small, we have

$$\phi(L_{0,\epsilon}) = L_{-k\epsilon, (1-k)\epsilon}.$$

and hence (via Φ)

$$\phi(T_{b, \sqrt{b^2 - \frac{\epsilon}{\pi}}}) = T_{\sqrt{b^2 - \frac{k\epsilon}{\pi}}, \sqrt{b^2 - \frac{(1-k)\epsilon}{\pi}}}.$$

It then implies that, for all $\epsilon > 0$ small enough, the Clifford tori $T_{b, \sqrt{b^2 - \frac{\epsilon}{\pi}}}$ and $T_{\sqrt{b^2 - \frac{k\epsilon}{\pi}}, \sqrt{b^2 - \frac{(1-k)\epsilon}{\pi}}}$ are Hamiltonian isotopic, which cannot be possible by Proposition 3.1, unless $k = 0$. Thus $\tilde{g}_k \notin \mathcal{H}_b$ for any $k \neq 0$. Hence $t(T_{b,b}) = 0$.

Note that since $\tilde{f}_k \tilde{f}_l = \tilde{g}_{k-l}$, \mathcal{H}_b can contain at most one involution. In fact there exists a Hamiltonian self-isotopy of $T_{b,b}$ with monodromy $\tilde{f}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. To see this, first let us consider the path in the unitary group $U(2)$ defined by

$$A_t := \begin{pmatrix} \cos \frac{\pi t}{2} & -\sin \frac{\pi t}{2} \\ \sin \frac{\pi t}{2} & \cos \frac{\pi t}{2} \end{pmatrix} \in GL(2, \mathbb{C}), \quad 0 \leq t \leq 1.$$

A_t acts on \mathbb{C}^2 , is the time t map of the Hamiltonian vector field $X = \frac{\pi}{2}(x_1 \partial_{x_2} - x_2 \partial_{x_1} + y_1 \partial_{y_2} - y_2 \partial_{y_1})$, $\omega(X, \cdot) = -dH$, $H = \frac{\pi}{2}(x_2 y_1 - x_1 y_2)$. Observe that $A_1(T_{a,b}) = T_{b,a}$, $(A_1)_* = \tilde{f}_1$ on $H_1(T_{b,b}, \mathbb{Z})$. Fix $b > 0$ and modify H to get a C^∞ function \tilde{H} with compact support such that $\tilde{H} = H$ on $\{|z_1| \leq 2b, |z_2| \leq 2b\}$. Let ϕ_t be the time t map of the flow of the Hamiltonian vector field associated to \tilde{H} . Then $\phi_1(T_{b,b}) = (T_{b,b})$, and $(\phi_1)_* = (A_1)_* = \tilde{f}_1$ on $H_1(T_{b,b}, \mathbb{Z})$. Hence \mathcal{H}_b is a group of order 2 generated by the involution \tilde{f}_1 . Hence $s(T_{b,b}) = 2$ by Remark 2.5, Proposition 2.10 and Remark 2.11. \square

Chekanov tori. Now we consider another type of monotone Lagrangian tori in \mathbb{R}^4 : the Chekanov tori (called *special* tori in [1]). Consider the diffeomorphism $\rho : T^*S^1 = S^1 \times \mathbb{R} \rightarrow E := \mathbb{R}_{x_1, x_2}^2 \setminus \{(0, 0)\}$ defined by $\rho(\theta, s) = (e^s \cos \theta, e^s \sin \theta)$. The corresponding map $\rho^* : T^*E \subset \mathbb{R}^4 \rightarrow T^*(T^*S^1)$ is a symplectomorphism between two cotangent bundles. Let

$$\Psi := (\rho^*)^{-1} : T^*(T^*S^1) = (T^*S^1) \times \mathbb{R}^2 \rightarrow T^*E = E \times \mathbb{R}_{y_1, y_2}^2 \subset \mathbb{R}^4$$

be the inverse symplectic map. Let $(\theta, s) \in S^1 \times \mathbb{R}$ be coordinates for T^*S^1 , (θ^*, s^*) be the dual coordinates for the fiber of $T^*(T^*S^1)$. Let (x_1, x_2, y_1, y_2) be coordinates for $E \times \mathbb{R}^2$. Then

$$\Psi(\theta, s, \theta^*, s^*) = (e^s \cos \theta, e^s \sin \theta, e^{-s}(-\theta^* \sin \theta + s^* \cos \theta), e^{-s}(\theta^* \cos \theta + s^* \sin \theta)).$$

From now on, we will identify $T^*(T^*S^1)$ with its image in \mathbb{R}^4 via Ψ .

For $a \in \mathbb{R}$ and $b > 0$ the torus

$$T'_{a,b} := \{\theta^* = a, s^2 + (s^*)^2 = b^2\}$$

is Lagrangian. Moreover, for $a \neq 0$, $T'_{a,b}$ is Hamiltonian isotopic to the Clifford torus $T_{b,b+|a|}$ by Chekanov [1]. The $a = 0$ case is special. We call the special torus $T'_{0,b}$ a *Chekanov torus*. $T'_{0,b}$ is monotone, is Lagrangian isotopic to $T_{b,b}$ but not Hamiltonian isotopic to $T_{b,b}$ [1, 2].

Lemma 3.3. *The Hamiltonian monodromy group \mathcal{H}'_b of $T'_{0,b}$ is a group of order 2, i.e., it is generated by a single involution, hence $t(T'_{0,b}) = 0$. Moreover $s(T'_{0,b}) = 1$.*

Proof. Let $\gamma \in H_1(T'_{0,b}, \mathbb{Z})$ be represented by the curve $(\theta, s, \theta^*, s^*) = (0, b \cos t, 0, b \sin t)$, $\sigma \in H_1(T'_{0,b}, \mathbb{Z})$ be represented by the curve $(\theta, s, \theta^*, s^*) = (t, b, 0, 0)$. It can be verified that $\mu(\gamma) = 2$ and $\mu(\sigma) = 0$. With this understood we adapt the notations f_k, g_k for monodromies from the proof of Proposition 2.3.

We identify $T_{b,b}$ with $L = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with coordinates (t_1, t_2) so that, for $t \in \mathbb{R}/\mathbb{Z}$,

$$(4) \quad \{(t, 0)\} \text{ represents the class } \gamma, \text{ and } \{(0, t)\} \text{ represents the class } \sigma.$$

Also let (s_1, s_2) be the dual coordinates for the fiber of T^*L . The cotangent bundle $(T^*L, -d\lambda_{\text{can}})$ is symplectic, where λ_{can} is the canonical 1-form (see [8]), $\lambda_{\text{can}} = s_1 dt_1 + s_2 dt_2$.

Consider the map $\Phi' : T^*L \rightarrow T^*(S^1 \times \mathbb{R}) = \{(\theta, s, \theta^*, s^*)\}$,

$$\Phi'(t_1, t_2, s_1, s_2) := (t_2, \sqrt{b^2 - \frac{s_1}{\pi}} \cos 2\pi t_1, s_2, \sqrt{b^2 - \frac{s_1}{\pi}} \sin 2\pi t_1).$$

The map Φ' is defined on the domain $U'_L := \{s_1 < \pi b^2\} \subset T^*L$, and is a symplectic embedding from U'_L into $T^*(S^1 \times \mathbb{R})$. Also $T'_{0,b} \subset \Phi'(U'_L)$.

Let $L_{c_1, c_2} := \{s_1 = c_1, s_2 = c_2\} \subset U'_L$, then for $c_2 \neq 0$,

$$(5) \quad \Psi \circ \Phi'(L_{c_1, c_2}) = \Psi\left(T'_{c_2, \sqrt{b^2 - \frac{c_1}{\pi}}}\right) = T_{\sqrt{b^2 - \frac{c_1}{\pi}}, \sqrt{b^2 - \frac{c_1}{\pi}} + |c_2|},$$

where the second equality is up to a Hamiltonian isotopy.

Assume there exists $\phi \in \text{Ham}(\mathbb{R}^4, T'_{0,b})$ with monodromy g_k for some $k \in \mathbb{Z}$. Modifying ϕ by a L -preserving Hamiltonian isotopy if necessary we may assume that, for some $\delta > 0$ small enough,

$$\phi(\theta_1, \theta_2, s_1, s_2) = (\theta_1, k\theta_1 + \theta_2, s_1 - ks_2, s_2) \quad \text{on } U'_\delta.$$

Then $\phi(L_{c_1, c_2}) = L_{c_1 - kc_2, c_2}$.

Now, by taking $c_1 = c_2 = \epsilon > 0$ very small, we have

$$\phi(L_{\epsilon, \epsilon}) = L_{(1-k)\epsilon, \epsilon}.$$

and hence (via $\Psi \circ \Phi'$, see (5))

$$\phi(T_{\sqrt{b^2 - \frac{\epsilon}{\pi}}, \sqrt{b^2 - \frac{\epsilon}{\pi}} + \epsilon}) = T_{\sqrt{b^2 - \frac{(1-k)\epsilon}{\pi}}, \sqrt{b^2 - \frac{(1-k)\epsilon}{\pi}} + \epsilon},$$

It then implies that the Clifford tori $T_{\sqrt{b^2 - \frac{\epsilon}{\pi}}, \sqrt{b^2 - \frac{\epsilon}{\pi}} + \epsilon}$ and $T_{\sqrt{b^2 - \frac{(1-k)\epsilon}{\pi}}, \sqrt{b^2 - \frac{(1-k)\epsilon}{\pi}} + \epsilon}$ are Hamiltonian isotopic for all $\epsilon > 0$ small enough, which cannot be possible by Proposition 3.1, unless $k = 0$. Thus $g_k \notin \mathcal{H}'_b$ for any $k \neq 0$. Hence $t(T'_{0,b}) = 0$.

Then \mathcal{H}'_b can contain at most one involution. Below we will construct $\phi \in \text{Ham}(\mathbb{R}^4, T'_{0,b})$ with monodromy $\phi_* = f_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. First observe that $T'_{0,b}$ is contained in the hyper-surface $\{\theta^* = 0\} \subset T^*(S^1 \times \mathbb{R})$ and hence in (via Ψ)

$$(6) \quad \{(x_1, y_1, x_2, y_2) = (e^s \cos \theta, e^{-s} s^* \cos \theta, e^s \sin \theta, e^{-s} s^* \sin \theta)\} \subset \mathbb{R}^4.$$

For $t \in [0, 1]$ the symplectomorphism $A_t : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $A_t(z_1, z_2) = (e^{i\pi t} z_1, z_2)$, is the time t map of the Hamiltonian vector field $X = \pi(x_1 \partial_{y_1} - y_1 \partial_{x_1})$ whose Hamiltonian function is $H = \frac{\pi}{2} |z_1|^2$. Observe that A_1 preserves the hyper-surface in (6) and send the point in (6) to the point

$$(7) \quad \begin{aligned} (x_1, y_1, x_2, y_2) &= (-e^s \cos \theta, -e^{-s} s^* \cos \theta, e^s \sin \theta, e^{-s} s^* \sin \theta) \\ &= (e^s \cos(\pi - \theta), e^{-s} s^* \cos(\pi - \theta), e^s \sin(\pi - \theta), e^{-s} s^* \sin(\pi - \theta)). \end{aligned}$$

Hence $A_1(T'_{0,b}) = T'_{0,b}$. Moreover, $(A_1)_* \gamma = \gamma$ and $(A_1)_* \sigma = -\sigma$, i.e., $(A_1)_* = f_0$ on $H_1(T'_{0,b}, \mathbb{Z})$.

We modify H to get $\tilde{H} \in C^\infty(\mathbb{R}^4)$ with compact support, such that $\tilde{H} = H$ on $\{|z_1| \leq e^{2b}, |z_2| \leq e^{2b}\}$. Let ϕ_t be the time t map of the flow of the Hamiltonian vector field associated to \tilde{H} . Then $\phi_1(T'_{0,b}) = T'_{0,b}$ and $(\phi_1)_* = (A_1)_* = f_0$ on $H_1(T'_{0,b}, \mathbb{Z})$. So \mathcal{H}_b is generated by the involution f_0 and $s(T'_{0,b}) = 1$ by Proposition 2.10, Remark 2.11 and Definition 2.12. \square

Lemma 3.2 and Lemma 3.3 together imply Theorem 1.1.

Final discussion. We end this note with the following open questions.

Question 3.4. Let $L \subset \mathbb{R}^4$ be any monotone Lagrangian torus. Is it true that $\mathcal{H}_L \cong \mathbb{Z}_2$?

Question 3.5. Let $L \subset (\mathbb{R}^4, \omega)$ be either a monotone Clifford torus $T_{b,b}$ or a Chekanov torus $T'_{0,b}$. Let $B \subset \mathbb{R}^4$ be an open 4-ball containing L . Assume there is a symplectic embedding $\phi : (B, \omega) \rightarrow (M, \omega_M)$, where M is symplectic with $c_1(M) = 0$ and $H^1(M, \mathbb{R}) = 0$. Let \mathcal{H}^B (resp. \mathcal{H}^M) denote the Hamiltonian monodromy group of L in B (resp. in M). Via the inclusion ϕ , $\mathcal{H}^B \cong \mathbb{Z}_2$ is a subgroup of \mathcal{H}^M . Is it possible that \mathcal{H}^B is a proper subgroup of \mathcal{H}^M ?

Question 3.6. How to extend the constructions of $t(L), s(L)$ to higher dimensional cases, to distinguish monotone Lagrangian tori in \mathbb{R}^{2n} and beyond?

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