Math. Res. Lett. **16** (2009), *no.* 5, 753–760

ON HOFMANN'S BILINEAR ESTIMATE

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Abstract. Using the framework of a previous article joint with Axelsson and McIntosh, we extend to systems two results of S. Hofmann for real symmetric equations and their perturbations going back to a work of B. Dahlberg for Laplace's equation on Lipschitz domains. The first one is a certain bilinear estimate for a class of weak solutions and the second is a criterion which allows to identify the domain of the generator of the semi-group yielding such solutions.

1. Introduction

S. Hofmann proved in [10] that weak solutions of

$$\operatorname{div}_{t,x} A(x)\nabla_{t,x} U(t,x) = \sum_{i,j=0}^{n} \partial_i A_{i,j}(x)\partial_j U(t,x) = 0 \tag{1}$$

on the upper half space $\mathbf{R}^{1+n}_+ := \{(t,x) \in \mathbf{R}\times\mathbf{R}^n \ ;\ t>0\}$, $n\geq 1$, where the matrix $A = (A_{i,j}(x))_{i,j=0}^n \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{1+n}))$ is assumed to be t-independent and within some small L_∞ neighborhood of a real symmetric strictly elliptic t-independent matrix, obey the following bilinear estimate

$$\left| \iint_{\mathbf{R}^{1+n}_+} \nabla_{t,x} U \cdot \overline{\mathbf{v}}\, dtdx \right| \leq C\|U_0\|_2 (|||t\nabla \mathbf{v}||| + \|N_* \mathbf{v}\|_2)$$

for all $\mathbf{C}^{1+n}$-valued field $\mathbf{v}$ such that the right-hand side is finite. See below for the definition of the square-function $|||\ |||$ and the non-tangential maximal operator N_*. The trace of U at $t=0$ is assumed to be in the sense of non-tangential convergence a.e. and in $L_2(\mathbf{R}^n)$.

In addition, he proves that the solution operator $U_0 \to U(t,\cdot)$ defines a bounded C_0 semi-group on $L_2(\mathbf{R}^n)$ whose infinitesimal generator $\mathcal{A}$ has domain $W^{1,2}(\mathbf{R}^n)$ with $\|\mathcal{A}f\|_2 \sim \|\nabla f\|_2$.

Such results were first proved by B. Dahlberg [8] for harmonic functions on a Lipschitz domain. A version of the bilinear estimate for Clifford-valued monogenic functions was proved by Li-McIntosh-Semmes [15]. A short proof of Dahlberg's estimate for harmonic functions and some applications appear in Mitrea's work [16]. L^p versions are recently discussed by Varopoulos [19].

Hofmann's arguments for variable coefficients rely on the deep results of [1], and in particular Theorem 1.11 there where the boundedness and invertibility of the layer potentials are obtained from a $T(b)$ theorem, Rellich estimates in the case of real

Received by the editors December 16, 2008.
2000 *Mathematics Subject Classification.* 35J25, 35J55, 47N20, 42B25.
Key words and phrases. elliptic systems; Dirichlet problem; quadratic estimates; Carleson measures.

symmetric matrices and perturbation. This also generalizes somehow the case where $A_{0,i} = A_{i,0} = 0$ for $i = 1, \dots, n$ corresponding to the Kato square root problem.

The recent works [5, 4], pursuing ideas in [3], allow us to extend this further to systems, making clear in particular that specificities of real symmetric coefficients and their perturbations and of equations - in particular the De Giorgi-Nash-Moser estimates - are not needed: it only depends on whether the Dirichlet problem is solvable. We use the solution operator constructed in [5] and the proof using $P_t - Q_t$ techniques of Coifman-Meyer from [7] makes transparent the para-product like character of this bilinear estimate. We also establish a necessary and sufficient condition telling when the domain of the infinitesimal generator $\mathcal{A}$ of the Dirichlet semi-group is $W^{1,2}$.

We apologize to the reader for the necessary conciseness of this note and suggests he (or she) has (at least) the references [3, 5, 4] handy. In Section 2, we try to extract from them the relevant information. The proof or the bilinear estimate for variable coefficients systems is in Section 3. Section 4 contains the discussion on the domain of the Dirichlet semi-group.

2. Setting

We begin by giving a precise definition of well-posedness of the Dirichlet problem for systems. Throughout this note, we use the notation $X \approx Y$ and $X \lesssim Y$ for estimates to mean that there exists a constant $C > 0$, independent of the variables in the estimate, such that $X/C \le Y \le CX$ and $X \le CY$, respectively.

We write (t, x) for the standard coordinates for $\mathbf{R}^{1+n} = \mathbf{R} \times \mathbf{R}^n$, t standing for the vertical or normal coordinate. For vectors $\mathbf{v} = (\mathbf{v}_i^\alpha)_{0\le i\le n}^{1\le\alpha\le m} \in \mathbf{C}^{(1+n)m}$, we write $\mathbf{v}_0 \in \mathbf{C}^m$ and $\mathbf{v}_\parallel \in \mathbf{C}^{nm}$ for the normal and tangential parts of $\mathbf{v}$, i.e. $\mathbf{v}_0 = (\mathbf{v}_0^\alpha)^{1\le\alpha\le m}$ whereas $\mathbf{v}_\parallel = (\mathbf{v}_i^\alpha)_{1\le i\le n}^{1\le\alpha\le m}$.

For systems, gradient and divergence act as $(\nabla_{t,x} U)_i^\alpha = \partial_i U^\alpha$ and $(\mathrm{div}_{t,x}\mathbf{F})^\alpha = \sum_{i=0}^n \partial_i \mathbf{F}_i^\alpha$, with correponding tangential versions $\nabla_x U = (\nabla_{t,x} U)_\parallel$ and $(\mathrm{div}_x \mathbf{F})^\alpha = \sum_{i=1}^n \partial_i \mathbf{F}_i^\alpha$. With $\mathrm{curl}_x \mathbf{F}_\parallel = 0$, we understand $\partial_j \mathbf{F}_i^\alpha = \partial_i \mathbf{F}_j^\alpha$, for all $i,\ j = 1, \dots, n$, $\alpha = 1, \dots, m$.

We consider divergence form second order elliptic systems

$$\sum_{i,j=0}^{n} \sum_{\beta=1}^{m} \partial_i A_{i,j}^{\alpha,\beta}(x) \partial_j U^\beta(t,x) = 0, \qquad \alpha = 1, \dots, m, \tag{2}$$

on the half space $\mathbf{R}_+^{1+n} := \{(t,x) \in \mathbf{R} \times \mathbf{R}^n \ ; \ t > 0\}$, $n \ge 1$, where the matrix

$$A = (A_{ij}^{\alpha,\beta}(x))_{i,j=0,\dots,n}^{\alpha,\beta=1,\dots,m} \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$$

is assumed to be t-independent with complex coefficients, and *strictly accretive* on $\mathsf{N}(\mathrm{curl}_\parallel)$ in the sense that there exists $\kappa > 0$ such that

$$\sum_{i,j=0}^{n} \sum_{\alpha,\beta=1}^{m} \int_{\mathbf{R}^n} \mathrm{Re}(A_{i,j}^{\alpha,\beta}(x) \mathbf{f}_j^\beta(x) \overline{\mathbf{f}_i^\alpha(x)}) dx \ge \kappa \sum_{i=0}^{n} \sum_{\alpha=1}^{m} \int_{\mathbf{R}^n} |\mathbf{f}_i^\alpha(x)|^2 dx, \tag{3}$$

for all $\mathbf{f} \in \mathsf{N}(\mathrm{curl}_\parallel) := \{\mathbf{g} \in L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}) \ ; \ \mathrm{curl}_x(\mathbf{g}_\parallel) = 0\}$. This is nothing but ellipticity in the sense of Gårding. See the discussion in [5]. By changing m to $2m$ we could assume that the coefficients are real-valued. But this does not simplify matters and we need the complex hermitean structure of our L_2 space anyway.

Definition 2.1. The Dirichlet problem (Dir-A) is said to be *well-posed* if for each $u \in L_2(\mathbf{R}^n; \mathbf{C}^m)$, there is a unique function

$$U_t(x) = U(t,x) \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m))$$

such that $\nabla_x U \in C^0(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{nm}))$, where U satisfies (2) for $t > 0$, $\lim_{t\to 0} U_t = u$, $\lim_{t\to\infty} U_t = 0$, $\lim_{t\to\infty} \nabla_{t,x} U_t = 0$ in L_2 norm, and $\int_{t_0}^{t_1} \nabla_x U_s \, ds$ converges in L_2 when $t_0 \to 0$ and $t_1 \to \infty$. More precisely, by U satisfying (2), we mean that $\int_t^\infty ((A\nabla_{s,x} U_s)_\parallel, \nabla_x v) ds = -((A\nabla_{t,x} U_t)_0, v)$ for all $v \in C_0^\infty(\mathbf{R}^n; \mathbf{C}^m)$.

Restricting to real symmetric equations and their perturbations, this definition is not the one taken in [10]. However, a sufficient condition is provided in [5] to insure that the two methods give rise to the same solution. See also [1, Corollary 4.28]. It covers the matrices listed in Theorem 2.4 below. This definition is more akin to well-posedness for a Neumann problem[1] (see Section 4).

Remark 2.2. In the case of block matrices, ie $A_{0,i}^{\alpha,\beta}(x) = 0 = A_{i,0}^{\alpha,\beta}(x)$, $1 \le i \le n, 1 \le \alpha, \beta \le m$, the second order system (2) can be solved using semi-group theory: $V(t,\cdot) = e^{-tL^{1/2}} u_0$ for $L = -A_{00}^{-1} \mathrm{div}_x A_{\parallel\parallel} \nabla_x$ acting as an unbounded operator on $L_2(\mathbf{R}^n, C^{nm})$ (See below for the notation). This solution satisfies $V_t = V(t,\cdot) \in C^2(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m)) \cap C^1(\mathbf{R}_+, \mathsf{D}(L^{1/2}))$, $\lim_{t\to 0} V_t = u_0$, $\lim_{t\to\infty} V_t = 0$ in L_2 norm, and (2) holds in the strong sense in $\mathbf{R}^n$ for all $t > 0$ (and in the sense of distributions in $\mathbf{R}_+^{1+n}$). Hence, the two notions of solvability are not *a priori* equivalent. That the solutions are the same follows indeed from the solution of the Kato square root problem for L: $\mathsf{D}(L^{1/2}) = W^{1,2}(\mathbf{R}^n, C^{nm})$ with $\|L^{1/2} f\|_2 \sim \|\nabla_x f\|_2$. See [6] where this is explicitly proved when $A_{00} \neq I$.

The following result is Corollary 3.4 of [5] (which, as we recall, furnishes a different proof of results obtained by combining [11] and [9] in the case of real symmetric matrices equations ($m = 1$)).

Theorem 2.3. *Let $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ be a t-independent, complex matrix function which is strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$ and assume that (Dir-A) is well-posed. Then any function $U_t(x) = U(t,x) \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m))$ solving (2), with properties as in Definition 2.1, has estimates*

$$\int_{\mathbf{R}^n} |u|^2 dx \approx \sup_{t>0} \int_{\mathbf{R}^n} |U_t|^2 dx \approx \int_{\mathbf{R}^n} |\widetilde{N}_*(U)|^2 dx \approx |||t\nabla_{t,x} U|||^2,$$

where $u = U|_{\mathbf{R}^n}$. If furthermore A is real (not necessarily symmetric) and $m = 1$, then Moser's local boundedness estimate [17] *gives the pointwise estimate $\widetilde{N}_*(U)(x) \approx N_*(U)(x)$, where the standard non-tangential maximal function is $N_*(U)(x) := \sup_{|y-x|<ct} |U(t,y)|$, for fixed $0 < c < \infty$.*

We use the square-function norm

$$|||F_t|||^2 := \int_0^\infty \|F_t\|_2^2 \, \frac{dt}{t} = \iint_{\mathbf{R}_+^{1+n}} |F(t,x)|^2 \, \frac{dtdx}{t}$$

[1] We showed with A. Axelsson [2] that well-posedness in the sense of Definition 2.1 is equivalent to well-posedness in the class of weak solutions $U \in W_{loc}^{1,2}(\mathbf{R}_+^{1+n})$ of (2) such that $U \in C^0(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m))$ and $|||t\nabla_{t,x} U||| < \infty$.

and the following version $\widetilde{N}_*(F)$ of the modified *non-tangential maximal function* introduced in [12]

$$\widetilde{N}_*(F)(x) := \sup_{t>0} t^{-(1+n)/2} \|F\|_{L_2(Q(t,x))},$$

where $Q(t,x) := [(1-c_0)t, (1+c_0)t] \times B(x; c_1 t)$, for some fixed constants $c_0 \in (0,1)$, $c_1 > 0$.

Next is Theorem 3.2 of [5], specialized to the Dirichlet problem.

Theorem 2.4. *The set of matrices A for which (Dir-A) is well-posed is an open subset of $L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$. Furthermore, it contains*

(i) *all Hermitean matrices $A(x) = A(x)^*$ (and in particular all real symmetric matrices),*
(ii) *all block matrices where $A_{0,i}^{\alpha,\beta}(x) = 0 = A_{i,0}^{\alpha,\beta}(x)$, $1 \le i \le n, 1 \le \alpha, \beta \le m$, and*
(iii) *all constant matrices $A(x) = A$.*

More importantly is the solution algorithm using an "infinitesimal generator" T_A. Write $\mathbf{v} \in \mathbf{C}^{(1+n)m}$ as $\mathbf{v} = [\mathbf{v}_0, \mathbf{v}_\parallel]^t$, where $\mathbf{v}_0 \in \mathbf{C}^m$ and $\mathbf{v}_\parallel \in \mathbf{C}^{nm}$, and introduce the auxiliary matrices

$$\overline{A} := \begin{bmatrix} A_{00} & A_{0\parallel} \\ 0 & I \end{bmatrix}, \quad \underline{A} := \begin{bmatrix} 1 & 0 \\ A_{\parallel 0} & A_{\parallel\parallel} \end{bmatrix}, \qquad \text{if } A = \begin{bmatrix} A_{00} & A_{0\parallel} \\ A_{\parallel 0} & A_{\parallel\parallel} \end{bmatrix}$$

in the normal/tangential splitting of $\mathbf{C}^{(1+n)m}$. The strict accretivity of A on $\mathsf{N}(\mathrm{curl}_\parallel)$, as in (3), implies the pointwise strict accretivity of the diagonal block A_{00}. Hence A_{00} is invertible, and consequently $\overline{A}$ is invertible [This is not necessarily true for $\underline{A}$]. We define

$$T_A = \overline{A}^{-1} D \underline{A}$$

as an unbounded operator on $L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$ with D the first order self-adjoint operator given in the normal/tangential splitting by

$$D = \begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix}.$$

Proposition 2.5. *Let $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ be a t-independent, complex matrix function which is strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$.*

(1) *The operator T_A has quadratic estimates and a bounded holomorphic functional calculus on $L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$. In particular, for any holomorphic function ψ on the left and right open half planes, with $z\psi(z)$ and $z^{-1}\psi(z)$ qualitatively bounded, one has*

$$|||\psi(tT_A)\mathbf{f}||| \lesssim \|\mathbf{f}\|_2.$$

(2) *The Dirichlet problem (Dir-A) is well-posed if and only if the operator*

$$\mathcal{S} : \overline{\mathsf{R}(\chi_+(T_A))} \to L_2(\mathbf{R}^n, \mathbf{C}^m), \mathbf{f} \mapsto \mathbf{f}_0$$

is invertible. Here, $\chi_+ = 1$ on the right open half plane and 0 on the left open half plane.

Item (1) is [5, Corollary 3.6] (and see [4] for an explicit direct proof) and item (2) can be found in [5, Section 4, proof of Theorem 2.2].

Lemma 2.6. *Assume that (Dir-A) is well-posed. Let* $u_0 \in L_2(\mathbf{R}^n, \mathbf{C}^m)$*. Then the solution* U *of (Dir-A) in the sense of Definition 2.1 is given by*

$$U(t,\cdot) = (e^{-tT_A}\mathbf{f})_0, \quad \mathbf{f} = \mathcal{S}^{-1}u_0 \in \overline{\mathsf{R}(\chi_+(T_A))}$$

and furthermore

$$\nabla_{t,x}U(t,\cdot) = \partial_t e^{-tT_A}\mathbf{f}.$$

Proof. [5, Lemma 4.2] (See also [3, Lemma 2.55] with a slightly different formulation of the Dirichlet problem). □

3. The bilinear estimate

We are now in position to state and prove the generalisation of Hofmann's result.

Theorem 3.1. *Assume that (Dir-A) is well-posed. Let* $u_0 \in L_2(\mathbf{R}^n, \mathbf{C}^m)$ *and* U *be the solution to (Dir-A) in the sense of Definition 2.1. Then for all* $\mathbf{v}\colon \mathbf{R}^{1+n}_+ \to \mathbf{C}^{(1+n)m}$ *such that the right-hand side is finite,*

$$\left| \iint_{\mathbf{R}^{1+n}_+} \nabla_{t,x}U \cdot \overline{\mathbf{v}}\, dtdx \right| \le C\|u_0\|_2 (|||t\nabla_{t,x}\mathbf{v}||| + \|N_*\mathbf{v}\|_2).$$

The pointwise values of $\mathbf{v}(t,x)$ in the non-tangential control $N_*\mathbf{v}$ can be slightly improved to L^1 averages on balls having radii $\sim t$ for each fixed t. See the end of proof.

Proof. It follows from the previous result that there exists $\mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}$ such that $U(t,\cdot) = (e^{-tT_A}\mathbf{f})_0$ and

$$\nabla_{t,x}U(t,\cdot) = \partial_t\mathbf{F} = -T_A e^{-tT_A}\mathbf{f}, \quad \mathbf{F} = e^{-tT_A}\mathbf{f}.$$

Integrating by parts with respect to t, we find

$$\iint_{\mathbf{R}^{1+n}_+} \nabla U \cdot \overline{\mathbf{v}}\, dtdx = -\iint_{\mathbf{R}^{1+n}_+} t\partial_t\mathbf{F} \cdot \overline{\partial_t\mathbf{v}}\, dtdx - \iint_{\mathbf{R}^{1+n}_+} t\partial_t^2\mathbf{F} \cdot \overline{\mathbf{v}}\, dtdx.$$

The boundary term vanishes because $t\partial_t\mathbf{F}$ goes to 0 in L_2 when $t \to 0, \infty$ (this uses $\mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}$) and $\sup_{t>0}\|\mathbf{v}(t,\cdot)\|_2 < \infty$ from $\|N_*\mathbf{v}\|_2 < \infty$.

For the first term, we use Cauchy-Schwarz inequality and that $|||t\partial_t\mathbf{F}||| \lesssim \|u_0\|_2$ from Theorem 2.3.

For the second term, we use the following identity: $T_A = \overline{A}^{-1}DB\overline{A}$ with $B = \underline{A}\overline{A}^{-1}$ which, by [5, Proposition 3.2], is strictly accretive on $\mathsf{N}(\mathrm{curl}_\parallel)$, and observe that

$$\begin{aligned} t^2\partial_t^2\mathbf{F} &= \overline{A}^{-1}(tDB)^2 e^{-tDB}(\overline{A}\mathbf{f}) \\ &= \overline{A}^{-1}(tDB)(I + (tDB)^2)^{-1}\psi(tDB)(\overline{A}\mathbf{f}) \\ &= \overline{A}^{-1}(tDB)(I + (tDB)^2)^{-1}\overline{A}\psi(tT_A)(\mathbf{f}) \end{aligned}$$

with

$$\psi(z) = z(1+z^2)e^{-(\mathrm{sgnRe}\, z)z}.$$

Thus,

$$\iint_{\mathbf{R}^{1+n}_+} t\partial_t^2\mathbf{F} \cdot \overline{\mathbf{v}}\, dtdx = \iint_{\mathbf{R}^{1+n}_+} \overline{A}\psi(tT_A)(\mathbf{f}) \cdot \overline{Q_t\mathbf{v}_t}\, \frac{dtdx}{t}$$

with $Q_t = \Theta_t \overline{A}^{-1^*}$ and $\Theta_t = (tB^*D)(I + (tB^*D)^2)^{-1}$ acting on $\mathbf{v}_t \equiv \mathbf{v}(\mathbf{t}, \cdot)$ for each fixed t [The notation $\overline{A}$ has nothing to do with complex conjugate and we apologize for any conflict this may cause]. It follows from the quadratic estimates of Proposition 2.5 that

$$|||\psi(tT_A)(\mathbf{f})||| \lesssim \|\mathbf{f}\|_2.$$

It remains to estimate $|||Q_t \mathbf{v}_t|||$. To do that we follow the principal part approximation of [4] - which is an elaboration of the so-called Coifman-Meyer trick [7] - applied to Q_t instead of Θ_t there. That is, we write

$$(4) \qquad Q_t \mathbf{v}_t = Q_t \left(\frac{I - P_t}{t(-\Delta)^{1/2}} \right) t(-\Delta)^{-1/2} \mathbf{v}_t + (Q_t P_t - \gamma_t S_t P_t)\mathbf{v}_t + \gamma_t S_t P_t \mathbf{v}_t$$

where Δ is the Laplacian on $\mathbf{R}^n$, P_t is a nice scalar approximation to the identity acting componentwise on $L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$ and γ_t is the element of $L^2_{\text{loc}}(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ given by

$$\gamma_t(x)\mathbf{w} := (Q_t \mathbf{w})(x)$$

for every $\mathbf{w} \in \mathbf{C}^{(1+n)m}$. We view $\mathbf{w}$ on the right-hand side of the above equation as the constant function valued in $\mathbf{C}^{(1+n)m}$ defined on $\mathbf{R}^n$ by $\mathbf{w}(x) := \mathbf{w}$. We identify $\gamma_t(x)$ with the (possibly unbounded) multiplication operator $\gamma_t : f(x) \mapsto \gamma_t(x) f(x)$. Finally, the *dyadic averaging operator* $S_t : L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m}) \to L_2(\mathbf{R}^n, \mathbf{C}^{(1+n)m})$ is given by

$$S_t \mathbf{u}(x) := \frac{1}{|Q|} \int_Q \mathbf{u}(y)\, dy$$

for every $x \in \mathbf{R}^n$ and $t > 0$, where Q is the unique dyadic cube in $\mathbf{R}^n$ that contains x and has side length ℓ with $\ell/2 < t \le \ell$.

With this in hand, we apply the triple bar norm to (4).

Using the uniform L_2 boundedness of Q_t and that of $\frac{1-P_t}{t(-\Delta)^{1/2}}$, the first term in the RHS is bounded by $|||t(-\Delta)^{1/2} \mathbf{v}_t||| \le |||t \nabla_x \mathbf{v}_t|||$.

Following exactly the computation of Lemma 3.6 in [4], the second term in the RHS is bounded by $C|||t\nabla_x P_t \mathbf{v}_t||| \le C|||t\nabla_x \mathbf{v}_t|||$ using the uniform L_2 boundedness of P_t. This computation makes use of the off-diagonal estimates of Θ_t, hence of Q_t, proved in [4, Proposition 3.11].

For the third term in the RHS, we observe that $\gamma_t(x)\mathbf{w} = \Theta_t(\overline{A}^{-1^*}\mathbf{w})(x)$. Hence, the square-function estimate on Θ_t proved in [4, Theorem 1.1], the off-diagonal estimates of Θ_t and the fact that $\overline{A}^{-1}$ is bounded imply that $|\gamma_t(x)|^2 \frac{dtdx}{t}$ is a Carleson measure. Hence, from Carleson embedding theorem the third term contributes $\|N_*(S_t P_t \mathbf{v})\|_2$, which is controlled pointwise by the non-tangential maximal function in the statement with appropriate opening. □

4. The domain of the Dirichlet semi-group

Assume (Dir-A) in the sense of Definition 2.1 is well-posed. If we set

$$\mathcal{P}_t u_0 = (e^{-tT_A}\mathbf{f})_0, \quad \mathbf{f} = \mathcal{S}^{-1} u_0 \in \overline{\mathsf{R}(\chi_+(T_A))}$$

for all $t > 0$, then Lemma 2.6 implies that $(\mathcal{P}_t)_{t>0}$ is a bounded C_0-semigroup on $L_2(\mathbf{R}^n, \mathbf{C}^m)$ [Recall that well-posedness includes uniqueness and this allows to prove the semigroup property].

Furthermore, with our definition of well-posedness of the Dirichlet problem, the domain of the infinitesimal generator $\mathcal{A}$ of this semi-group is contained in the Sobolev space $W^{1,2}(\mathbf{R}^n, \mathbf{C}^m)$ and $\|\nabla_x u_0\|_2 \lesssim \|\mathcal{A}u_0\|_2$. Indeed, from Lemma 2.6 we have for all $t > 0$, $\partial_t e^{-tT_A}\mathbf{f} = \nabla_{t,x} U(t,\cdot)$. Also $\partial_t e^{-tT_A}\mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}$ and the invertibility of $\mathcal{S}$ tells that $\nabla_{t,x} U(t,\cdot) = \mathcal{S}^{-1}(\partial_t U(t,\cdot))$. Therefore

$$\|\nabla_x U(t,\cdot))\|_2 \lesssim \|\partial_t U(t,\cdot)\|_2.$$

By definition of $\mathcal{A}$, $\partial_t U(t,\cdot) = \mathcal{A}U(t,\cdot)$, thus we have for all $t > 0$

$$\|\nabla_x U(t,\cdot))\|_2 \lesssim \|\mathcal{A}U(t,\cdot)\|_2.$$

The conclusion for the domain follows easily.

The question of whether this domain coincides with $W^{1,2}(\mathbf{R}^n, \mathbf{C}^m)$ is answered by the following theorem

Theorem 4.1. *Assume that (Dir-A) and (Dir-A**) *are well-posed. Then the domain of the infinitesimal generator* $\mathcal{A}$ *of* $(\mathcal{P}_t)_{t>0}$ *coincides with the Sobolev space* $W^{1,2}(\mathbf{R}^n, \mathbf{C}^m)$ *and* $\|\nabla_x u_0\|_2 \sim \|\mathcal{A}u_0\|_2$.

This theorem applies to the three situations listed in Theorem 2.4.

Proof. Combining [4, Lemma 4.2] (which says that (Dir-A^*) is equivalent to an auxiliary Neumann problem for A^*), [3, Proposition 2.52] (which says that this auxiliary Neumann problem is equivalent to a regularity problem for A: this is non trivial) with the proof of Theorem 2.2 in [4] (giving the necessary and sufficient condition below for well-posedness of the regularity problem for A), we have that (Dir-A^*) is well-posed if and only if

$$\mathcal{R} : \overline{\mathsf{R}(\chi_+(T_A))} \to \{\mathbf{g} \in L_2(\mathbf{R}^n; \mathbf{C}^{nm}) \ ; \ \mathrm{curl}_x(\mathbf{g}) = 0\}, \mathbf{f} \mapsto \mathbf{f}_\parallel$$

is invertible. This implies that for $\mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}$, we have that

$$\|\mathbf{f}\|_2 \sim \|\mathbf{f}_\parallel\|_2.$$

Therefore, the conjunction of well-posedness for (Dir-A) and (Dir-A^*) gives

$$\|\mathbf{f}_0\|_2 \sim \|\mathbf{f}_\parallel\|_2, \quad \mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}.$$

From this, it is easy to identify the domain of $\mathcal{A}$ by an argument as before. □

We have seen that invertibility of $\mathcal{S}$ reduces to that of $\mathcal{R}$ (up to taking adjoints). The only known way to prove it in such a generality (except for constant coefficients) is via a continuity method and the Rellich estimates showing that $\|\mathbf{f}_\parallel\|_2 \sim \|(A\mathbf{f})_0\|_2$ for all $\mathbf{f} \in \overline{\mathsf{R}(\chi_+(T_A))}$. This method was first used in the context of Laplace equation on Lipschitz domains by Verchota [20]. This depends strongly of A. Various relations between Dirichlet, regularity and Neumann problems for L^p data in the sense of non tangential approach for second order real symmetric equations are studied in [12, 13] and more recently in [14, 18].

References

[1] M. Alfonseca, P. Auscher, A. Axelsson, S. Hofmann, and S. Kim, *Analyticity of layer potentials and L^2 solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients*, Preprint at arXiv:0705.0836v1 [math.AP].

[2] P. Auscher and A. Axelsson, *Weighted maximal regularity estimates and solvability of non-smooth elliptic systems.*, Preprint.

[3] P. Auscher, A. Axelsson, and S. Hofmann, *Functional calculus of Dirac operators and complex perturbations of Neumann and Dirichlet problems*, J. Funct. Anal. **255** (2008), no. 2, 374–448.

[4] P. Auscher, A. Axelsson, and A. McIntosh, *On a quadratic estimate related to the Kato conjecture and boundary value problems*, Proceedings of the 8th International Conference on Harmonic Analysis and Partial Differential Equations El Escorial 2008, to appear in the AMS series of Contemporary Mathematics.

[5] ———, *Solvability of elliptic systems with square integrable boundary data*, To appear in ArKiv för Matematik, online DOI: 10.1007/s11512-009-0108-2.

[6] A. Axelsson, S. Keith, and A. McIntosh, *Quadratic estimates and functional calculi of perturbed Dirac operators*, Invent. Math. **163** (2006), no. 3, 455–497.

[7] R. R. Coifman and Y. Meyer, *Nonlinear harmonic analysis, operator theory and P.D.E*, in Beijing lectures in harmonic analysis (Beijing, 1984), Vol. 112 of *Ann. of Math. Stud.*, 3–45, Princeton Univ. Press, Princeton, NJ (1986).

[8] B. E. J. Dahlberg, *Poisson semigroups and singular integrals*, Proc. Amer. Math. Soc. **97** (1986), no. 1, 41–48.

[9] B. E. J. Dahlberg, D. S. Jerison, and C. E. Kenig, *Area integral estimates for elliptic differential operators with nonsmooth coefficients*, Ark. Mat. **22** (1984), no. 1, 97–108.

[10] S. Hofmann, *Dahlberg's bilinear estimate for solutions of divergence form complex elliptic equations*, Proc. Amer. Math. Soc. **136** (2008), no. 12, 4223–4233.

[11] D. S. Jerison and C. E. Kenig, *The Dirichlet problem in nonsmooth domains*, Ann. of Math. (2) **113** (1981), no. 2, 367–382.

[12] C. E. Kenig and J. Pipher, *The Neumann problem for elliptic equations with nonsmooth coefficients*, Invent. Math. **113** (1993), no. 3, 447–509.

[13] ———, *The Neumann problem for elliptic equations with nonsmooth coefficients. II*, Duke Math. J. **81** (1995), no. 1, 227–250 (1996). A celebration of John F. Nash, Jr.

[14] J. Kilty and Z. Shen, *The L^p regularity problem on Lipschitz domains*, arXiv:0904.3906v2 [math.AP].

[15] C. Li, A. McIntosh, and S. Semmes, *Convolution singular integrals on Lipschitz surfaces*, J. Amer. Math. Soc. **5** (1992), no. 3, 455–481.

[16] M. Mitrea, *On Dahlberg's Lusin area integral theorem*, Proc. Amer. Math. Soc. **123** (1995), no. 5, 1449–1455.

[17] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961) 577–591.

[18] Z. Shen, *A relationship between the Dirichlet and regularity problems for elliptic equations*, Math. Res. Lett. **14** (2007), no. 2, 205–213.

[19] N. T. Varopoulos, *Singular integrals and potential theory*, Milan J. Math. **75** (2007) 1–60.

[20] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal. **59** (1984), no. 3, 572–611.

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