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POLYNOMIAL DECAY FOR DAMPED WAVE EQUATIONS ON PARTIALLY RECTANGULAR DOMAINS

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Abstract. We consider the energy decay of a damped wave equation on the partially rectangular domain. By using integration by parts, assuming that the damping only exists near the boundary of non-rectangular part, we can prove the polynomial type energy decay. We also give some examples which are not available in the literature.

1. Introduction

In this article, we consider the energy decay of the damped wave equation on the partially rectangular domains. First we recall the setting of the problem. We consider the following equation,

$$\begin{cases} (\partial_t^2 - \Delta + 2a(x)\partial_t)u(t,x) = 0, \ (t,x) \in \mathbf{R}\times\Omega, \\ u = 0 \text{ on } \mathbf{R}\times\partial\Omega, \\ u|_{t=0} = u_0 \in H_0^1(\Omega), \ \partial_t u|_{t=0} = u_1 \in L^2(\Omega). \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbf{R}^n$ is a bounded domain with $C^{1,1}$ boundary; $C^{h,\theta}$ means that the boundary is C^h and the hth derivative is Hölder continuous with exponent θ. The damping term a is a non-negative function on Ω and continuous up to the boundary. $\Delta = \sum_{i=1}^n \partial_i^2$ denotes the Laplacian where we write $\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}$. We assume that the Hilbert space of the Cauchy data $\mathcal{H} = H_0^1(\Omega)\times L^2(\Omega)$ is equipped with the following norm,

$$\left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_{\mathcal{H}}^2 = \|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2.$$

We rewrite (1.1) as an evolution equation by using

$$A = \begin{pmatrix} 0 & Id \\ \Delta & -2a \end{pmatrix} : \ \mathcal{H}\to\mathcal{H}, \ \ \mathcal{D}(A) = (H_0^1 \cap H^2)\times H_0^1. \tag{1.2}$$

Then we can write (1.1) as

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = A \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad \left.\begin{pmatrix} u \\ \partial_t u \end{pmatrix}\right|_{t=0} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \tag{1.3}$$

Then applying the Hille-Yosida's theorem, we know (1.1) has a unique solution $u(t,x) \in C^0(\mathbf{R}; H_0^1(\Omega)) \cap C^1(\mathbf{R}; L^2(\Omega))$. For the solution u, we define its energy $E(u,t)$ by

$$E(u,t) := \frac{1}{2}\int_\Omega |\partial_t u|^2 + |\nabla u|^2 dx = \frac{1}{2}\left\| \begin{pmatrix} u \\ \partial_t u \end{pmatrix}(t,\cdot) \right\|_{\mathcal{H}}^2.$$

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The property of A is related to the energy of solutions. For example, we have $E(u,t) \leq E(u,0)$, $t \geq 0$ since A is dissipative operator;

$$\mathrm{Re}(Av, v) \leq 0, \quad \text{for all } v \in \mathcal{D}(A).$$

Moreover, by [5], it is known that $E(u,t) \to 0$ as $t \to \infty$ if $a(x) \not\equiv 0$. So we are interested in the energy decay rate for the damped wave equation (1,1). To this problem, Bardos, Lebeau and Rauch 's work [1] is important, which says that the energy decays exponentially with respect to the initial data under the geometric control condition; there exists $T_0 > 0$ such that any billiard trajectory in Ω of length$\geq T_0$ meets the set $\{x; a(x) > 0\}$, with some additional assumptions on $a(x)$ and Ω. On the other hand, Lebeau [5] shows that if $a \not\equiv 0$, then the energy decays logarithmically provided the initial data is sufficiently regular. The purpose of this article is to show the polynomial type energy decay for some damped wave equation to which the geometric control condition may not hold.

We assume that Ω is a bounded domain in $\mathbf{R}^2$ with $C^{1,1}$ boundary and Ω is the partially rectangular domain. This means that Ω can be written $\Omega = R \cup W$. Here R is a rectangle $\{(x,y); x \in [-\alpha, \alpha], y \in (-\beta, \beta)\}$ and the boundary of W can be written as $\partial W = \{(x,y); x = \{-\alpha, \alpha\},\ y \in [-\beta, \beta]\} \cup \Gamma$, $\Gamma \cap R = \emptyset$. In what follows, we write $x_1 = x, x_2 = y$.

A famous example of partially rectangular domain is the Bunimovich stadium S. This is a domain given by the union of rectangle $R = \{(x,y); x \in [-\alpha, \alpha], y \in [-\beta, \beta]\}$ and wing W; the two semicircular domain centered at $(\pm\alpha, 0)$ with radius β which lie outside R.

To the partially rectangular domain, Burq and Hitrik [4] show that if $a(x,y) > 0$ on $\overline{W}$ then the energy decays of polynomial order assuming the initial data is sufficiently regular. Our result is an improvement of this fact. The first result is the following one.

Theorem 1.1. *Let Ω be a partially rectangular domain with $C^{1,1}$boundary, $\Omega = R \cup W$. Assume $a(x,y) > 0$ on $\partial W \cap \partial\Omega$. Then for any $k > 0$ there exists a constant $C_k > 0$ such that*

$$E(u,t)^{\frac{1}{2}} \leq C_k \left(\frac{\log t}{t}\right)^{\frac{k}{2}} (\log t) \|(u_0, u_1)\|_{\mathcal{D}(A^k)} \tag{1.4}$$

for all $(u_0, u_1) \in \mathcal{D}(A^k)$.

In three dimension, Phung [7] obtains a polynomial decay estimate on a partially cubic domain assuming $a > 0$ near the boundary of non-cubic part though Phung does not specify exact decay rate. Theorem 1.1 is correspondent to the Phung's result and the bound (1.4) is similar to the result of [4].

Remark 1. In fact, from our proof of Theorem 1.1, we can prove the polynomial energy decay under the different damping assumption. We give two examples of this.

Example 1. On the Bunimovich stadium, by only assuming that $a(x,y) > 0$ near the boundary of one side of the wing part, similar energy decay holds.

Example 2 . On the Bunimovich stadium, we assume that $a(x,y) > 0$ on some band; $\overline{R}_0 = \{(x,y); x \in [\tilde{\alpha}, \hat{\alpha}], y \in [-\beta, \beta]\}$ for some $-\alpha < \tilde{\alpha} < \hat{\alpha} < \alpha$ then similar energy decay holds.

Example2 is correspondent to the result of Liu and Rao [6] to the polynomial energy decay on the square.

In Theorem 1.1, the damping term exists in the rectangular part. Next we consider the case that the damping term may vanish on the rectangular part. We introduce distance function from rectangular part; $w = \max\{|x| - \alpha, 0\}$.

Theorem 1.2. *Let Ω be a partially rectangular domain with $C^{h,\theta}$ boundary, $\Omega = R \cup W$. Assume $T = h + \theta - 1 \geq 1$, and $a(x, y) > Cw^m\chi$ for some $m > 0, C > 0$. Here $\chi \in C^\infty(\overline{\Omega})$ is a non-negative function satisfying $\chi = 1$ near $\partial\Omega$. Then for any $k > 0$ there exists a constant $C_k > 0$ such that*

$$E(u,t)^{\frac{1}{2}} \leq C_k \left(\frac{\log t}{t}\right)^{\frac{k}{2M}} (\log t)\|(u_0, u_1)\|_{\mathcal{D}(A^k)} \tag{1.5}$$

for all $(u_0, u_1) \in \mathcal{D}(A^k)$. Here $M = \max\{\frac{m+1}{T}, \frac{m+3-T}{T}, 1\}$.

Remark 2. The methods of proving above theorems are refinements of the methods of [3]. In fact, we can improve the result of [3] from the calculus to deriving this theorem.

We state about this. On the Bunimovich stadium $\Omega = R \cup W$, we consider the following equation

$$\begin{aligned} &(-\Delta - \lambda^2)u = g, \\ &u|_{\partial\Omega} = 0. \end{aligned} \tag{1.6}$$

Here $\lambda \in \mathbf{R}$, $u \in H_0^1(\Omega)$ and $g \in L^2(\Omega)$. In [3], for any u, g satisfying (1.6), they show following two bounds

$$\|u\|_{L^2(\Omega)} \leq C(\lambda^4\|u\|_{L^2(W)} + \lambda^2\|g\|^2_{L^2(\Omega)})$$
$$\|u\|_{L^2(\Omega)} \leq C(\lambda^8\|u\|_{L^2(W)} + \|g\|^2_{L^2(\Omega)})$$

where $C > 0$ depends only on α and β. We can improve these estimates. We shall show the estimate

$$\|u\|_{L^2(\Omega)} \leq C(\lambda^4\|u\|_{L^2(W)} + \|g\|^2_{L^2(\Omega)}) \tag{1.7}$$

for u, g satisfying (1.6). Here $C > 0$ depends only on α and β.

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2. Preliminary to the proof of theorems

Following, [4], [5], We adapt the stationary approach and we will estimate the resolvent $(i\lambda - A)^{-1}$. First by simple computation, we note following relation

$$(i\lambda - A)^{-1} = \begin{pmatrix} R(\lambda)(2a + i\lambda) & -R(\lambda) \\ R(\lambda)(2ia\lambda - \lambda^2) - 1 & -i\lambda R(\lambda) \end{pmatrix} \quad \lambda \in \mathbf{R}. \tag{2.1}$$

Here we write $R(\lambda) = (-\Delta + 2i\lambda a - \lambda^2)^{-1}$, $L^2(\Omega) \to L^2(\Omega)$. From a result of [5], we know that $i\lambda$, $\lambda \in \mathbf{R}$ is not the spectra of A if $a(x) \not\equiv 0$. From the bound of $R(\lambda)$, we can deduce a bound of the resolvent $(i\lambda - A)^{-1}$ by using the following proposition.

Proposition 2.1. *Assume that*

$$R(\lambda) = \mathcal{O}(|\lambda|^l): \quad L^2(\Omega) \to L^2(\Omega), \quad |\lambda| > 1, \lambda \in \mathbf{R}, \; l \geq 0.$$

Then we have

$$(i\lambda - A)^{-1} = \mathcal{O}(|\lambda|^{l+1}): \quad \mathcal{H} \to \mathcal{H}, \quad |\lambda| > 1, \lambda \in \mathbf{R} \; l \geq 0.$$

Proof. This can be proved by similar argument as in [4]. Seeing (2.1), we can prove this by showing the following bounds,

$$\begin{aligned} &R(\lambda) = \mathcal{O}(|\lambda|^{l+1}): \; L^2 \to H_0^1, \quad R(\lambda)(2ia\lambda - \lambda^2) - 1 = \mathcal{O}(|\lambda|^{l+1}): \; H_0^1 \to L^2, \\ &R(\lambda)(2a + i\lambda) = \mathcal{O}(|\lambda|^{l+1}): \; H_0^1 \to H_0^1. \end{aligned} \tag{2.2}$$

Since $(-\Delta + 2i\lambda a - \lambda^2)R(\lambda) = \mathrm{Id}$ we have

$$\begin{aligned} \|u\|_{L^2}^2 + \|R(\lambda)u\|_{L^2}^2 &\geq |((-\Delta + 2i\lambda a - \lambda^2)R(\lambda)u, R(\lambda)u)| \\ &\geq \|\nabla R(\lambda)u\|_{L^2}^2 - \lambda^2\|R(\lambda)u\|_{L^2}^2 - 2\lambda(aR(\lambda)u, R(\lambda)u). \end{aligned}$$

thus we have

$$\|R(\lambda)u\|_{H_0^1}^2 \leq \lambda^2\|R(\lambda)u\|_{L^2}^2 + \|u\|_{L^2}^2 \tag{2.3}$$

for $|\lambda| > 1, \lambda \in \mathbf{R}$. This implies

$$R(\lambda) = \mathcal{O}(|\lambda|^{l+1}) : L^2 \to H_0^1, \tag{2.4}$$

and by duality,

$$R(\lambda) = \mathcal{O}(|\lambda|^{l+1}) : H^{-1} \to L^2.$$

Then we have

$$R(\lambda)(2ia\lambda - \lambda^2) - 1 = R(\lambda)\Delta = \mathcal{O}(|\lambda|^{l+1}): \; H_0^1 \to L^2 \tag{2.5}$$

since $\Delta: \; H_0^1 \to H^{-1}$ is continuous. Inserting $u = (2a + i\lambda)v$, $v \in H_0^1$ in (2.3), we obtain

$$\begin{aligned} \|R(\lambda)(2a + i\lambda)v\|_{H_0^1}^2 &\leq \lambda^2\|R(\lambda)(2a + i\lambda)v\|_{L^2}^2 + \|(2a + i\lambda)v\|_{L^2}^2 \\ &\leq \|R(\lambda)(2i\lambda a - \lambda^2)v\|_{L^2}^2 + \lambda^2\|v\|_{L^2}^2 \end{aligned}$$

for $|\lambda| > 1, \lambda \in \mathbf{R}$. Using (2.5) and the Poincaré's inequality, we have

$$\|R(\lambda)(2a + i\lambda)v\|_{H_0^1}^2 \leq C|\lambda|^{2l+2}\|v\|_{H_0^1}^2. \tag{2.6}$$

From (2.4), (2.5) and (2.6), we have proved (2.2). □

From this type resolvent estimate, we can prove Theorems by using the following result. See [4] and [6].

Theorem 2.2. *Let $\mathcal{A}$ be a linear operator on Hilbert space $\mathcal{H}$. Assume that*
(H1). $\mathcal{A}$ generates a bounded C^0 semigroup $e^{t\mathcal{A}}$ on $\mathcal{H}$.
(H2). $i\mathbf{R} \cap \sigma(\mathcal{A}) = \emptyset$.
(H3). For some $l > 0$, $(i\lambda - \mathcal{A})^{-1} = \mathcal{O}(|\lambda|^l): \quad \mathcal{H} \to \mathcal{H}, \quad |\lambda| > 1, \lambda \in \mathbf{R}$.
Then for any $k > 0$ there exist a constant $C_k > 0$ such that

$$\|e^{t\mathcal{A}}x\|_{\mathcal{H}} \leq C_k \left(\frac{\log t}{t}\right)^{\frac{k}{l}} (\log t)\|x\|_{\mathcal{D}(\mathcal{A}^k)} \tag{2.7}$$

for all $x \in \mathcal{D}(\mathcal{A}^k)$.

To obtain the bound of $R(\lambda)$, we consider the following equation,

$$(-\Delta + 2i\lambda a - \lambda^2)u = f, \quad u|_{\partial\Omega} = 0. \tag{2.8}$$

Here $f \in L^2$, and $u \in H_0^1$. We write $g = -2i\lambda a u + f$ and we have

$$(-\Delta - \lambda^2)u = g, \quad \lambda \in \mathbf{R}. \tag{2.9}$$

We will estimate (2.9) by using similar method to [3]. The following commutator identity is mainly used.

Lemma 2.3. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and satisfy $(-\Delta - \lambda^2)u = g$, $\lambda \in \mathbf{R}$. Then for any vector field B whose coefficients are real and smooth up to boundary, we have*

$$(u, [(-\Delta - \lambda^2), B]u)_{L^2} = 2\mathrm{Re}(Bu, g)_{L^2} + ((\mathrm{div}B)u, g)_{L^2} + \int_{\partial\Omega} (\partial_N u)\overline{Bu}\ dS. \tag{2.10}$$

Here we compute $[(-\Delta - \lambda^2), B]$ in the sense of distribution and this is second order operator.

Proof. This is proved by direct computation. First by using Green's identity, we obtain

$$(-\Delta u, Bu)_{L^2} = (\nabla u, \nabla Bu)_{L^2} - \int_{\partial\Omega} \partial_N u \overline{Bu} dS. \tag{2.11}$$

From now on we write $B = b_1(x)\partial_{x_1} + \cdots + b_n(x)\partial_{x_n}$, $b = (b_1, \cdots, b_n)$ and we compute

$$\begin{aligned}
2\mathrm{Re}(\nabla u, \nabla Bu)_{L^2} &= (\nabla u, \nabla Bu)_{L^2} + (\nabla Bu, \nabla u)_{L^2} \\
&= (\nabla u, \sum_j \partial_j(\nabla b_j u))_{L^2} - (\nabla u, \nabla(\mathrm{div}Bu))_{L^2} + (\nabla Bu, \nabla u)_{L^2} \\
&= \int_{\partial\Omega} b \cdot \nu |\nabla u|^2 dS - \sum_j (\partial_j \nabla u, \nabla(b_j u))_{L^2} \\
&\quad - (\nabla u, \nabla(\mathrm{div}Bu))_{L^2} + (\nabla Bu, \nabla u)_{L^2}.
\end{aligned}$$

For the final identity, we use integration by parts and the Dirichlet boundary condition. We continue the computation

$$
\begin{aligned}
&-\sum_j(\partial_j \nabla u, \nabla(b_j u))_{L^2} - (\nabla u, \nabla(\mathrm{div} Bu))_{L^2} + (\nabla Bu, \nabla u)_{L^2} \\
&= -\sum_j((\nabla b_j)\partial_j \nabla u, u)_{L^2} - \sum_j(\partial_j \nabla u, b_j \nabla u)_{L^2} \\
&+ (\nabla Bu, \nabla u)_{L^2} - (\nabla u, \nabla(\mathrm{div} Bu))_{L^2} \\
&= -\sum_j((\nabla b_j)\partial_j \nabla u, u)_{L^2} + \sum_j((\nabla b_j)\partial_j u, \nabla u)_{L^2} - (\nabla u, \nabla(\mathrm{div} Bu))_{L^2} \\
&= \sum_j(\nabla u, \partial_j(\nabla b_j)u)_{L^2} + \sum_j(\partial_j u, (\nabla b_j)\nabla u)_{L^2} - (\nabla u, \nabla(\mathrm{div} Bu))_{L^2} \\
&= \sum_j(\nabla u, (\nabla b_j)\partial_j u)_{L^2} + \sum_j(\partial_j u, (\nabla b_j)\nabla u)_{L^2} - (\nabla u, (\mathrm{div} B)\nabla u)_{L^2} \\
&= -\sum_j(u, \nabla(\nabla b_j)\partial_j u)_{L^2} - \sum_j(u, (\nabla b_j)\nabla \partial_j u)_{L^2} - (u, -(\mathrm{div} B)\Delta u)_{L^2} \\
&= (u, [-\Delta, B]u)_{L^2} - ((\mathrm{div} B)u, -\Delta u)_{L^2}.
\end{aligned}
$$

By using the Dirichlet boundary condition, we note

$$\int_{\partial\Omega} b \cdot \nu |\nabla u|^2 dS = \int_{\partial\Omega} (\partial_N u)\overline{Bu} dS.$$

By using Integration by parts and the Dirichlet boundary condition, we have

$$2\mathrm{Re}(u, Bu)_{L^2} = \int_\Omega B|u|^2 dx = ((\mathrm{div} B)u, u)_{L^2}.$$

Summing up these calculation, we have proved Lemma 2.3. $\square$

The key lemma to estimating (2.9) is the following one.

Lemma 2.4. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and satisfy $(-\Delta - \lambda^2)u = g$, $\lambda \in \mathbf{R}$. Then we have*

$$\|u_{x_1}\|_{L^2}^2 \le C \int_{\partial\Omega} x_1 \omega_N |\partial_N u|^2 dS + C\|g\|_{L^2}^2. \tag{2.12}$$

Here ω_N is the x_1 component of the unit outer-ward normal.

Proof. In Lemma 2.3, we take $B = x_1\partial_{x_1}$. Then we note $[\Delta, x_1\partial_{x_1}] = 2\partial_{x_1}^2$ and we have

$$
\begin{aligned}
(u_{x_1}, u_{x_1}) = -(u, \partial_{x_1}^2 u) &= \frac{1}{2}(u, [-\Delta - \lambda^2, x_1\partial_{x_1}]u) \\
&= \frac{1}{2}\left(\int_{\partial\Omega} \partial_N u \overline{x_1\partial_{x_1} u}\, dS + 2\mathrm{Re}(x_1\partial_{x_1} u, g)_{L^2} + (u, g)_{L^2}\right)
\end{aligned}
$$

By using the Cauchy's inequality, We obtain

$$|2\mathrm{Re}(x_1\partial_{x_1} u, g)_{L^2} + (u, g)_{L^2}| \le \epsilon(\|u\|_{L^2}^2 + \|u_{x_1}\|_{L^2}^2) + C\|g\|_{L^2}^2$$

We write $\partial_{x_1} = \omega_T\partial_T + \omega_N\partial_N$ at each boundary point. Here ∂_T means the tangential differential to the boundary and ∂_N is the normal differential. From the Dirichlet condition, we note $\omega_T\partial_T u = 0$ at boundary. So we have the following inequality,

$$\|u_{x_1}\|_{L^2}^2 \le C\int_{\partial\Omega} x_1\omega_N|\partial_N u|^2 dS + \epsilon(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2) + C\|g\|_{L^2}^2$$

By the Poincaré's inequality we have $\|u\|_{L^2}^2 \le C\|u_{x_1}\|_{L^2}^2$. So by taking ϵ sufficiently small, we deduce

$$\|u_{x_1}\|_{L^2}^2 \le C\int_{\partial\Omega} x_1\omega_N|\partial_N u|^2 dS + C\|g\|_{L^2}^2$$

□

In the following section, we will give some bound to $R(\lambda)$ by estimating $\int_{\partial\Omega} x\omega_N|\partial_N u|^2 dS$ part by using Lemma 2.3 again. Next we need the following lemma.

Lemma 2.5. *Let $u \in H^2(\Omega)\cap H_0^1(\Omega)$ and satisfy $(-\Delta-\lambda^2)u = g$, $\lambda \in \mathbf{R}$. Then we have*

$$\|u\|_{H_0^1}^2 \le C\left(|\lambda|^2\|u\|_{L^2}^2 + |\lambda|^{-2}\|g\|_{L^2}^2\right). \tag{2.13}$$

Proof. This can be proved by easy computation;

$$\|\nabla u\|_{L^2}^2 = (-\Delta u, u)_{L^2} = (\lambda^2 u + g, u)_{L^2} \le C(\lambda^2\|u\|_{L^2}^2 + |\lambda|^{-2}\|g\|_{L^2}^2)$$

□

3. Proof of Theorem 1.1

We will show the following proposition,

Proposition 3.1. *Let Ω be a partially rectangular domain with $C^{1,1}$boundary, $\Omega = R\cup W$. Assume $a(x,y) > 0$ on $\partial W\cap\partial\Omega$. Then we have*

$$R(\lambda) = \mathcal{O}(|\lambda|): \quad L^2(\Omega)\to L^2(\Omega), \quad |\lambda| > 1, \lambda\in\mathbf{R} \tag{3.1}$$

where $R(\lambda) = (-\Delta + 2i\lambda a - \lambda^2)^{-1}$.

We can deduce this proposition from the following estimate,

$$\|u\|_{L^2}^2 \le C(\lambda^2\|a^{\frac{1}{2}}u\|_{L^2} + \|f\|_{L^2}^2), \quad |\lambda| > 1 \tag{3.2}$$

for $u \in H_0^1$, $f \in L^2$ satisfying (2.8).

First we will prove this by using Lemma 2.3. We assume that $g \in L^2(\Omega)$, $u \in H_0^1(\Omega)\cap H^2(\Omega)$ are satisfy (2.9).In Lemma 2.3, we take a vector field B as the following form $B = \psi(x,y)\partial_x$. Here we choose $\psi \in C^\infty(\bar\Omega)$ as a real-valued function. Then by using Lemma 2.3, we have

$$\int_{\partial\Omega}\psi\omega_N|\partial_N u|^2 dS = (u, [(-\Delta-\lambda^2), B]u)_{L^2} - 2\mathrm{Re}(Bu, g)_{L^2} - ((\mathrm{div}B)u, g)_{L^2}.$$

The second term and the third term of the right hand side can be estimated by using the Cauchy's inequality and the Poincaré's inequality;

$$|2\mathrm{Re}(Bu,g)_{L^2}| \le \epsilon\|u_x\|_{L^2}^2 + C\|g\|_{L^2}^2, \; |((\mathrm{div}B)u,g)_{L^2}| \le \epsilon\|u_x\|_{L^2}^2 + C\|g\|_{L^2}^2.$$

To estimate the first term, we note

$$[\Delta, B]u = 2\partial_x(\psi_x u_x) + 2\partial_y(\psi_y u_x) - (\psi_{xx} + \psi_{yy})u_x.$$

By using integration by parts and the Dirichlet boundary condition, we have

$$|(u, [\Delta, B]u)_{L^2}| \leq 2|(u_x, \psi_x u_x)_{L^2} + (u_y, \psi_y u_x)_{L^2}| + |(u, (\psi_{xx} + \psi_{yy})u_x)_{L^2}|$$

We obtain

$$\begin{aligned}|(u_x, \psi_x u_x)_{L^2} + (u_y, \psi_y u_x)_{L^2}| &= |((\partial_x\psi)\partial_x u + (\partial_y\psi)\partial_y u, \partial_x u)_{L^2}| \\ &\leq \epsilon\|u_x\|_{L^2}^2 + C\|\nabla\psi \cdot \nabla u\|_{L^2}^2\end{aligned}$$

and

$$|(u, (\psi_{xx} + \psi_{yy})u_x)_{L^2}| \leq \epsilon(\|u_x\|_{L^2}^2 + C\{\|\psi_{xx}u\|_{L^2}^2 + \|\psi_{yy}u\|_{L^2}^2).$$

Next we will estimate $\|\nabla\psi \cdot \nabla u\|_{L^2}^2$. First we note

$$\|\nabla\psi \cdot \nabla u\|_{L^2}^2 \leq 2\sum_{i=1}^{2} \|\partial_{x_i}\psi\partial_{x_i}u\|_{L^2}^2 \leq 2\sum_{i=1}^{2} \|\psi_{x_i}\nabla u\|_{L^2}^2.$$

So we will estimate each $\|\psi_{x_i}\nabla u\|_{L^2}^2$. By using integration by parts and the Dirichlet boundary condition, we compute

$$\begin{aligned}\|\psi_{x_i}\nabla u\|_{L^2}^2 =&(\psi_{x_i}^2\nabla u, \nabla u)_{L^2} = -(\nabla \cdot (\psi_{x_i}^2\nabla u), u)_{L^2} \\ =&-(\psi_{x_i}^2\Delta u, u)_{L^2} - ((\nabla\psi_{x_i}^2) \cdot \nabla u, u)_{L^2} \\ =&-(\psi_{x_i}^2\Delta u, u)_{L^2} - 2(\psi_{x_i}(\nabla\psi_{x_i}) \cdot \nabla u, u)_{L^2}\end{aligned}$$

We can estimate the first term by using $\Delta u = \lambda^2 u - g$,

$$|-(\psi_{x_i}^2\Delta u, u)_{L^2}| \leq \|\psi_{x_i}g\|_{L^2}^2 + (\lambda^2 + 1)\|\psi_{x_i}u\|_{L^2}^2.$$

The second term is estimated as follows,

$$\begin{aligned}|(\psi_{x_i}(\nabla\psi_{x_i}) \cdot \nabla u, u)_{L^2}| &= |\sum_{j=1}^{2}(\psi_{x_i}(\partial_{x_j}\psi_{x_i})u_{x_j}, u)_{L^2}| = \sum_{j=1}^{2}|(\psi_{x_i}u_{x_j}, (\partial_{x_j}\psi_{x_i}u)_{L^2}| \\ &\leq \sum_{j=1}^{2}(\epsilon\|\psi_{x_i}u_{x_j}\|_{L^2}^2 + C\|(\partial_{x_j}\psi_{x_i})u\|_{L^2}^2) = \epsilon\|\psi_{x_i}\nabla u\|_{L^2}^2 + C\sum_{j=1}^{2}\|(\partial_{x_j}\psi_{x_i})u\|_{L^2}^2.\end{aligned}$$

Taking ϵ sufficiently small, we deduce

$$\|\psi_{x_i}\nabla u\|_{L^2}^2 \leq C(\|\psi_{x_i}g\|_{L^2}^2 + (\lambda^2 + 1)\|\psi_{x_i}u\|_{L^2}^2 + \sum_{j=1}^{2}\|(\partial_{x_j}\psi_{x_i})u\|_{L^2}^2).$$

Summing up these estimates, we have

Lemma 3.2. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and let $\psi \in C^\infty(\overline{\Omega})$ is real-valued. Assume that u satisfy $(-\Delta - \lambda^2)u = g$, $\lambda \in \mathbf{R}$. Then for any $\epsilon > 0$ and some constant $C > 0$, the following inequality holds*

$$\begin{aligned}\int_{\partial\Omega}\psi\omega_N|\partial_N u|^2 dS \leq& \epsilon\|u_x\|_{L^2}^2 + C(\|g\|_{L^2}^2 + \|(\nabla\psi)g\|_{L^2}^2 \\ &+ (\lambda^2 + 1)\|(\nabla\psi)u\|_{L^2}^2 + \sum_{i,j=1}^{2}\|(\partial_{x_i,x_j}^2\psi)u\|_{L^2}^2).\end{aligned} \tag{3.3}$$

Here ω_N is the x component of the unit outer-ward normal.

We will prove (3.2). Assume $a(x, y) > 0$ on $\partial W \cap \partial\Omega$. Then we take $\psi(x, y) = x\phi(x, y)$. Here $\phi \in C^\infty(\overline{\Omega})$ is a real-valued function satisfying $\mathrm{supp}(\phi) \subset \{(x, y) \in \Omega; a(x, y) > C\}$ for some $C > 0$. Moreover we can assume $\phi|_{\partial W \cap \partial\Omega} = 1$ since $a(x, y) > 0$ on $\partial W \cap \partial\Omega$. We note $\omega_N = 0$ on the boundary of rectangular part and $\phi = 1$ on the boundary of the wing part. We have

$$\int_{\partial\Omega} x\phi\omega_N|\partial_N u|^2 dS = \int_{\partial\Omega} x\omega_N|\partial_N u|^2 dS. \tag{3.4}$$

In Lemma 3.2, we can replace $\nabla\psi, \partial^2_{i,j}\psi$ by $a^{\frac{1}{2}}$ using $a > C$ on $\mathrm{supp}(\phi) \supset \mathrm{supp}\psi$ for some $C > 0$.

$$\int_{\partial\Omega} x\phi\omega_N|\partial_N u|^2 dS \leq C(\|g\|^2_{L^2} + (\lambda^2 + 1)\|a^{\frac{1}{2}}u\|^2_{L^2}) + \epsilon\|u_x\|^2_{L^2}. \tag{3.5}$$

From Lemma 2.4, (3.4) and (3.5), for $u \in H^1_0(\Omega) \cap H^2(\Omega)$, $g \in L^2(\Omega)$ satisfying (2.9), following inequality holds,

$$\|u_x\|^2_{L^2} \leq C(\|g\|^2_{L^2} + (\lambda^2 + 1)\|a^{\frac{1}{2}}u\|^2_{L^2}).$$

By using the Poincaré's inequality, we obtain

$$\|u\|^2_{L^2} \leq C((\lambda^2 + 1)\|a^{\frac{1}{2}}u\|^2_{L^2} + \|g\|^2_{L^2}).$$

For $u \in H^1_0(\Omega)$, $g \in L^2(\Omega)$ satisfying (2.9), using global regularity of the Laplace equation, since the boundary is $C^{1,1}$, we have $u \in H^2(\Omega)$ and the above inequarity holds. Recall $g = -2i\lambda au + f$, so we have (3.2).

We will prove Proposition 3.1 from (3.2). To estimate $\lambda^2\|a^{\frac{1}{2}}u\|^2$, we multiplying (2.8) by $\bar{u}$ and integrate in Ω. By using integration by parts and taking the imaginary part, we have

$$2\lambda\|a^{\frac{1}{2}}u\|^2_{L^2} = \mathrm{Im}(f, u)_{L^2}.$$

This gives us

$$|\lambda|\|a^{\frac{1}{2}}u\|^2_{L^2} \leq C\|f\|_{L^2}\|u\|_{L^2}. \tag{3.6}$$

So we have

$$\|u\|^2_{L^2} \leq C(|\lambda|\|f\|_{L^2}\|u\|_{L^2} + \|f\|^2_{L^2}) \leq C|\lambda|^2\|f\|^2_{L^2} + \epsilon\|u\|^2_{L^2}, \quad |\lambda| > 1, \lambda \in \mathbf{R}.$$

This shows Proposition 3.1. Applying Proposition 2.1 and Theorem 2.2 to this proposition, we have Theorem1.1

We remark some improvement of Theorem 1.1.

Remark 1. In the estimate of $\int_{\partial\Omega} x\omega_N|\partial_N u|^2 dS$, we can abandon $x\omega_N \leq 0$ part. So in the proof of Proposition 2.4, we take $\phi \geq 0$ satisfies $\phi = 1$ on $\{(x, y) \in \partial W \cap \partial\Omega; x\omega_N(x, y) \geq 0\}$. Then the same proof hold and we can deduce the polynomial energy decay if we assume $a(x, y) > 0$ on $\{(x, y) \in \overline{\Omega}; x\omega_N(x, y) \geq 0\}$.

For an application of this fact, we have Example 1 of the introduction. This is because, by parallel transformation to x axis, we can assume $x\omega_N \leq 0$ on the other side of boundary where the damping term may vanish.

Remark 2. We use $a(x, y) > C$, for some $C > 0$ on $\mathrm{supp}(\psi)$ to estimate (3.3) but the same estimate holds if we assume $a(x, y) > C$ on $\mathrm{supp}(\nabla\psi)$.

Using this we can prove the polynomial energy decay under the different type damping assumption. For example, we will give a proof of Example 2 of the introduction

Proof. We can assume $S \subset \{(x,y); x > 0\}$ by parallel transformation to x axis. Then $x\omega_N \le 0$ on the left hand side of the boundary of W. We choose real-valued smooth function ψ such that ψ is constant function except $\overline{R}_0$ and $\psi \equiv 1$ on $x\omega_N > 0$ region and $\psi \equiv 0$ on $x\omega_N < 0$ region. Then, by Remark 1, we have

$$\int_{\partial\Omega} x\omega_N |\partial_N u|^2 dS \le C \int_{\partial\Omega} \psi\omega_N |\partial_N u|^2 dS$$

and from Remark 2, we can prove the polynomial energy decay. □

4. Proof of Theorem 1.2

To prove Theorem 1.2, we will estimate $\int_{\partial\Omega} x\omega_N|\partial_N u|^2 dS$ more carefully. By using the parallel transformation, we can assume $\Omega = R \cup W$, $R = \{(x,y); x \in [-2\alpha, 0], y \in [-\beta, \beta]\}$. Then $\int_{\partial\Omega} x\omega_N|\partial_N u|^2 dS$ becomes $\int_{\partial\Omega}(x+\alpha)\omega_N|\partial_N u|^2 dS$. Let $\partial\Omega$ is $C^{h,\theta}$, $h+\theta \ge 2$ and $a(x,y) \ge Cx^m\chi$, $m \ge 0$. Here χ is a non-negative function satisfying $\chi = 1$ near $\partial\Omega$. We will estimate $\int_{\partial\Omega}(x+\alpha)\omega_N|\partial_N u|^2 dS$ assuming that u is a smooth function, equal to zero on $\partial\Omega$, and satisfy (2.9).

We only estimate on the right hand side of the region $W_0 = W \cap \{(x,y); x \ge 0\}$ and the other side is same. As in [3], we decompose W_0 to three regions; Region I: $0 \le x \le \left(\frac{\delta}{|\lambda|}\right)^l$, Region II: $\left(\frac{\delta}{|\lambda|}\right)^l \le x \le \frac{1}{\tilde{C}}$, Region III: $\frac{1}{\tilde{C}} \le x$ for some $\delta > 0$, $\tilde{C} > 0$ and $l > 0$. We will estimate $\int x\omega_N|\partial_N u|^2 dS$ on each regions.

We start from Region III. We take ϕ as a real-valued smooth function satisfying $\phi \equiv 1$ on $\partial W \cap \{(x,y), x \ge \frac{1}{\tilde{C}}\}$ and $\mathrm{supp}(\phi) \subset \{(x,y) \in \Omega; a(x,y) > C\}$ for some $C > 0$. Then by similar argument in the proof of theorem 1.1, we have

$$\begin{aligned}(4.1)\quad &\int_{\partial\Omega\cap III}(x+\alpha)\omega_N|\partial_N u|^2 dS \\ &\qquad \le C\Big(\int_{\partial\Omega\cap(I\cup II)}(x+\alpha)\omega_N|\partial_N u|^2 dS + \|g\|^2_{L^2} + (\lambda^2+1)\|a^{\frac{1}{2}}u\|^2_{L^2}\Big).\end{aligned}$$

Next we discuss on Region I. We take B as following form $B = \phi(x)\psi(y)\partial_y$. Here $\phi(x)$, $\psi(y)$ are real-valued smooth function on $\mathbf{R}$ satisfies $0 \le \phi(x) \le 1$, $\phi(x) \equiv 1$ on $0 \le x \le \frac{1}{2\tilde{C}}$, $\mathrm{supp}(\phi(x)) \subset \{x, ; \alpha \le x \le \frac{1}{\tilde{C}}\}$ and $\psi(\beta) \equiv 1$, $\psi(-\beta) \equiv -1$. We apply Lemma 2.3,

(4.2)

$$\int_{\partial\Omega}(\partial_N u)\overline{\phi(x)\psi(y)\partial_y u}\, dS = (u, [(-\Delta-\lambda^2), B]u)_{L^2} - 2\mathrm{Re}(Bu, g)_{L^2} - ((\mathrm{div}B)u, g)_{L^2}.$$

Since $\partial_y = \partial_N$ on $(x,y) = (0,\beta)$, $\partial_y = -\partial_N$ on $(x,y) = (0,-\beta)$ and the boundary is $C^{1,1}$, we have

$$(4.3)\qquad \int_{\partial\Omega}(\partial_N u)\overline{\phi(x)\psi(y)\partial_y u}\, dS \ge \frac{1}{C}\int_{\partial\Omega\cap \mathrm{I}}|\partial_N u|^2 dS$$

by taking $\tilde{C}$ sufficient large. Since $[(-\Delta-\lambda^2),B]$ is a second order operator using the global regularity and $-\Delta u=\lambda^2 u+g$, we have

$$(u,[(-\Delta-\lambda^2),B]u)_{L^2}\le C\|(-\Delta+1)u\|_{L^2}\|u\|_{L^2}\le C(\lambda^2\|u\|_{L^2}^2+\|g\|_{L^2}^2) \tag{4.4}$$

for $|\lambda|>1$. By the Cauchy inequality and Lemma 2.5, we have

$$\begin{aligned}|2\mathrm{Re}(Bu,g)_{L^2}+((\mathrm{div}B)u,g)_{L^2}|&\le C(\|u\|_{L^2}^2+\|Bu\|_{L^2}^2+\|g\|_{L^2}^2)\\&\le C(\lambda^2\|u\|_{L^2}^2+\|g\|_{L^2}^2)\end{aligned} \tag{4.5}$$

for $|\lambda|>1$. Since the boundary is $C^{h,\theta}$ and $\omega_N=0$ on rectangle part, we have $\omega_N=\mathcal{O}(|x|^T)$ and we have $\omega_N=\mathcal{O}\left((\frac{\delta}{|\lambda|})^{lT}\right)$ on Region I. Here we write $T=h+\theta-1$. So from (4.2), (4.3) ,(4.4) and (4.5), the following inequality folds

$$\int_{\partial\Omega\cap\mathrm{I}}\omega_N|\partial_N u|^2dS\le C(\delta^{lT}|\lambda|^{2-lT}\|u\|_{L^2}^2+\delta^{lT}|\lambda|^{-lT}\|g\|_{L^2}^2).$$

We choose $lT\ge 2$ and $\delta\ll 1$, we obtain

$$\int_{\partial\Omega\cap\mathrm{I}}\omega_N|\partial_N u|^2dS\le \epsilon(\|u\|_{L^2}^2+\|g\|_{L^2}^2). \tag{4.6}$$

Finally we estimate on Region II. We use λ depend vector field $B=|x|^N\phi_\lambda(x)\psi(y)\partial_y=\tilde{\phi}(x)\psi(y)\partial_y$. Here $\phi_\lambda(x)$, $\psi(y)$ are real-valued smooth functions on $\mathbf{R}$. Assume that $\phi_\lambda(x)$ and $\psi(y)$ satisfy $0\le\phi_\lambda(x)\le 1$, $\phi_\lambda(x)\equiv 1$ on $x\ge\left(\frac{\delta}{|\lambda|}\right)^l$, $\phi_\lambda(x)\equiv 0$ on $x\le\frac{1}{2}\left(\frac{\delta}{|\lambda|}\right)^l$, $\partial_x\phi_\lambda\ge 0$ and $\psi(\beta)\equiv 1$, $\psi(-\beta)\equiv -1$, $\partial_y\psi(y)\ge 0$, $|\psi(y)|\in C^\infty$ and $\mathrm{supp}(\phi_\lambda\psi)\cap\bar{\Omega}\subset\{\chi(x,y)=1\}$. By using $|x|\sim|\lambda|^{-l}$ on $\mathrm{supp}(\partial_x\phi_\lambda)$, we note $\partial_x^n\tilde{\phi}(x)=\mathcal{O}(|x|^{N-n})$. We take N satisfying $N\ge m+3-\frac{2}{l}$ and $N\ge m+1$. We obtain by applying lemma 2.3,

$$\int_{\partial\Omega}(\partial_N u)\overline{\tilde{\phi}(x)\psi(y)\partial_y u}\,dS=(u,[(-\Delta-\lambda^2),B]u)_{L^2}-2\mathrm{Re}(Bu,g)_{L^2}-((\mathrm{div}B)u,g)_{L^2}.$$

As in the proof of Lemma 2.4, we note that the left hand side is real-valued. So we have

$$\begin{aligned}&\int_{\partial\Omega}(\partial_N u)\overline{\tilde{\phi}(x)\psi(y)\partial_y u}\,dS\\&\quad=\mathrm{Re}(u,[(-\Delta-\lambda^2),B]u)_{L^2}-2\mathrm{Re}(Bu,g)_{L^2}-\mathrm{Re}((\mathrm{div}B)u,g)_{L^2}.\end{aligned} \tag{4.7}$$

Since $\omega_N=\mathcal{O}(|x|^T)$ and $\left(\frac{\delta}{\lambda}\right)^l\le|x|\le C$ on Region II, we have

$$\begin{aligned}\int_{\partial\Omega}(\partial_N u)\overline{\tilde{\phi}(x)\psi(y)\partial_y u}\,dS&\ge\frac{1}{C}\int_{\partial\Omega\cap\mathrm{II}}|x|^{N-T}x\omega|\partial_N u|^2dS\\&\ge\frac{1}{C}\mathrm{Min}\{(|\lambda|)^{-l(N-T)},1\}\int_{\partial\Omega\cap\mathrm{II}}x\omega|\partial_N u|^2dS\end{aligned} \tag{4.8}$$

We will estimate $\int_{\partial\Omega}(\partial_N u)\overline{\tilde{\phi}(x)\psi(y)\partial_y u}\,dS$ by using (4.7). Because $\mathrm{div}B=\tilde{\phi}\psi_y$, we have

$$|((\mathrm{div}B)u,g)_{L^2}|\le\lambda^2\|\tilde{\phi}\psi_y u\|_{L^2}^2+\lambda^{-2}\|g\|_{L^2}^2.$$

Since $N\ge m+1$, we can exchange $|\tilde{\phi}\psi_y|^2$ to $a(x,y)$

$$|((\mathrm{div}B)u,g)_{L^2}|\le C\lambda^2\|a^{\frac{1}{2}}u\|_{L^2}^2+\lambda^{-2}\|g\|_{L^2}^2. \tag{4.9}$$

We compute

$$[(-\Delta-\lambda^2),B]u=-2\partial_x(\tilde{\phi}_x\psi u_y)-2\partial_y(\tilde{\phi}\psi_y u_y)+\tilde{\phi}_{xx}\psi u_y+\tilde{\phi}\psi_{yy}u_y.$$

By integration by parts, we have

$$\begin{aligned}\mathrm{Re}(u,[(-\Delta-\lambda^2),B]u)_{L^2}=2\mathrm{Re}(\tilde{\phi}_x\psi u_y,u_x)_{L^2}+2(\tilde{\phi}\psi_y u_y,u_y)_{L^2}\\ +\mathrm{Re}(u,\tilde{\phi}_{xx}\psi u_y)_{L^2}+\mathrm{Re}(u,\tilde{\phi}\psi_{yy}u_y)_{L^2}.\end{aligned}\tag{4.10}$$

We will estimate second part of the right hand side. Since $\tilde{\phi}\psi_y\geq 0$, we add $(\tilde{\phi}\psi_y u_x,u_x)$ and using integration by parts, we obtain

$$\begin{aligned}(\tilde{\phi}\psi_y u_y,u_y)_{L^2}&\leq(\tilde{\phi}\psi_y u_y,u_y)_{L^2}+(\tilde{\phi}\psi_y u_x,u_x)_{L^2}\\ &=-(\tilde{\phi}\psi_y\Delta u,u)_{L^2}-(\tilde{\phi}\psi_{yy}u_y,u)_{L^2}-(\tilde{\phi}_x\psi_y u_x,u)_{L^2}\\ &\leq C(|\lambda|^2\|(\tilde{\phi}\psi_y)^{\frac{1}{2}}u\|_{L^2}^2+|\lambda|^{-2}\|g\|_{L^2}^2)-\mathrm{Re}\{(\tilde{\phi}\psi_{yy}u_y,u)_{L^2}+(\tilde{\phi}_x\psi_y u_x,u)_{L^2}\}.\end{aligned}\tag{4.11}$$

Here we use $-\Delta u=\lambda^2u+g$. The first part of right hand side of (4.11) can be estimated by using $N\geq m+1$ and exchange $|\tilde{\phi}\psi_y|^2$ to $a(x,y)$

$$\|(\tilde{\phi}\psi_y)^{\frac{1}{2}}u\|_{L^2}^2\leq C\|a^{\frac{1}{2}}u\|_{L^2}^2.$$

To estimate the third part and forth part of (4.11), we use the following identity

$$\begin{aligned}2\mathrm{Re}(\tilde{\phi}\psi_{yy}u_y,u)_{L^2}&=(\tilde{\phi}\psi_{yy}u_y,u)_{L^2}+(\tilde{\phi}\psi_{yy}u,u_y)_{L^2}=-(\tilde{\phi}\psi_{yyy}u,u)_{L^2}\\ 2\mathrm{Re}(\tilde{\phi}_x\psi_y u_x,u)_{L^2}&=(\tilde{\phi}_x\psi_y u_x,u)_{L^2}+(\tilde{\phi}_x\psi_y u,u_x)_{L^2}=-(\tilde{\phi}_{xx}\psi_y u,u)_{L^2}.\end{aligned}$$

Since $|x|^{-\frac{2}{l}}\leq C|\lambda|^2$ on $\mathrm{supp}(\phi_\lambda)$ and $N+\frac{2}{l}-2\geq m$, we have

$$\begin{aligned}|(\tilde{\phi}\psi_{yyy}u,u)_{L^2}|&=|(|x|^{-\frac{2}{l}}|x|^{\frac{2}{l}}\tilde{\phi}\psi_{yyy}u,u)_{L^2}|\leq C|\lambda|^2\|a^{\frac{1}{2}}u\|_{L^2}^2\\ |(\tilde{\phi}_{xx}\psi_y u,u)_{L^2}|&=|(|x|^{-\frac{2}{l}}|x|^{\frac{2}{l}}\tilde{\phi}_{xx}\psi_y u,u)_{L^2}|\leq C|\lambda|^2\|a^{\frac{1}{2}}u\|_{L^2}^2\end{aligned}$$

The same estimate can be applied to the third part and forth part of (4.10).

$$\begin{aligned}2\mathrm{Re}(u,\tilde{\phi}_{xx}\psi u_y)_{L^2}&=-(u,\tilde{\phi}_{xx}\psi_y u)_{L^2}\leq C|\lambda|^2\|a^{\frac{1}{2}}u\|_{L^2}^2\\ 2\mathrm{Re}(u,\tilde{\phi}\psi_{yy}u_y)_{L^2}&=-(u,\tilde{\phi}\psi_{yyy}u)_{L^2}\leq C\|a^{\frac{1}{2}}u\|_{L^2}^2\end{aligned}$$

Next we estimate the first part of (4.10),

$$\begin{aligned}|(\tilde{\phi}_x\psi u_y,u_x)_{L^2}|&=|((\tilde{\phi}_x\psi)_+^{\frac{1}{2}}u_y,(\tilde{\phi}_x\psi)_+^{\frac{1}{2}}u_x)_{L^2}-((\tilde{\phi}_x\psi)_-^{\frac{1}{2}}u_y,(\tilde{\phi}_x\psi)_-^{\frac{1}{2}}u_x)_{L^2}|\\ &\leq\||\tilde{\phi}_x\psi|^{\frac{1}{2}}u_x\|_{L^2}^2+\||\tilde{\phi}_x\psi|^{\frac{1}{2}}u_y\|_{L^2}^2.\end{aligned}$$

Here we write $f_+=\max\{f,0\}$ and $f_-=\max\{-f,0\}$. By using integration by parts, we obtain

$$\begin{aligned}\||\tilde{\phi}_x\psi|^{\frac{1}{2}}u_x\|_{L^2}^2+\||\tilde{\phi}_x\psi|^{\frac{1}{2}}u_y\|_{L^2}^2&=(\tilde{\phi}_x|\psi|u_x,u_x)_{L^2}+(\tilde{\phi}_x|\psi|u_y,u_y)_{L^2}\\ &\leq C(\lambda^2\||\tilde{\phi}_x\psi|^{\frac{1}{2}}u\|_{L^2}^2+|\lambda|^{-2}\|g\|_{L^2}^2-\mathrm{Re}\{(\tilde{\phi}_x|\psi|_y u_y,u)_{L^2}+(\tilde{\phi}_{xx}|\psi|u_x,u)_{L^2}\}.\end{aligned}$$

We have

$$\lambda^2\||\tilde{\phi}_x\psi|^{\frac{1}{2}}u\|_{L^2}^2\leq C\lambda^2\|a^{\frac{1}{2}}u\|_{L^2}^2,$$

$$|2\mathrm{Re}(\tilde{\phi}_x|\psi|_y u_y,u)_{L^2}|=|(\tilde{\phi}_x|\psi|_{yy}u,u)_{L^2}|=|(|x|^{-\frac{2}{l}}|x|^{\frac{2}{l}}\tilde{\phi}_x|\psi|_{yy}u,u)_{L^2}|\leq C\lambda^2\|a^{\frac{1}{2}}u\|_{L^2}^2$$

by using $N \geq m+1$ and $N + \frac{2}{l} - 2 \geq m$. We also have

$$|2\mathrm{Re}(\tilde{\phi}_{xx}|\psi|u_x, u)_{L^2}| = |(\tilde{\phi}_{xxx}|\psi|u, u)_{L^2}| \\ = |(|x|^{-\frac{2}{l}}|x|^{\frac{2}{l}}\tilde{\phi}_{xxx}|\psi|u, u)_{L^2}| \leq C\lambda^2\|a^{\frac{1}{2}}u\|_{L^2}^2. \tag{4.12}$$

Here we use $N + \frac{2}{l} - 3 \geq m$. Summing up above estimates, we have

$$\mathrm{Re}(u, [(-\Delta - \lambda^2), B]u)_{L^2} \leq C(|\lambda|^2\|a^{\frac{1}{2}}u\|_{L^2}^2 + \|g\|_{L^2}^2). \tag{4.13}$$

$\mathrm{Re}(Bu, g)_{L^2}$ can be estimate by the same argument. First we obtain

$$\mathrm{Re}(Bu, g)_{L^2} \leq \|\tilde{\phi}\psi u_y\|_{L^2}^2 + \|g\|_{L^2}^2.$$

To estimate $\|\tilde{\phi}\psi u_y\|_{L^2}^2$, adding $\|\tilde{\phi}\psi u_x\|_{L^2}^2$ and using integration by parts, we have,

$$\|\tilde{\phi}\psi u_y\|_{L^2}^2 \leq C(\lambda^2\|\tilde{\phi}\psi u\|_{L^2}^2 + \lambda^{-2}\|g\|_{L^2}^2 - (\tilde{\phi}^2(\psi^2)_y u_y, u)_{L^2} - ((\tilde{\phi}^2)_x\psi^2 u_x, u)_{L^2}).$$

Applying the same argument to estimate (4.11), using $N \geq m+1$ and $N + \frac{2}{l} - 2 \geq m$, we have

$$\|\tilde{\phi}\psi u\|_{L^2}^2 \leq C\|a^{\frac{1}{2}}u\|_{L^2}^2,$$
$$-(\tilde{\phi}^2(\psi^2)_y u_y, u)_{L^2} - ((\tilde{\phi}^2)_x\psi^2 u_x, u)_{L^2} \leq C\lambda^2\|a^{\frac{1}{2}}u\|_{L^2}^2.$$

So we obtain

$$2\mathrm{Re}(Bu, g)_{L^2} \leq C\|a^{\frac{1}{2}}u\|_{L^2}^2 + \|g\|_{L^2}^2. \tag{4.14}$$

From (4.7), (4.9), (4,13), and (4.14), we have

$$\int_{\partial\Omega} (\partial_N u)\overline{\tilde{\phi}(x)\psi(y)\partial_y u}\; dS \leq C(|\lambda|^2\|a^{\frac{1}{2}}u\|_{L^2}^2 + \|g\|_{L^2}^2). \tag{4.15}$$

So we get from (4.1), (4.6), (4.8) and (4.15)

$$\int_{\partial\Omega\cap W_0} x\omega|\partial_N u|^2 dS \leq C\mathrm{Max}\{(|\lambda|)^{l(N-T)}, 1\}(|\lambda|^2\|a^{\frac{1}{2}}u\|_{L^2}^2 + \|g\|_{L^2}^2) + \epsilon\|u\|_{L^2}^2. \tag{4.16}$$

We can apply the same estimate to the other side of wings and by using lemma 2.4 and by the same argument as in proof of theorem 1.1, we have

$$\|u\|_{L^2} \leq C|\lambda|^{\mathrm{Max}\{l(N-T)+1, 1\}}\|f\|_{L^2} \tag{4.17}$$

for $u \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$ satisfying (2.8). Here $N \geq m + 3 - \frac{2}{l}$, $N \geq m+1$ and $lT \geq 2$. Taking $l = \frac{2}{T}$ and by using Proposition 2.1, we have

$$(i\lambda - A)^{-1} = \mathcal{O}(|\lambda|^K): \quad \mathcal{H} \to \mathcal{H}, \quad |\lambda| > 1, \lambda \in \mathbf{R}\; l \geq 0. \tag{4.18}$$

Here $K = \max\{2\frac{N}{T}, 2\}$ and N satisfies $N \geq m+1$ and $N \geq m + 3 - T$. From this estimate, we obtain Theorem 1.2 by using Theorem 2.2.

Remark. From above calculus, we can improve the result of [3]. In above estimate, except (4.12), the same estimate holds only assuming $N + \frac{2}{l} - 2 \geq m$ and $N \geq m+1$. We replace the estimate (4.12) by assuming $N + \frac{4}{l} - 3 \geq m$ as following

$$|2\mathrm{Re}(\tilde{\phi}_{xx}|\psi|u_x, u)_{L^2}| = |(\tilde{\phi}_{xxx}|\psi|u, u)_{L^2}| \\ = |(|x|^{-\frac{4}{l}}|x|^{\frac{4}{l}}\tilde{\phi}_{xxx}|\psi|u, u)_{L^2}| \leq C\lambda^4\|a^{\frac{1}{2}}u\|_{L^2}^2.$$

Here we use $|x|^{-\frac{1}{l}} \leq C|\lambda|$ on $\mathrm{supp}(\tilde{\phi})$.

On the Bunimovich stadium $\Omega = R \cup W$, since $\partial\Omega$ is $C^{1,1}$, we can take $m = 0$, $l = 2$ and $N = 1$. We take χ as $\chi \equiv 1$ on W and $\chi \equiv 0$ on R. In above estimate, we can exchange $\|au\|_{L^2}$ to $\|\chi u\|_{L^2}$ and we have the estimate (1.7) for u, g satisfying (1.6).

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