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# INTEGRALLY CLOSED IDEALS ON LOG TERMINAL SURFACES ARE MULTIPLIER IDEALS

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Abstract. We show that all integrally closed ideals on log terminal surfaces are multiplier ideals by extending an existing proof for smooth surfaces.

## 1. Introduction

Consider a scheme $X = \operatorname{Spec} \mathcal{O}_X$, where $\mathcal{O}_X$ is a two-dimensional local normal domain essentially of finite type over $\mathbb{C}$. Our purpose is to partially address the following question, raised in [6]:

**Question.** If $X$ has a rational singularity, is every integrally closed ideal which is contained in $\mathcal{J}(X, \mathcal{O}_X)$ a multiplier ideal?

Here, $\mathcal{J}(X, \mathfrak{a}^\lambda)$ denotes the multiplier ideal corresponding to an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ with coefficient $\lambda \in \mathbb{Q}_{>0}$. When $X$ is regular, an affirmative answer was given concurrently by [8] and [3]. Our main result is to generalize their methods to prove the following:

**Theorem 1.1.** *Suppose $X$ has log terminal singularities. Then every integrally closed ideal is a multiplier ideal.*

Log terminal singularities satisfy $\mathcal{J}(X, \mathcal{O}_X) = \mathcal{O}_X$ by definition, and are necessarily rational (see Theorem 5.22 in [4]). Thus, Theorem 1.1 gives a complete answer to the above question in this case.

There are several difficulties in trying to extend the techniques used in [8]. One must show that successful choices can be made in the construction (specifically, the choice of $\epsilon$ and $N$ in Lemma 2.2 of [8]). Here, it is essential that $X$ has log terminal singularities. Further problems arise from the failure of unique factorization to hold for integrally closed ideals. As $X$ is not necessarily factorial, we may no longer reduce to the finite colength case. In addition, the crucial contradiction argument which concludes the proof in [8] does not apply. These nontrivial difficulties are overcome by using a relative numerical decomposition for divisors on a resolution over $X$. Further, appropriately interpreted, the proof of Theorem 1.1 applies over an algebraically closed field of arbitrary characteristic.

Our presentation is self-contained and elementary. Section 2 contains background material covering the relative numerical decomposition, antinef closures, and some computations using generic sequences of blowups. Section 3 is dedicated to the constructions and arguments in the proof of Theorem 1.1.

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## 2. Background

**2.1. Relative Numerical Decomposition.** For the remainder, we will consider a scheme $X = \operatorname{Spec} \mathcal{O}_X$, where $\mathcal{O}_X$ is a two-dimensional local normal domain essentially of finite type over an algebraically closed field of arbitrary characteristic. Let $x \in X$ be the unique closed point, and suppose $f\colon Y \to X$ is a projective birational morphism such that $Y$ is regular and $f^{-1}(x)$ is a simple normal crossing divisor. Let $E_1, \ldots, E_u$ be the irreducible components of $f^{-1}(x)$, and $\Lambda = \oplus_i \mathbb{Z} E_i \subset \operatorname{Div}(Y)$ the lattice they generate.

The intersection pairing $\operatorname{Div}(Y) \times \Lambda \to \mathbb{Z}$ induces a negative definite $\mathbb{Q}$-bilinear form on $\Lambda_{\mathbb{Q}}$ (see [1] for an elementary proof). Consequently, there is a dual basis $\check{E}_1, \ldots, \check{E}_u$ for $\Lambda_{\mathbb{Q}}$ defined by the property that

$$\check{E}_i \cdot E_j = -\delta_{ij} = \begin{cases} -1 & i = j \\ 0 & i \neq j \end{cases} .$$

Recall that a divisor $D \in \operatorname{Div}_{\mathbb{Q}}(Y)$ is said to be $f$-antinef if $D \cdot E_i \leq 0$ for all $i = 1, \ldots, u$. In this case, $D$ is effective if and only if $f_* D$ is effective (see Lemma 3.39 in [4]). In particular, $\check{E}_1, \ldots, \check{E}_u$ are effective.

If $C \in \operatorname{Div}_{\mathbb{Q}}(X)$, we define the numerical pullback of $C$ to be the unique $\mathbb{Q}$-divisor $f^* C$ on $Y$ such that $f_* f^* C = C$ and $f^* C \cdot E_i = 0$ for all $i = 1, \ldots, u$. Note that, when $C$ is Cartier or even $\mathbb{Q}$-Cartier, this agrees with the standard pullback of $C$. If $D \in \operatorname{Div}_{\mathbb{Q}}(Y)$, we have

$$D = f^* f_* D + \sum_i (-D \cdot E_i) \check{E}_i. \tag{1}$$

We shall refer to this as a relative numerical decomposition for $D$. Note that, even when $D$ is integral, both $f^* f_* D$ and $\check{E}_1, \ldots, \check{E}_u$ are likely non-integral. The fact that $f^* f_* D$ and $\check{E}_1, \ldots, \check{E}_u$ are always integral divisors when $X$ is smooth and $D$ is integral is equivalent to the unique factorization of integrally closed ideals. See [7] for further discussion.

**2.2. Antinef Closures and Global Sections.** Suppose now that $D' = \sum_E a'_E E$ and $D'' = \sum_E a''_E E$ are $f$-antinef divisors, where the sums range over the prime divisors $E$ on $Y$. It is easy to check that $D' \wedge D'' = \sum_E \min\{a'_E, a''_E\} E$ is also $f$-antinef. Further, any integral $D \in \operatorname{Div}(Y)$ is dominated by some integral $f$-antinef divisor (e.g. $(f^{-1}{}_*) f_* D + M(\check{E}_1 + \cdots + \check{E}_u)$ for sufficiently large and divisible $M$). In particular, there is a unique smallest integral $f$-antinef divisor $D^\sim$, called the $f$-antinef closure of $D$, such that $D^\sim \geq D$. One can verify that $f_* D = f_* D^\sim$, and in addition the following important lemma holds (see Lemma 1.2 of [8]). The proof also gives an effective algorithm for computing $f$-antinef closures.

**Lemma 2.1.** *For any $D \in \operatorname{Div}(Y)$, we have $f_* \mathcal{O}_Y(-D) = f_* \mathcal{O}_Y(-D^\sim)$.*

*Proof.* Let $s_D \in \mathbb{N}$ be the sum of the coefficients of $D^\sim - D$ when written in terms of $E_1, \ldots, E_u$. If $s_D = 0$, then $D = D^\sim$ is $f$-antinef and the statement follows trivially. Else, there is an index $i$ such that $D \cdot E_i > 0$. As $E_i \cdot E_j \geq 0$ for $j \neq i$, we must have

$$D \leq D + E_i \leq D^\sim = (D + E_i)^\sim.$$

Thus, $s_{D+E_i} = s_D - 1$. By induction, we may assume

$$f_*\mathcal{O}_Y(-(D+E_i)) = f_*\mathcal{O}_Y(-(D+E_i)^\sim) = f_*\mathcal{O}_Y(-D^\sim)$$

and it is enough to show $f_*\mathcal{O}_Y(-D) = f_*\mathcal{O}_Y(-(D+E_i))$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-(D+E_i)) \longrightarrow \mathcal{O}_Y(-D) \longrightarrow \mathcal{O}_{E_i}(-D) \longrightarrow 0.$$

Since $\deg(\mathcal{O}_{E_i}(-D)) = -D \cdot E_i < 0$, we have $f_*\mathcal{O}_{E_i}(-D) = 0$; applying $f_*$ yields the desired result. $\square$

**2.3. Generic Sequences of Blowups.** In the proof of Theorem 1.1, we will make use of the following auxiliary construction. Suppose $x^{(i)}$ is a closed point of $E_i$ with $x^{(i)} \notin E_j$ for $j \neq i$. A generic sequence of $n$-blowups over $x^{(i)}$ is:

$$Y = Y_0 \xleftarrow{\sigma_1} Y_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_{n-1}} Y_{n-1} \xleftarrow{\sigma_n} Y_n$$

where $\sigma_1 \colon Y_1 \to Y_0$ is the blowup of $Y_0 = Y$ at $x_1 := x^{(i)}$, and $\sigma_k \colon Y_k \to Y_{k-1}$ is the blowup of $Y_{k-1}$ at a generic closed point $x_k$ of $(\sigma_{k-1})^{-1}(x_{k-1})$ for $k = 2, \ldots, n$. Let $\sigma \colon Y_n \to Y$ be the composition $\sigma_n \circ \cdots \circ \sigma_1$. We will denote by $E(1), \ldots, E(u)$ the strict transforms of $E_1, \ldots, E_u$ on $Y_n$. Also, let $E(i, x^{(i)}, k)$, $k = 1, \ldots, n$, be the strict transforms of the $n$ new $\sigma$-exceptional divisors created by the blowups $\sigma_1, \ldots, \sigma_n$, respectively.

**Lemma 2.2.** *(a.) Let $\sigma \colon Y_n \to Y$ be a generic sequence of blowups over $x^{(i)} \in E_i$. Then one has*

$$\check{E}(i) \leq \check{E}(i, x^{(i)}, 1) \leq \cdots \leq \check{E}(i, x^{(i)}, n).$$

*(b.) Suppose $D \in \operatorname{Div}(Y_n)$ is an integral $(f \circ \sigma)$-antinef divisor such that $E_i$ is the unique component of $\sigma_* D$ containing $x^{(i)}$. If $\operatorname{ord}_{E(i)} D = a_0$ and $\operatorname{ord}_{E(i,x^{(i)},k)} D = a_k$ for $k = 1, \ldots, n$, then*

$$a_0 \leq a_1 \leq \cdots \leq a_n.$$

*Further, $a_0 < a_n$ if and only if*

$$\left( \sum_{k=1}^{n} (-D \cdot E(i, x^{(i)}, k)) \check{E}(i, x^{(i)}, k) \right) \geq \check{E}(i).$$

*Proof.* If $n = 1$, we have

$$\check{E}(i, x^{(i)}, 1) = \left( \sigma^* \check{E}_i + E(i, x^{(i)}, 1) \right) \geq \sigma^* \check{E}_i = \check{E}(i)$$

$$D = \sigma^* \sigma_* D + (-D \cdot E(i, x^{(i)}, 1)) \check{E}(i, x^{(i)}, 1).$$

The general case of both statments follows easily by induction. $\square$

## 3. Main Theorem

**3.1. Log Terminal Singularities and Multiplier Ideals.** Once more, suppose $x \in X$ is the unique closed point and $f\colon Y \to X$ is a projective birational morphism such that $Y$ is regular and $f^{-1}(x)$ is a simple normal crossing divisor. Let $E_1, \ldots, E_u$ be the irreducible components of $f^{-1}(x)$, and let $K_Y$ be a canonical divisor on $Y$. Then $K_X := f_* K_Y$ is a canonical divisor on $X$. If we write the relative canonical divisor as

$$K_f := K_Y - f^* K_X = \sum_i b_i E_i$$

then $X$ has (numerically) log terminal singularities if and only if $b_i > -1$ for all $i = 1, \ldots, u$. In this case, when working over $\mathbb{C}$, $X$ is automatically $\mathbb{Q}$-factorial (see Proposition 4.11 in [4], as well as [2] for recent developments).

If $\mathfrak{a} \subseteq \mathcal{O}$ is an ideal, recall that $f\colon Y \to X$ as above is said to be a log resolution of $\mathfrak{a}$ if $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-G)$ for an effective divisor $G$ such that $\mathrm{Ex}(f) \cup \mathrm{Supp}(G)$ has simple normal crossings. In this case, we can define the multiplier ideal of $(X, \mathfrak{a})$ with coefficient $\lambda \in \mathbb{Q}_{>0}$ as

$$\mathcal{J}(X, \mathfrak{a}^\lambda) = f_* \mathcal{O}_Y(\lceil K_f - \lambda G \rceil).$$

See [9] for an introduction in a similar setting, or [5] for a more comprehensive overview. Also recall that $\mathfrak{a}$ is integrally closed if and only if

$$\mathfrak{a} = f_* \mathcal{O}_Y(-G).$$

**3.2. Choosing $\mathfrak{a}$ and $\lambda$.** We now begin the proof of Theorem 1.1. For the remainder, assume $X$ is log terminal, and let $I \subseteq \mathcal{O}_X$ be an integrally closed ideal. In this section, we construct another ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ along with a coefficient $\lambda \in \mathbb{Q}_{>0}$; and in the following section it will be shown that $\mathcal{J}(X, \mathfrak{a}^\lambda) = I$. Let $f\colon Y \to X$ a log resolution of $I$ with exceptional divisors $E_1, \ldots, E_u$. Suppose $I\mathcal{O}_Y = \mathcal{O}_Y(-F^0)$, and write

$$K_f = \sum_{i=1}^{u} b_i E_i$$

$$F^0 = (f^{-1}{}_*) f_*(F^0) + \sum_{i=1}^{u} a_i E_i.$$

Choose $0 < \epsilon < 1/2$ such that $\lfloor \epsilon (f^{-1}{}_*) f_*(F^0) \rfloor = 0$ and

$$\epsilon(a_i + 1) < 1 + b_i$$

for $i = 1, \ldots, u$. Note that, since $X$ is log terminal, $1 + b_i > 0$ and any sufficiently small $\epsilon > 0$ will do. Let $n_i := \lfloor \frac{1+b_i}{\epsilon} - (a_i + 1) \rfloor \geq 0$, and $e_i := (-F^0 \cdot E_i)$. Choose $e_i$ distinct closed points $x_1^{(i)}, \ldots, x_{e_i}^{(i)}$ on $E_i$ such that $x_j^{(i)} \notin \mathrm{Supp}\left((f^{-1}{}_*) f_*(F^0)\right)$ and $x_j^{(i)} \notin E_l$ for $l \neq i$. Denote by $g : Z \to Y$ the composition of $n_i$ generic blowups at each of the points $x_j^{(i)}$ for $j = 1, \ldots, e_i$ and $i = 1, \ldots, u$. As in Section 2.3, denote by $E(1), \ldots, E(u)$ the strict transforms of $E_1, \ldots, E_u$, and $E(i, x_j^{(i)}, 1), \ldots, E(i, x_j^{(i)}, n_i)$ the strict transforms of the $n_i$ exceptional divisors over $x_j^{(i)}$.

Let $h := f \circ g$, $F = g^*(F^0)$, and choose an effective $h$-exceptional integral divisor $A$ on $Z$ such that $-A$ is $h$-ample. It is easy to see that

$$K_g = \sum_{i=1}^{u} \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} k\, E(i, x_j^{(i)}, k)$$

and one checks

$$K_g \cdot E(i) = e_i \qquad K_g \cdot E(i, x_j^{(i)}, k) = \begin{cases} 0 & k \neq n_i \\ -1 & k = n_i \end{cases} .$$

It follows immediately that $F + K_g$ is $h$-antinef. Choose $\mu > 0$ sufficiently small that

$$\lfloor (1+\epsilon)(F + K_g + \mu A) - K_h \rfloor = \lfloor (1+\epsilon)(F + K_g) - K_h \rfloor. \tag{2}$$

As $-(F + K_g + \mu A)$ is $h$-ample, there exists $N >> 0$ such that $G := N(F + K_g + \mu A)$ is integral and $-G$ is relatively globally generated.[1] In other words, $\mathfrak{a} := h_*\mathcal{O}_Z(-G)$ is an integrally closed ideal such that $\mathfrak{a}\mathcal{O}_Z = \mathcal{O}_Z(-G)$. Set $\lambda = \frac{1+\epsilon}{N}$.

**3.3. Conclusion of Proof.** Here, we will show $\mathcal{J}(X, \mathfrak{a}^\lambda) = I = h_*\mathcal{O}_Z(-F)$. Since

$$\mathcal{J}(X, \mathfrak{a}^\lambda) = h_*\mathcal{O}_Z(\lceil K_h - \lambda G \rceil) = h_*\mathcal{O}_Z(-\lfloor \lambda G - K_h \rfloor),$$

by Lemma 2.1, it suffices to show $F' := \lfloor \lambda G - K_h \rfloor^\sim = F$. In particular, we have reduced to showing a purely numerical statement.

**Lemma 3.1.** *We have $F' \leq F$ and $h_*F' = h_*F$. In addition, for $i = 1, \ldots, u$ and $j = 1, \ldots, e_i$,*

$$\mathrm{ord}_{E(i, x_j^{(i)}, n_i)}(F') = \mathrm{ord}_{E(i, x_j^{(i)}, n_i)}(F) = \mathrm{ord}_{E(i)}(F).$$

*Proof.* Since $F' = \lfloor \lambda G - K_h \rfloor^\sim$ and $F$ is $h$-antinef ($-F$ is relatively globally generated), it suffices to show these statements with $\lfloor \lambda G - K_h \rfloor$ in place of $F'$. By (2), we have

$$\begin{aligned} \lfloor \lambda G - K_h \rfloor &= \lfloor (1+\epsilon)(F + K_g) - K_h \rfloor \\ &= F + \lfloor \epsilon(F + K_g) - g^* K_f \rfloor. \end{aligned}$$

Since $\lfloor \epsilon ({f^{-1}}_*) f_* F^0 \rfloor = 0$, it follows immediately that $h_* \lfloor \lambda G - K_h \rfloor = h_* F$. For the remaining two statements, consider the coefficients of $\epsilon(F + K_g) - g^* K_f$. Along $E(i)$, we have $\epsilon a_i - b_i$, which is less than one by choice of $\epsilon$. Along $E(i, x_j^{(i)}, k)$, we have $\epsilon(a_i + k) - b_i$. This expression is greatest when $k = n_i$, where our choice of $n_i$ guarantees

$$0 \leq \epsilon(a_i + n_i) - b_i < 1.$$

It follows that $\lfloor \lambda G - K_h \rfloor \leq F$, with equality along $E(i, x_j^{(i)}, n_i)$. □

**Lemma 3.2.** *For each $i = 1, \ldots, u$,*

$$(-F' \cdot E(i))\check{E}(i) + \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(i, x_j^{(i)}, k))\check{E}(i, x_j^{(i)}, k) \quad \geq \quad (-F \cdot E(i))\check{E}(i).$$

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[1]Over $\mathbb{C}$, as $X$ is log terminal, it also has rational singularities and by Theorem 12.1 of [7] it follows that $-(F + K_g)$ is already globally generated without the addition of $-A$. However, the above approach seems more elementary, and avoids unnecessary reference to these nontrivial results.

*Proof.* If $\operatorname{ord}_{E(i)} F' = \operatorname{ord}_{E(i)} F$, as $F' \leq F$ we have $F' \cdot E(i) \leq F \cdot E(i)$ and the conclusion follows as $\check{E}(i)$ and $\check{E}(i, x_j^{(i)}, k)$ are effective and $F'$ is $h$-antinef. Otherwise, if $\operatorname{ord}_{E(i)} F' < \operatorname{ord}_{E(i)} F = \operatorname{ord}_{E(i,x_j^{(i)},n_i)} F'$, then for each $j = 1, \ldots, e_i$ we saw in Lemma 2.2(b) that

$$\sum_{k=1}^{n_i} (-F' \cdot E(i, x_j^{(i)}, k)) \check{E}(i, x_j^{(i)}, k) \quad \geq \quad \check{E}(i).$$

Summing over all $j$ gives the desired conclusion. □

We now finish the proof by showing that $F' \geq F$. Using the relative numerical decomposition (1) and the previous two Lemmas, we compute

$$\begin{aligned} F' &= h^* h_* F' + \sum_{i=1}^{u} (-F' \cdot E(i)) \check{E}(i) + \sum_{i=1}^{u} \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(i, x_j^{(i)}, k)) \check{E}(i, x_j^{(i)}, k) \\ &= h^*(h_* F) + \sum_{i=1}^{u} \left( (-F' \cdot E(i)) \check{E}(i) + \sum_{j=1}^{e_i} \sum_{k=1}^{n_i} (-F' \cdot E(i, x_j^{(i)}, k)) \check{E}(i, x_j^{(i)}, k) \right) \\ &\geq h^* h_* F + \sum_{i=1}^{u} (-F \cdot E(i)) \check{E}(i) = F. \end{aligned}$$

This concludes the proof of Theorem 1.1.

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