

ODD KHOVANOV HOMOLOGY IS MUTATION INVARIANT

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ABSTRACT. We prove that odd Khovanov homology is mutation invariant over \mathbb{Z} , and therefore that Khovanov homology is mutation invariant over $\mathbb{Z}/2\mathbb{Z}$. We also establish mutation invariance for the entire Ozsváth-Szabó spectral sequence from reduced Khovanov homology to the Heegaard Floer homology of the branched double-cover.

1. Introduction

To an oriented link $L \subset S^3$, Khovanov associates a bigraded homology group $Kh(L)$ whose graded Euler characteristic is the unnormalized Jones polynomial [9]. This invariant also has a reduced version $\overline{Kh}(L, K)$, which depends on a choice of marked component K . While the Jones polynomial itself is insensitive to Conway mutation, Khovanov homology generally detects mutations that swap strands between link components [16]. Whether the theory is invariant under component-preserving mutation, and in particular for knots, remains an interesting open question, explored in [4], [10], and [17]. No counterexamples exist with fewer than 14 crossings, although Khovanov homology does distinguish knots related by genus 2 mutation [7], whereas the (colored) Jones polynomial does not.

In 2003, Ozsváth and Szabó introduced a link surgery spectral sequence whose E^2 term is $\overline{Kh}(L; \mathbb{Z}/2\mathbb{Z})$ and which converges to $\widehat{HF}(-\Sigma(L))$, the Heegaard Floer homology of the branched double-cover with reversed orientation [13]. In search of a candidate for the E^2 page over \mathbb{Z} , Ozsváth, Rasmussen and Szabó developed odd Khovanov homology $Kh'(L)$, a theory whose mod 2 reduction coincides with that of Khovanov homology [12]. While the reduced version $\overline{Kh}'(L)$ also categorifies the Jones polynomial, it is independent of the choice of marked component and determines $Kh'(L)$ according to the equation

$$(1) \quad Kh'_{t,q}(L) \cong \overline{Kh}'_{t,q-1}(L) \oplus \overline{Kh}'_{t,q+1}(L).$$

In contrast to Khovanov homology for links, we prove:

Theorem 1. *Odd Khovanov homology is mutation invariant. Indeed, connected mutant link diagrams give rise to isomorphic odd Khovanov complexes.*

Corollary 2. *Khovanov homology over $\mathbb{Z}/2\mathbb{Z}$ is mutation invariant.*

It is not known if these results extend to genus 2 mutation [7]. Wehrli announced a proof of Corollary 2 for component-preserving Conway mutation in 2007, using an approach outlined by Bar-Natan in 2005 [4]. Shortly after our paper first appeared, Wehrli posted his proof, which is completely independent and extends to the case of Lee homology over $\mathbb{Z}/2\mathbb{Z}$ [17].

Received by the editors April 1, 2009. The author was supported by NSF grant DMS-0739392.

Mutant links in S^3 have homeomorphic branched double-covers. It follows that the E^∞ page of the link surgery spectral sequence is also mutation invariant. Building on work of Roberts [14], Baldwin has shown that all pages E^i with $i \geq 2$ are link invariants, as graded vector spaces [2]. To this, we add:

Theorem 3. *The E^i page of the link surgery spectral sequence is mutation invariant for $i \geq 2$. Indeed, connected mutant link diagrams give rise to isomorphic filtered complexes.*

Note that Khovanov homology, even over $\mathbb{Z}/2\mathbb{Z}$, is not an invariant of the branched double-cover itself [15].

1.1. Organization. In Section 2, to a connected, decorated link diagram \mathcal{D} we associate a set of numerical data that is determined by (and, in fact, determines) the equivalence class of \mathcal{D} modulo diagrammatic mutation (and planar isotopy). From this data alone, we construct a complex $(\tilde{\mathcal{C}}(\mathcal{D}), \tilde{\partial}_\epsilon)$ which is *a priori* invariant under mutation of \mathcal{D} . In Section 3, we recall the construction of odd Khovanov homology and prove:

Proposition 4. *The complex $(\tilde{\mathcal{C}}(\mathcal{D}), \tilde{\partial}_\epsilon)$ is canonically isomorphic to the reduced odd Khovanov complex $(\bar{\mathcal{C}}(\mathcal{D}), \bar{\partial}_\epsilon)$.*

This establishes Theorem 1 and verifies that our construction leads to a well-defined (in fact, previously-defined) link invariant.

In Section 4, we consider a surgery diagram for the branched double-cover of \mathcal{D} given by a framed link $\mathbb{L} \subset S^3$ with one component for each crossing. Ozsváth and Szabó associate a filtered complex to \mathcal{D} by applying the Heegaard Floer functor to a hypercube of 3-manifolds and cobordisms associated to various surgeries on \mathbb{L} (see [13] for details). The link surgery spectral sequence is then induced by standard homological algebra. In Proposition 7, we prove that the framed isotopy type of \mathbb{L} is determined by the mutation equivalence class of \mathcal{D} , establishing Theorem 3. Note that Corollary 2 may be viewed as the E^2 page of Theorem 3.

We conclude Section 4 with a remark on our original motivation for the construction of the complex $(\tilde{\mathcal{C}}(\mathcal{D}), \tilde{\partial}_\epsilon)$. An essential observation in [13] is that one recovers the reduced Khovanov complex over $\mathbb{Z}/2\mathbb{Z}$ by first branched double-covering the Khovanov hypercube of 1-manifolds and 2-dimensional cobordisms and then applying the Heegaard Floer TQFT to the resulting hypercube of 3-manifolds and 4-dimensional cobordisms. As we observe in recent work, a similar relationship holds between reduced odd Khovanov homology and the monopole Floer TQFT [5]. From this perspective, our results may be viewed as immediate consequences of the topological fact that branched double-covering destroys all evidence of mutation.

2. A thriftier construction of reduced odd Khovanov homology

Given an oriented link L , fix a connected, oriented link diagram \mathcal{D} with crossings c_1, \dots, c_n . Let n_+ and n_- be the number of positive and negative crossings, respectively. We use $\mathcal{V}(\mathcal{D})$ and $\mathcal{E}(\mathcal{D})$ to denote the sets of vertices and edges, respectively, of the hypercube $\{0, 1\}^n$, with edges oriented in the direction of increasing weight. Decorate each crossing c_i with an arrow x_i , which may point in one of two parallel

directions and appears in each complete resolution $\mathcal{D}(I)$ as an oriented arc between circles according to the conventions in Figure 1.

Recall that a planar link diagram \mathcal{D} admits a checkerboard coloring with white exterior, as illustrated at left in Figure 3. The black graph $\mathcal{B}(\mathcal{D})$ is formed by placing a vertex in each black region and drawing an edge through each crossing. The edge through c_i connects the vertices in the black regions incident to c_i . Given a spanning tree $\mathcal{T} \subset \mathcal{B}(\mathcal{D})$, we may form a resolution of \mathcal{D} consisting of only one circle by merging precisely those black regions which are incident along \mathcal{T} . In particular, all connected diagrams admit at least one such resolution.

With these preliminaries in place, we now give a recipe for associating a bigraded chain complex $(\tilde{\mathcal{C}}(\mathcal{D}), \tilde{\partial}_\epsilon)$ to the decorated diagram \mathcal{D} . The key idea is simple. Think of each resolution of \mathcal{D} as a connected, directed graph whose vertices are the circles and whose edges are the oriented arcs. While the circles merge and split from one resolution to the next, the arcs are canonically identified throughout. So we use the exact sequence

$$\{\text{cycles}\} \hookrightarrow \mathbb{Z}\langle \text{arcs} \rangle \xrightarrow{d} \mathbb{Z}\langle \text{circles} \rangle \rightarrow \mathbb{Z} \rightarrow 0$$

of free Abelian groups to suppress the circles entirely and instead keep track of the cycles in each resolution, thought of as relations between the arcs themselves.

We begin the construction by fixing a vertex $I^* = (m_1^*, \dots, m_n^*) \in \mathcal{V}(\mathcal{D})$ such that the resolution $\mathcal{D}(I^*)$ consists of only one circle S . To each pair of oriented arcs (x_i, x_j) in $\mathcal{D}(I^*)$ we associate a linking number $a_{ij} \in \{0, \pm 1\}$ according to the symmetric convention in Figure 2. We set $a_{ii} = 0$. Note that arcs on the same side of S cannot link. For each $I \in \mathcal{V}(\mathcal{D})$, we have an Abelian group

$$\tilde{V}(\mathcal{D}(I)) = \mathbb{Z}\langle x_1, \dots, x_n \mid r_1^I, \dots, r_n^I \rangle$$

presented by relations

$$(2) \quad r_i^I = \begin{cases} x_i - \sum_{\{j \mid m_j \neq m_j^*\}} (-1)^{m_j^*} a_{ij} x_j & \text{if } m_i = m_i^* \\ - \sum_{\{j \mid m_j \neq m_j^*\}} (-1)^{m_j^*} a_{ij} x_j & \text{if } m_i \neq m_i^*. \end{cases}$$

Indeed, these relations generate the cycles in the graph of circles and arcs at $\mathcal{D}(I)$ (see Lemma 6).

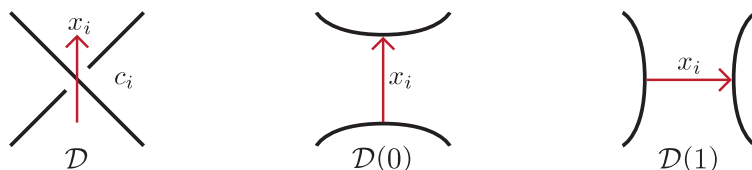


FIGURE 1. **Oriented resolution conventions.** The arrow x_i at crossing c_i remains fixed in a 0-resolution and rotates 90° clockwise in a 1-resolution. To see the other choice for the arrow at c_i , rotate the page 180° . Mutation invariance of the Jones polynomial follows from the rotational symmetries of the above tangles in \mathbb{R}^3 .

To an edge $e \in \mathcal{E}(\mathcal{D})$ from I to J given by an increase in resolution at c_i , we associate a map

$$\tilde{\partial}_J^I : \Lambda^* \tilde{V}(\mathcal{D}(I)) \rightarrow \Lambda^* \tilde{V}(\mathcal{D}(J))$$

of exterior algebras, which is defined in the case of a split and a merge, respectively, by

$$\tilde{\partial}_J^I(u) = \begin{cases} x_i \wedge u & \text{if } x_i = 0 \in \tilde{V}(\mathcal{D}(I)) \\ u & \text{if } x_i \neq 0 \in \tilde{V}(\mathcal{D}(I)). \end{cases}$$

Extending by zero, we may view each of these maps as an endomorphism of the group

$$\tilde{C}(\mathcal{D}) = \bigoplus_{I \in \mathcal{V}(\mathcal{D})} \Lambda^* \tilde{V}(\mathcal{D}(I)).$$

Consider a 2-dimensional face of the hypercube from I to J corresponding to an increase in resolution at c_i and c_j . The two corresponding composite maps in $\tilde{C}(\mathcal{D})$ commute up to sign, and they vanish identically if and only if the arcs x_i and x_j in $\mathcal{D}(I)$ are in one of the two configurations in Figure 2, denoted Type X and Type Y. Note that we can distinguish a Type X face from a Type Y face without reference to the diagram by checking which of the relations $x_i \pm x_j = 0$ holds in $\tilde{C}(\mathcal{D})$ over the two vertices of the face that are strictly between I and J .

A Type Y edge assignment on $\tilde{C}(\mathcal{D})$ is a map $\epsilon : \mathcal{E}(\mathcal{D}) \rightarrow \{\pm 1\}$ such that the product of signs around a face of Type X or Type Y agrees with the sign of the linking convention in Figure 2 and such that, after multiplication by ϵ , every face of $\tilde{C}(\mathcal{D})$ anticommutes. Such an assignment defines a differential $\tilde{\partial}_\epsilon : \tilde{C}(\mathcal{D}) \rightarrow \tilde{C}(\mathcal{D})$ by

$$\tilde{\partial}_\epsilon(v) = \sum_{\{e \in \mathcal{E}(\mathcal{D}), J \in \mathcal{V}(\mathcal{D}) \mid e \text{ goes from } I \text{ to } J\}} \epsilon(e) \cdot \tilde{\partial}_J^I(v)$$

for $v \in \Lambda^* \tilde{V}(\mathcal{D}(I))$. Type Y edge assignments always exist and any two yield isomorphic complexes, as do any two choices for the initial arrows on \mathcal{D} . We can equip $(\tilde{C}(\mathcal{D}), \tilde{\partial}_\epsilon)$ with a bigrading that descends to homology and is initialized using n_\pm just as in [12]. The bigraded group $\tilde{C}(\mathcal{D})$ and maps $\tilde{\partial}_J^I$ are constructed entirely from the numbers a_{ij} , m_i^* , and n_\pm . Thus, up to isomorphism:



FIGURE 2. **Linking number conventions.** Two arcs are linked in $\mathcal{D}(I^*)$ if and only if their endpoints are interleaved on the circle. Otherwise, $a_{ij} = 0$. Any linked configuration is isotopic to one of the above on the 2-sphere $\mathbb{R}^2 \cup \{\infty\}$.

Proposition 5. *The bigraded complex $(\tilde{\mathcal{C}}(\mathcal{D}), \tilde{\partial}_\epsilon)$ is determined by the linking matrix A of any one-circle resolution of \mathcal{D} , the vertex of this resolution, and the number of positive and negative crossings.*

Proof of Theorem 1. The following argument is illustrated in Figure 3. Given oriented, mutant links L and L' , fix a corresponding pair of oriented, connected diagrams \mathcal{D} and \mathcal{D}' for which there is a circle C exhibiting the mutation. This circle crosses exactly two black regions of \mathcal{D} , which we connect by a path Γ in $\mathcal{B}(\mathcal{D})$. To simplify the exposition, we will assume there is a crossing between the two strands of \mathcal{D} in C , so that Γ may be chosen in C . Extend Γ to a spanning tree \mathcal{T} to obtain a resolution $\mathcal{D}(I^*)$ with one circle. The natural pairing of the crossings of \mathcal{D} and \mathcal{D}' induces an identification $\mathcal{V}(\mathcal{D}) \cong \mathcal{V}(\mathcal{D}')$. The resolution $\mathcal{D}'(I^*)$ may be obtained directly from $\mathcal{D}(I^*)$ by mutation and also consists of one circle S . We can partition the set of arcs inside C into those which go across S (in dark blue) and those which have both endpoints on the same side of S (in red). Since this division is preserved by mutation, the mod 2 linking matrix is preserved as well.

Arrows at the crossings of \mathcal{D} orient the arcs of $\mathcal{D}(I^*)$, which in turn orient the arcs of $\mathcal{D}'(I^*)$ via the mutation. To preserve the linking matrix at I^* with sign, we modify the arcs of $\mathcal{D}'(I^*)$ as follows. Let $A = \{\text{arcs in } C \text{ and in } S\}$ and $B = \{\text{arcs in } C \text{ and not in } S\}$. We reverse those arcs of $\mathcal{D}'(I^*)$ that lie in A , B , or $A \cup B$, according to whether the mutation is about the z -, y -, or x -axis, respectively (as represented at right in Figure 3). We then select a corresponding set of arrows on \mathcal{D}' . Note that it may also be necessary to switch the orientations of both strands inside C so that \mathcal{D}' will be consistently oriented. In any case, the number of positive and negative crossings is unchanged. Propositions 4 and 5 now imply the theorem for $(\tilde{\mathcal{C}}(\mathcal{D}), \tilde{\partial}_\epsilon)$. The unreduced odd Khovanov complex is isomorphic to two copies of the reduced complex, just as in (1). \square

Remark. We have seen that if connected diagrams \mathcal{D} and \mathcal{D}' are related by diagrammatic mutation (and planar isotopy), then there is an identification of their crossings and a vertex I^* such that $\mathcal{D}(I^*)$ and $\mathcal{D}'(I^*)$ have the same mod 2 linking matrix. Remarkably, the converse holds as well, i.e., I^* and A mod 2 together determine \mathcal{D} up to diagrammatic mutation. This follows from a theorem of Chmutov and Lando: *Chord diagrams have the same intersection graph if and only if they are related by mutation* [6]. Here we view a one-circle resolution as a bipartite chord diagram, so that its mod 2 linking matrix is precisely the adjacency matrix of the corresponding intersection graph. Note that in the bipartite case, any combinatorial mutation as defined in [6] can be realized by a finite sequence of our diagrammatic ones.

Chmutov and Lando apply their result to the chord-diagram construction of finite type invariants. All finite type invariants of order ≤ 10 are insensitive to Conway mutation, whereas there exists an invariant of order 11 that distinguishes the knots in Figure 3 and one of order 7 that distinguishes genus 2 mutants (see [11] and [6]).

3. The original construction of reduced odd Khovanov homology

We now recall the original construction of reduced odd Khovanov homology, following [12]. Given an oriented link $L \subset S^3$, we fix a decorated, oriented diagram \mathcal{D} as before, though now it need not be connected. For each vertex $I \in \mathcal{V}(\mathcal{D})$, the

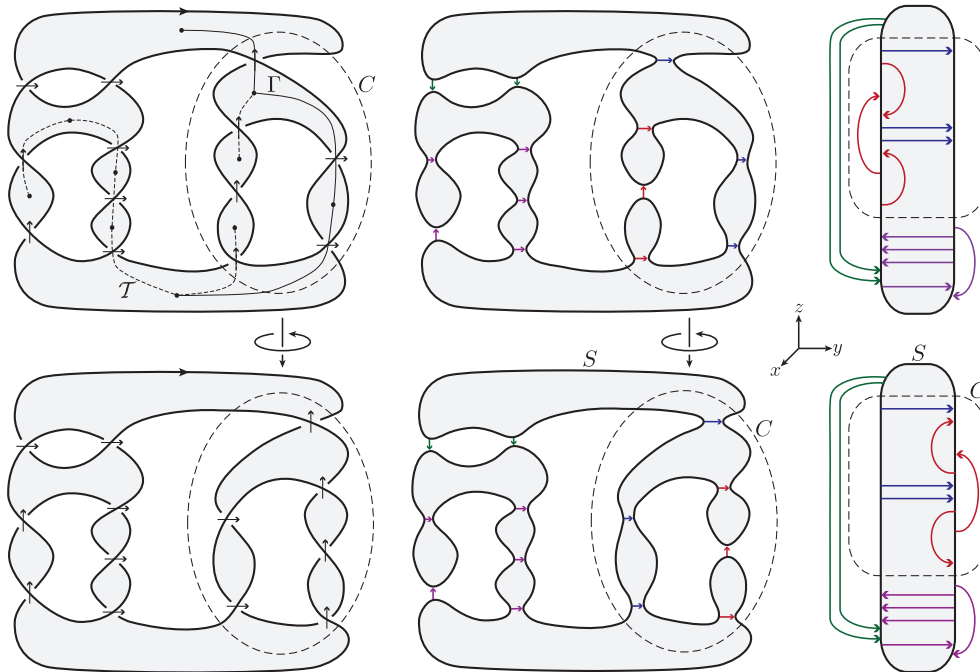


FIGURE 3. **The Kinoshita-Terasaka and Conway knots.** Orientations on the arcs in the upper-right resolution induce orientations on the arcs in the lower-right resolution via the mutation. In order to obtain the same linking data with sign, we have reversed the five arcs in the lower-right resolution that lie inside both S and C . We then work backwards to select arrows in the lower-left knot diagram.

resolution $\mathcal{D}(I)$ consists of a set of circles $\{S_i^I\}$. Let $V(\mathcal{D}(I))$ be the free Abelian group generated by these circles. The reduced group $\bar{V}(\mathcal{D}(I))$ is defined to be the kernel of the augmentation $\eta: V(\mathcal{D}(I)) \rightarrow \mathbb{Z}$ given by $\sum a_i S_i^I \mapsto \sum a_i$.

Now let $\mathbb{Z}\langle x_1, \dots, x_n \rangle$ denote the free Abelian group on n generators. For each $I \in \mathcal{V}(\mathcal{D})$, we have a boundary map

$$d^I: \mathbb{Z}\langle x_1, \dots, x_n \rangle \rightarrow V(\mathcal{D}(I))$$

given by $d^I x_i = S_j^I - S_k^I$, where x_i is directed from S_j^I to S_k^I in $\mathcal{D}(I)$. Consider an edge $e \in \mathcal{E}(\mathcal{D})$ from I to J corresponding to an increase in resolution at c_i . If two circles merge as we move from $\mathcal{D}(I)$ to $\mathcal{D}(J)$, then the natural projection map $\{S_i^I\} \twoheadrightarrow \{S_i^J\}$ induces a morphism of exterior algebras. Alternatively, if a circle splits into two descendants, the two reasonable inclusion maps $\{S_i^I\} \hookrightarrow \{S_i^J\}$ induce equivalent morphisms on exterior algebras after wedging with the ordered difference of the descendants in $\mathcal{D}(J)$. In other words, we have a well-defined map

$$\bar{\partial}_J^I: \Lambda^* \bar{V}(\mathcal{D}(I)) \rightarrow \Lambda^* \bar{V}(\mathcal{D}(J))$$

given by

$$\bar{\partial}_J^I(v) = \begin{cases} d^J x_i \wedge v & \text{if } d^I x_i = 0 \in \bar{V}(\mathcal{D}(I)) \\ v & \text{if } d^I x_i \neq 0 \in \bar{V}(\mathcal{D}(I)) \end{cases}$$

in the case of a split and a merge, respectively, along x_i .

As in Section 2, we now form a group $\bar{C}(\mathcal{D})$ over the hypercube and choose a Type Y edge assignment to obtain a differential $\bar{\partial}_\epsilon : \bar{C}(\mathcal{D}) \rightarrow \bar{C}(\mathcal{D})$. The reduced odd Khovanov homology $\bar{K}h^I(L) \cong H_*(\bar{C}(\mathcal{D}), \bar{\partial}_\epsilon)$ is independent of all choices and comes equipped with a bigrading that is initialized using n_\pm . The unreduced version is obtained by replacing $\bar{V}(\mathcal{D}(I))$ with $V(\mathcal{D}(I))$ above.

Proof of Proposition 4. Suppose that \mathcal{D} is connected. Then for each $I \in \mathcal{V}(\mathcal{D})$, the image of $d^I : \mathbb{Z}\langle x_1, \dots, x_n \rangle \rightarrow V(\mathcal{D}(I))$ is precisely $\bar{V}(\mathcal{D}(I))$. In fact, by Lemma 6 below, d^I induces an isomorphism $\tilde{V}(\mathcal{D}(I)) \cong \bar{V}(\mathcal{D}(I))$. The collection of maps d^I therefore induce a group isomorphism $\tilde{C}(\mathcal{D}) \cong \bar{C}(\mathcal{D})$ which is immediately seen to be equivariant with respect to the edge maps $\hat{\partial}_J^I$ and $\bar{\partial}_J^I$. After fixing a common Type Y edge assignment, the proposition follows. \square

Lemma 6. *The relations r_i^I generate the kernel of the map $d^I : \mathbb{Z}\langle x_1, \dots, x_n \rangle \rightarrow V(\mathcal{D}(I))$.*

Proof. To simplify notation, we assume that $m_i \neq m_i^*$ if and only if $i \leq k$, for some $1 \leq k \leq n$. Consider the $n \times n$ matrix M^I with column i given by the coefficients of $(-1)^{m_i^*} r_i^I$. Let A^I be the leading $k \times k$ minor, a symmetric matrix. We build an orientable surface F^I by attaching k 1-handles to the disk D^2 bounded by S so that the cores of the handles are given by the arcs x_1, \dots, x_k as they appear in $\mathcal{D}(I^*)$. We obtain a basis for $H_1(F^I)$ by extending each oriented arc to a loop using a chord through D^2 . The cocores of the handles are precisely x_1, \dots, x_k as they appear in $\mathcal{D}(I)$, so these oriented arcs form a basis for $H_1(F^I, \partial F^I)$. With respect to these bases, the homology long exact sequence of the pair $(F^I, \partial F^I)$ includes the segment

$$(3) \quad H_1(F^I) \xrightarrow{A^I} H_1(F^I, \partial F^I) \xrightarrow{d^I|_{\mathbb{Z}\langle x_1, \dots, x_k \rangle}} H_0(\partial F^I) \xrightarrow{\eta} \mathbb{Z} \rightarrow 0.$$

Furthermore, for each $i > k$, the oriented chord in D^2 between the endpoints of x_i is represented in $H_1(F^I, \partial F^I)$ by the first k entries in column i of M^I . We can therefore enlarge (3) to an exact sequence

$$\mathbb{Z}\langle x_1, \dots, x_n \rangle \xrightarrow{M^I} \mathbb{Z}\langle x_1, \dots, x_n \rangle \xrightarrow{d^I} V(\mathcal{D}(I)) \xrightarrow{\eta} \mathbb{Z} \rightarrow 0,$$

which implies the lemma. \square

We can reduce the number of generators and relations in our construction by using the smaller presentation in (3). Namely, for each $I = (m_1, \dots, m_n) \in \mathcal{V}(\mathcal{D})$, we let $\hat{V}(\mathcal{D}(I))$ be the group generated by $\{x_j \mid m_j \neq m_j^*\}$ and presented by A^I . By (2), the edge map $\bar{\partial}_J^I$ at c_i is replaced by

$$\hat{\partial}_J^I(u) = \begin{cases} x_i \wedge u & \text{if } x_i = 0 \in \hat{V}(\mathcal{D}(I)) \text{ and } m_i = m_i^* \\ -r_i^I \wedge u & \text{if } x_i = 0 \in \hat{V}(\mathcal{D}(I)) \text{ and } m_i \neq m_i^* \\ u & \text{if } x_i \neq 0 \in \hat{V}(\mathcal{D}(I)), \end{cases}$$

where it is understood that $x_i \mapsto 0$ when $m_i \neq m_i^*$. While the definition of $\hat{\partial}_I^J$ is more verbose, the presentations A^I are simply the 2^n principal minors of a single, symmetric matrix: $\{(-1)^{m_i^*+m_j^*} a_{ij}\}$. The resulting complex $(\hat{C}(\mathcal{D}), \hat{\partial}_\epsilon)$ sits in between $(\tilde{C}(\mathcal{D}), \tilde{\partial}_\epsilon)$ and $(\overline{C}(\mathcal{D}), \overline{\partial}_\epsilon)$ and is canonically isomorphic to both.

4. Branched double-covers and the link surgery spectral sequence

To a one-circle resolution $\mathcal{D}(I^*)$ of a connected diagram of a link $L \subset S^3$, we associate a framed link $\mathbb{L} \subset S^3$ that presents $-\Sigma(L)$ by surgery (see also [8]). Figure 4 illustrates the procedure starting from each resolution at right in Figure 3. We first cut open the circle S and stretch it out along the y -axis, dragging the arcs along for the ride. We then slice along the Seifert surface $\{x = 0, z < 0\}$ for S and pull the resulting two copies up to the xy -plane as though opening a book. This moves those arcs which started inside S to the orthogonal half-plane $\{z = 0, x > 0\}$, as illustrated in the second row. The double cover of S^3 branched over S is obtained by rotating a copy of the half-space $\{z \geq 0\}$ by 180° about the y -axis and gluing it back onto the

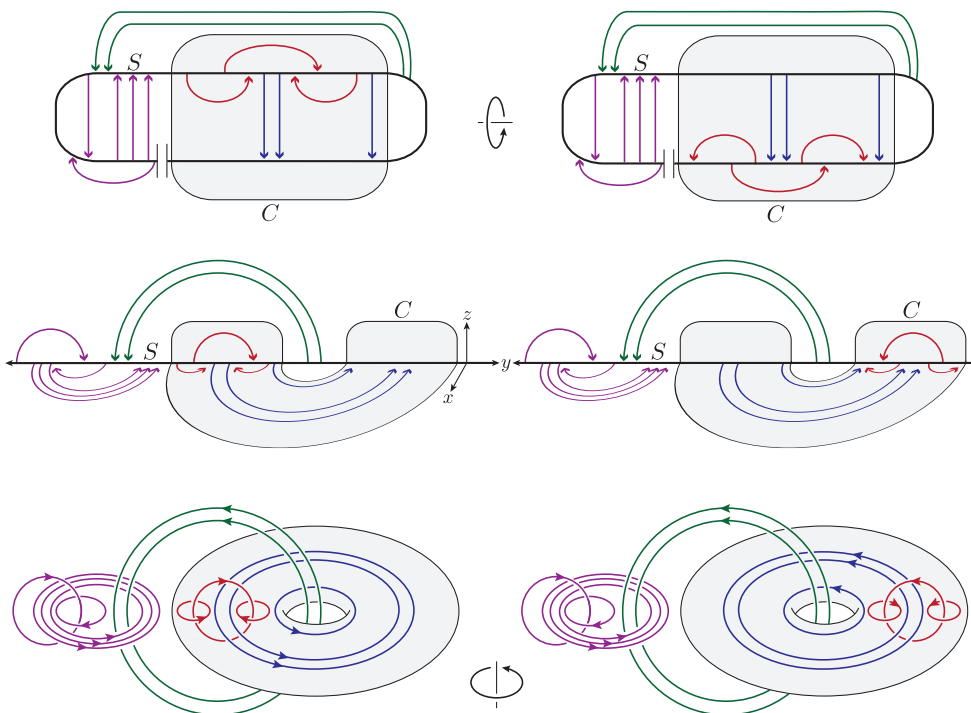


FIGURE 4. **Constructing a surgery diagram for the branched double-cover.** The resolutions in the first row are related by mutation along the Conway sphere formed by attaching disks to either side of C . The double cover of S^2 branched over its intersection with S is represented by each torus in the third row. Rotation of the torus about the z -axis yields a component-preserving isotopy from \mathbb{L} to \mathbb{L}' .

upper half-space. The arcs x_i lift to circles $\mathbb{K}_i \subset S^3$, which comprise \mathbb{L} . We assign \mathbb{K}_i the framing $(-1)^{m_i^*}$.

If \mathcal{D} is decorated, then \mathbb{L} may be oriented by the direction of each arc in the second row of Figure 4. The linking matrix \mathbb{A} of \mathbb{L} then coincides with A off the diagonal, with the diagonal itself encoding I^* . In fact, the geometric constraints on \mathbb{L} are so severe that it is determined up to framed isotopy by \mathbb{A} . This follows intuitively from hanging \mathbb{L} on a wall, and more rigorously from:

Proposition 7. *The isotopy type of $\mathbb{L} \subset S^3$ is determined by the intersection graph of $\mathcal{D}(I^*)$, whereas the framing of \mathbb{L} is determined by I^* .*

Proof. Suppose that $\mathcal{D}(I^*)$ and $\mathcal{D}'(J^*)$, thought of as bipartite chord diagrams, have the same intersection graph. Then by [6], $\mathcal{D}(I^*)$ is connected to $\mathcal{D}'(J^*)$ by a sequence of mutations (see Remark). Each mutation corresponds to a component-preserving isotopy of \mathbb{L} modeled on a half-integer translation of a torus $\mathbb{R}^2/\mathbb{Z}^2$ embedded in S^3 (see Figure 4). Therefore, the associated links \mathbb{L} and \mathbb{L}' are isotopic. The second statement is true by definition. \square

Proof of Theorem 3. From the construction of the spectral sequence in [13], it is clear that the filtered complex associated to a connected diagram \mathcal{D} only depends on \mathcal{D} through the framed isotopy type of the link \mathbb{L} associated to a one-circle resolution. We conclude that each page (E^i, d^i) for $i \geq 1$ is fully determined by the mutation invariant data in Proposition 7 (for further details, see [2] and [14]). \square

We finally come to the surgery perspective that first motivated the construction of the complex $(\tilde{C}(\mathcal{D}), \tilde{\partial}_\epsilon)$. For each $I = (m_1, \dots, m_n) \in \mathcal{V}(\mathcal{D})$, let \mathbb{L}_I consist of the same underlying link as \mathbb{L} , but with framing modified to

$$\lambda_i^I = \begin{cases} \infty & \text{if } m_i = m_i^* \\ 0 & \text{if } m_i \neq m_i^* \end{cases}$$

on \mathbb{K}_i . Then \mathbb{L}_I is a surgery diagram for $\Sigma(\mathcal{D}(I)) \cong \#^{k_I} S^1 \times S^2$, where $\mathcal{D}(I)$ consists of $k_I + 1$ circles. The linking matrix of \mathbb{L}_I then presents $H_1(\Sigma(\mathcal{D}(I)))$ with respect to fixed meridians $\{x_i \mid m_i \neq m_i^*\}$. By identifying $H_1(\Sigma(\mathcal{D}(I)))$ with $\widehat{V}(\mathcal{D}(I))$, we may construct $(\widehat{C}(\mathcal{D}), \widehat{\partial}_\epsilon)$, and therefore $\overline{Kh}'(L)$, completely on the level of branched double-covers. We elaborate on this perspective, and its relationship to the monopole Floer and Donaldson TQFT's, in [5]. We also show that the signature of L is given by $\sigma(L) = \sigma(\mathbb{A}) + w(I^*) - n_-$, where $w(I^*)$ is the number of ones in I^* .

Acknowledgements

I wish to thank John Baldwin, Josh Greene, Adam Levine, Ina Petkova, and my advisor Peter Ozsváth for insightful discussions. I was especially influenced by Greene's construction of Heegaard diagrams for branched double-covers [8] and Baldwin and Plamenevskaya's construction of the reduced Khovanov homology of open book decompositions [3]. I recently learned that Baldwin had programmed an algorithm [1] to compute the latter that is similar to the mod 2 reduction of the above construction.

References

- [1] J. Baldwin, *Kh*, <http://www.math.princeton.edu/~baldwinj/trans.html>.
- [2] ———, *On the spectral sequence from Khovanov homology to Heegaard Floer homology*. Preprint (2008), <http://arxiv.org/abs/0809.3293>.
- [3] J. Baldwin and O. Plamenevskaya, *Khovanov homology, open books, and tight contact structures*. Preprint (2008), <http://arxiv.org/abs/0808.2336>.
- [4] D. Bar-Natan, *Mutation Invariance of Khovanov homology*, The Knot Atlas (2005). http://katlas.org/drorbn/index.php?title=Mutation_Invariance_of_Khovanov_Homology.
- [5] J. Bloom, *A link surgery spectral sequence in monopole Floer homology*. Preprint (2009), <http://arxiv.org/abs/0909.0816>.
- [6] S. V. Chmutov and S. K. Lando, *Mutant knots and intersection graphs*. To appear, *Algebr. Geom. Topol.* Preprint (2007), <http://arxiv.org/abs/0704.1313>.
- [7] N. Dunfield, S. Garoufalidis, A. Shumakovitch, and M. Thistlethwaite, *Behavior of knot invariants under genus 2 mutation*. Preprint (2006), <http://arxiv.org/abs/math/0607258>.
- [8] J. Greene, *A spanning tree model for the Heegaard Floer homology of a branched double-cover*. Preprint (2008), <http://arxiv.org/abs/0805.1381>.
- [9] M. Khovanov, *A categorification of the Jones polynomial*, *Duke Math. J.* **101** (2000), no. 3, 359–426.
- [10] I. Kofman and A. Champanerkar, *On mutation and Khovanov homology*, *Communications in Contemporary Mathematics* **10** (2008) 973–992.
- [11] H. R. Morton and N. Ryder, *Invariants of genus 2 mutants*, *J. Knot Theory Ramifications* **18** (2009) 1423–1438.
- [12] P. Ozsváth, J. Rasmussen, and Z. Szabó, *Odd Khovanov homology*. Preprint (2007), <http://arxiv.org/abs/0710.4300>.
- [13] P. Ozsváth and Z. Szabó, *On the Heegaard Floer homology of branched double-covers*, *Adv. Math.* **194** (2005), no. 1, 1–33.
- [14] L. Roberts, *Notes on the Heegaard-Floer link surgery spectral sequence*. Preprint (2008), <http://arxiv.org/abs/0808.2817>.
- [15] L. Watson, *A remark on Khovanov homology and two-fold branched covers*. To appear, *Pacific J. Math.* Preprint (2008), <http://arxiv.org/abs/0808.2797>.
- [16] S. Wehrli, *Khovanov homology and Conway mutation*. Preprint (2003), <http://arxiv.org/abs/math/0301312>.
- [17] ———, *Mutation invariance of Khovanov homology over \mathbb{F}_2* . Preprint (2009), <http://arxiv.org/abs/0904.3401>.

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