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# TILTING INVARIANCE OF THE AUSLANDER-REITEN CONJECTURE

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Abstract. Let $R$ be an artin algebra and $T$ be a tilting or cotilting $R$-module with $S = \mathrm{End}_R T$. We show that $R$ satisfies the Auslander-Reiten conjecture if and only if so does $S$.

## 1. Introduction

Throughout this paper, we consider artin algebras and finitely generated left modules over them. Let $R$ be an artin algebra, we denote by $\mathrm{mod}R$ the category of finitely generated left $R$-modules. For an $X \in \mathrm{mod}R$, we denote by $\mathrm{pd}_R X$ (resp., $\mathrm{id}_R X$) the projective (resp., injective) dimension of $X$.

In studying the Nakayama conjecture, Auslander and Reiten [2] proposed the following conjecture, which is also listed as the 10th conjecture in the book [4].

**Auslander-Reiten Conjecture:** Let $R$ be an artin algebra and $X$ be an $R$-module. If $\mathrm{Ext}^i_R(X, X \oplus R) = 0$ for all $i \geq 1$, then $X$ is projective.

Auslander and Reiten [2] proved the conjecture over artin algebras such that every module $M$ has an ultimately closed projective resolution, that is, there is some syzygy $N$ of $M$ such that all indecomposable direct summands of $N$ already appear in earlier syzygies. This includes algebras of finite representation type, algebras with radical square zero and all torsionless-finite algebras. Hoshino [9] proved if $R$ is a self-injective artinian local ring with radical cube zero, then $\mathrm{Ext}^1_R(M, M) = 0$ implies that $M$ is free.

We note that the Auslander-Reiten Conjecture actually makes sense for any ring. In fact, there are already some results in the study of the Auslander-Reiten Conjecture for commutative algebras, see for instance [1, 5, 11, 12] etc.. Recently, Christensen and Holm proved that every left noetherian ring satisfy the Auslander-Reiten Conjecture if it satisfies the Auslander's condition on vanishing of cohomology [6]. Such rings contain group algebras of finite groups and artin algebras such that every module $M$ has an ultimately closed projective resolution [6, 16].

The Auslander-Reiten Conjecture is related to the Finitistic Dimension Conjecture which reads as follows.

**Finitistic Dimension Conjecture:** Let $R$ be an artin algebra. Then $\mathrm{fdim}R =$

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$\sup\{\mathrm{pd}_R X|\ \mathrm{pd}_R X < \infty\}$ is finite.

Indeed, if the Finitistic Dimension Conjecture holds for all artin algebras then the Auslander-Reiten Conjecture holds for all artin algebras. However, for an artin algebra satisfying the Finitistic Dimension Conjecture, we don't know if it also satisfies the Auslander-Reiten Conjecture. For instance, it is still a question if the Auslander-Reiten Conjecture holds for all self-injective artin algebras and in this case, it is just the Tachikawa conjecture [15]. However, there is a counterexample over QF-rings [14].

Our aim in this paper is to show that the Auslander-Reiten conjecture is in fact a tilting invariance. More precisely, we prove the following result.

**Main Theorem** Let $R$ be an artin algebra and $T \in \mathrm{mod}R$ be a tilting module with $S = \mathrm{End}_R T$. Then $R$ satisfies the Auslander-Reiten conjecture if and only if so does $S$.

Similar results for Finitistic Dimension Conjecture had been proved by Happel [7] using the techniques of derived categories. More recently, it was proved that the Finitistic Dimension Conjecture is stable under derived equivalences. It would be interesting to consider whether the Auslander-Reiten conjecture is also stable under derived equivalences.

## 2. Preliminaries

Let $R$ be an artin algebra and $M \in \mathrm{mod}R$. We denote by $R^o$ the opposite algebra and an $R^o$-module $M$ means the right $R$-module $M_R$.

Let $\mathcal{C}$ be a subcategory of $\mathrm{mod}R$, we denote by $\mathcal{C}$ the category of all $R$-modules $M$ such that there is an exact sequence $0 \to M \to C_0 \to \cdots \to C_m \to 0$ for some integer $m$ with each $C_i \in \mathcal{C}$. Let $M \in \mathcal{C}$, we denote by $\mathrm{codim}_{\mathcal{C}}(M)$ the minimal non-negative integer $m$ such that there is an exact sequence $0 \to M \to C_0 \to \cdots \to C_m \to 0$ with each $C_i \in \mathcal{C}$. We also denote by $(\mathcal{C})_n$ the category of all $M \in \mathcal{C}$ with $\mathrm{codim}_{\mathcal{C}}(M) \leq n$. Dually, the notion $\widehat{\mathcal{C}}$ denotes the category of all $R$-modules $M$ such that there is an exact sequence $0 \to C_m \to \cdots \to C_0 \to M \to 0$ for some integer $m$ with each $C_i \in \mathcal{C}$, and the notion $\dim_{\mathcal{C}}(M)$ denotes the minimal non-negative integer $m$ such that there is an exact sequence $0 \to C_m \to \cdots \to C_0 \to M \to 0$ with each $C_i \in \mathcal{C}$. Similarly, the notion $(\widehat{\mathcal{C}})_n$ is the category of all $M \in \widehat{\mathcal{C}}$ with $\dim_{\mathcal{C}}(M) \leq n$.

Let $M \in \mathrm{mod}R$. We denote by $\mathrm{add}_R M$ the category of modules isomorphic to direct summands of finite direct sums of $M$. The notion $M^{\perp}$ denotes the category of all modules $N \in \mathrm{mod}R$ such that $\mathrm{Ext}_R^{i\geq 1}(M,N) = 0$. Dually, the notion ${}^{\perp}M$ denotes the category of all modules $N$ such that $\mathrm{Ext}_R^{i\geq 1}(N,M) = 0$.

We denote by $\mathbf{D}$ the usual duality functor between $\mathrm{mod}R$ and $\mathrm{mod}R^o$. For an $M \in \mathrm{mod}R$ and a positive integer $t$, the notion $\Omega_R^t M$ denote the $t$-th syzygy of $M$.

We recall now some necessary basic tilting theory. The readers are suggested to refer to [3, 8, 13] for more details.

Let $R$ be an artin algebra and $n$ a non-negative integer. Recall that $T \in \mathrm{mod}R$ is called a tilting module of projective dimension at most $n$ if it satisfies the following three conditions:

(T1) $\mathrm{pd}T \leq n$, i.e., there is an exact sequence $0 \to R_n \to \cdots \to R_0 \to T \to 0$ with each $R_i$ projective.

(T2) $\mathrm{Ext}_R^i(T,T) = 0$ for all $i > 0$, and

(T3) there is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ for some $n$, where each $T_i \in \mathrm{add}T$.

The notion of tilting modules is left-right symmetric in the sense that if ${}_RT$ is a tilting module of projective dimension at most $n$, then $T_S$, where $S = \mathrm{End}_RT$, is also a tilting module of projective dimension at most $n$.

Dually, $C \in \mathrm{mod}R$ is a cotilting module of injective dimension at most $n$ if it satisfies

(C1) $\mathrm{id}T \leq n$,

(C2) $\mathrm{Ext}_R^i(T, T) = 0$ for all $i > 0$, and

(C3) there is an exact sequence $0 \to C_n \to \cdots \to C_0 \to \mathbf{D}(R_R) \to 0$ with each $C_i \in \mathrm{add}_RC$.

Note that $T \in \mathrm{mod}R$ is a tilting module of projective dimension at most $n$ if and only if $\mathbf{D}({}_RT) \in \mathrm{mod}R^o$ is a cotilting module of injective dimension at most $n$.

**Lemma 2.1.** *Let $R$ be an artin algebra and $T \in \mathrm{mod}R$ be a tilting module of projective dimension at most $n$. Then for any $M \in \mathrm{mod}R$, there is an exact sequence $0 \to M \to U_M \to V_M \to 0$ with $U_M \in T^\perp$ and $V_M \in (\mathrm{add}_RT)_{n-1}$. In particular, $V_M$ has the projective dimension at most $n$.*

*Proof.* The claimed exact sequence exists by for instance [3, Section 5]. □

The following is the well-known Brenner-Butler Theorem in the tilting theory, see for instance [8, 13].

**Lemma 2.2.** *Let $R$ be an artin algebra and $T \in \mathrm{mod}R$ be a tilting module with $S = \mathrm{End}_RT$. Denote $C = \mathbf{D}(T_S)$. Then there is an equivalence between $T^\perp$ and ${}^\perp C$, given by the functor $\mathrm{Hom}_R(T, -)$. Moreover, for any $U, W \in T^\perp$ and any $i \geq 0$, we have that $\mathrm{Ext}_R^i(U, W) \simeq \mathrm{Ext}_S^i(\mathrm{Hom}_R(T, U), \mathrm{Hom}_R(T, W))$ canonically.*

## 3. The proof of Main Theorem

Throughout this section, we fix an artin algebra $R$ and a tilting $R$-module $T$ with $S = \mathrm{End}_RT$. We set $n = \mathrm{pd}_RT$.

To prove the Main Theorem, we need some lemmas.

**Lemma 3.1.** *Assume that $M \in \mathrm{mod}R$ satisfies that $M \in {}^\perp(M \oplus R)$. Then $\Omega_R^n M \in {}^\perp(\Omega_R^n M \oplus R)$.*

*Proof.* Consider the exact sequence $0 \to \Omega^n M \to R_{n-1} \to \cdots \to R_0 \to M \to 0$ with each $R_i$ projective. By applying functors $\mathrm{Hom}_R(-, R)$, $\mathrm{Hom}_R(-, \Omega^n M)$ and $\mathrm{Hom}_R(M, -)$ in turn, we obtain for all $i \geq 1$ that firstly

$$\mathrm{Ext}_R^i(\Omega^n M, R) \simeq \mathrm{Ext}_R^{i+n}(M, R) = 0,$$

secondly

$$\mathrm{Ext}_R^i(\Omega^n M, \Omega^n M) \simeq \mathrm{Ext}_R^{i+n}(M, \Omega^n M),$$

and lastly

$$\mathrm{Ext}_R^{i+n}(M, \Omega^n M) \simeq \mathrm{Ext}_R^i(M, M) = 0,$$

by the dimension shift and the assumption. Hence the conclusion follows. □

**Lemma 3.2.** *Assume that $M \in {}^\perp T$. Then $\Omega_R^n M \in {}^\perp R$.*

*Proof.* Consider the exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ in the definition of tilting modules. Applying the functor $\mathrm{Hom}_R(M, -)$, we obtain that $\mathrm{Ext}_R^{i+n}(M, R) \simeq \mathrm{Ext}_R^i(M, T_n) = 0$ for all $i \geq 1$ by the dimension shift, since $M \in {}^\perp T$. It follows that $\mathrm{Ext}_R^i(\Omega_R^n M, R) \simeq \mathrm{Ext}_R^{i+n}(M, R) = 0$ for all $i \geq 1$, i.e., $\Omega_R^n M \in {}^\perp R$. □

The following lemma is important for the proof.

**Lemma 3.3.** *For any $M \in \mathrm{mod}R$, there is an exact sequence*

$$0 \to M \to V \to U \to 0 \qquad (\dagger)$$

*such that $U \in \mathrm{add}_R T$ and $V$ satisfies an exact sequence*

$$0 \to \Omega_R^n M \to T_{n-1} \to \cdots \to T_0 \to V \to 0 \qquad (\ddagger)$$

*with each $T_i \in \mathrm{add}_R T$.*

*Proof.* Clearly we have the exact sequence $0 \to \Omega^n M \to R_{n-1} \to \cdots \to R_0 \to M \to 0$ with each $R_i$ projective. Since $R \in (\mathrm{add}_R T)_n$, the conclusion then follows from [17, Lemma 2.3]. □

**Lemma 3.4.** *Assume that $M \in {}^\perp(T \oplus M) \bigcap T^\perp$. Then $\Omega_R^n M \in {}^\perp \Omega_R^n M$.*

*Proof.* We consider the two exact sequences in Lemma 3.3. Since $\mathrm{pd}_R T \leq n$ and $\mathrm{Ext}_R^i(T, T) = 0$ for all $i \geq 1$, we easily obtain that $V \in T^\perp$ from the exact sequence (‡). It follows that $U \in T^\perp$ from the exact sequence (†), since $M \in T^\perp$ too. Hence we obtain that $U \in T^\perp \bigcap \mathrm{add}_R T = \mathrm{add}_R T$. It turns out that the sequence (†) splits, and consequently $V \simeq M \oplus U \in \mathrm{add}_R(T \oplus M)$. Since $M \in {}^\perp(T \oplus M)$ by assumption, $M \in {}^\perp V$ too. Hence, applying the functor $\mathrm{Hom}_R(M, -)$ to the exact sequence (‡), we obtain that $\mathrm{Ext}_R^{i+n}(M, \Omega_R^n M) \simeq \mathrm{Ext}_R^i(M, V) = 0$ for all $i \geq 1$ by the dimension shift. It follows that $\mathrm{Ext}_R^i(\Omega_R^n M, \Omega_R^n M) \simeq \mathrm{Ext}_R^{i+n}(M, \Omega_R^n M) = 0$ for all $i \geq 1$, i.e., $\Omega_R^n M \in {}^\perp \Omega_R^n M$. □

**Lemma 3.5.** *Assume that $M \in {}^\perp(T \oplus M) \bigcap T^\perp$. If $R$ satisfies the Auslander-Reiten conjecture, then $M \in \mathrm{add}_R T$.*

*Proof.* Since $M \in {}^\perp(T \oplus M) \bigcap T^\perp$, we obtain that $\Omega_R^n M \in {}^\perp(\Omega_R^n M \oplus R)$ by Lemmas 3.2 and 3.4. If $R$ satisfies the Auslander-Reiten conjecture, then $\Omega_R^n M$ must be projective. It follows that $\mathrm{pd}_R M < \infty$. Combining with the assumption $M \in T^\perp$, we have that $M \in \widehat{\mathrm{add}_R T}$. Combining with the assumption $M \in {}^\perp T$, we easily obtain that $M \in \mathrm{add}_R T$. □

We can now prove the one-part of the Main Theorem.

**Proposition 3.6.** *If $R$ satisfies the Auslander-Reiten conjecture, then so does $S$.*

*Proof.* Take any $N \in \mathrm{mod}S$ such that $N \in {}^\perp(N \oplus S)$. Then $\Omega_S^n N \in {}^\perp(\Omega_S^n N \oplus S)$ by Lemma 3.1. Note that $\Omega_S^n N \in {}^\perp \mathbf{D}(T_S)$, so $\Omega_S^n N = \mathrm{Hom}_R(T, M)$ for some $M \in T^\perp$ by the tilting equivalence in Lemma 2.2. Since $S = \mathrm{Hom}_R(T, T)$ and $M \oplus T \in T^\perp$, we obtain that $\mathrm{Ext}_R^i(M, M \oplus T) \simeq \mathrm{Ext}_S^i(\mathrm{Hom}_R(T, M), \mathrm{Hom}_R(T, M) \oplus \mathrm{Hom}_R(T, T)) \simeq \mathrm{Ext}_S^i(\Omega_S^n N, \Omega_S^n N \oplus S) = 0$ for all $i \geq 1$ by assumption and Lemma 2.2 again. Hence we have that $M \in {}^\perp(T \oplus M) \bigcap T^\perp$. Since $R$ satisfies the Auslander-Reiten conjecture, we obtain from Lemma 3.5 that $M \in \mathrm{add}_R T$. It follows that $\Omega_S^n N (= \mathrm{Hom}_R(T, M))$ is projective. Consequently, $\mathrm{pd}_S N < \infty$. Since $N \in {}^\perp S$ too, it is easy to see that $N$ is projective. It follows that $S$ satisfies the Auslander-Reiten conjecture. □

*Proof of the Main Theorem*:

By the previous proposition, we need only to show that if $S$ satisfies the Auslander-Reiten conjecture, then so does $R$. To this end, let us take any $M \in \mathrm{mod}R$ such that $M \in {}^{\perp}(M \oplus R)$. Then we need to show that $M$ is projective.

By Lemma 2.1, there is exact sequence

$$0 \to M \to U_M \to V_M \to 0 \qquad (\sharp)$$

with $U_M \in T^{\perp}$ and $V_M \in (\mathrm{add}_R T)_{n-1}$. Note that for any $N \in \mathrm{mod}R$ with $\mathrm{pd}_R N < \infty$, we have that $\mathrm{Ext}^i_R(M, N) = 0$ for all $i \geq 1$ since $M \in {}^{\perp}R$. It follows that $M \in {}^{\perp}T$ and $M \in {}^{\perp}V_M$. Since clearly $V_M \in {}^{\perp}T$ too, we obtain that $U_M \in {}^{\perp}T$ from the sequence ($\sharp$). By assumption, $\mathrm{Ext}^i_R(M, M) = 0$ for all $i \geq 1$. It follows that $\mathrm{Ext}^i_R(M, U_M) = 0$ for all $i \geq 1$ by applying the functor $\mathrm{Hom}_R(M, -)$ to the sequence ($\sharp$). Note also that $\mathrm{Ext}^i_R(V_M, U_M) = 0$ for all $i \geq 1$ since $U_M \in T^{\perp}$ and $V_M \in \mathrm{add}_R T$, so applying the functor $\mathrm{Hom}_R(-, U_M)$ to the exact sequence ($\sharp$), we further obtain that $\mathrm{Ext}^i_R(U_M, U_M) = 0$ for all $i \geq 1$. It amounts to that $U_M \in {}^{\perp}(T \oplus U_M) \bigcap T^{\perp}$.

Denote $N = \mathrm{Hom}_R(T, U_M)$. Then by Lemma 2.2 and the above arguments, we obtain that $\mathrm{Ext}^i_S(N, N \oplus S) = \mathrm{Ext}^i_S(\mathrm{Hom}_R(T, U_M), \mathrm{Hom}_R(T, U_M) \oplus \mathrm{Hom}_R(T, T)) \simeq \mathrm{Ext}^i_R(U_M, U_M \oplus T) = 0$ for all $i \geq 1$. Hence if $S$ satisfies the Auslander-Reiten conjecture, then $N$ is projective. Consequently $U_M \in \mathrm{add}_R T$ by the tilting equivalence in Lemma 2.2. Thus from the exact sequence ($\sharp$) we obtain that $M \in \mathrm{add}_R T$. It follows that $\mathrm{pd}_R M < \infty$, and hence $M$ is projective. Thus $R$ satisfies the Auslander-Reiten conjecture.

**Corollary 3.7.** *Let $R$ be an artin algebra and $T \in \mathrm{mod}R$ with $S = \mathrm{End}_R T$.*

*(1) If $T$ is a tilting module, then $R^o$ satisfies the Auslander-Reiten conjecture if and only if so does $S^o$.*

*(2) If $T$ is a cotilting module, then $R$ (resp., $R^o$) satisfies the Auslander-Reiten conjecture if and only if so does $S$ (resp., $S^o$).*

*Proof.* (1) Note that $T$ is also a tilting $S^o$-module with $R^o \simeq \mathrm{End}_{S^o} T$, so the conclusion follows from the Main Theorem.

(2) Denote $C = \mathbf{D}(T_S)$. Then $C$ is a tilting $S$-module. Now the conclusion follows from the Main Theorem and the first part. □

Let $A, B$ be two artin algebras. We say that $A$ is tilting-cotilting equivalent to $B$, provided that there are some artin algebras $A_i$, $0 \leq i \leq n$, and some tilting or cotilting $A_i$-modules $T_i$ such that $A = A_0$, $B = A_n$, and $A_{i+1} \simeq \mathrm{End}_{A_i} T_i$ for $0 \leq i \leq n-1$. Clearly the tilting-cotilting equivalence is a kind of derived equivalences. Results in this paper show that if two artin algebras $A$ and $B$ are tilting-cotilting equivalent, then $A$ satisfies the Auslander-Reiten conjecture if and only if so does $B$.

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