

GERSTENHABER-BATALIN-VILKOVISKI STRUCTURES ON COISOTROPIC INTERSECTIONS

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ABSTRACT. Let Y, Z be a pair of smooth coisotropic subvarieties in a smooth algebraic Poisson variety X . We show that any data of first order deformation of the structure sheaf \mathcal{O}_X to a sheaf of noncommutative algebras and of the sheaves \mathcal{O}_Y and \mathcal{O}_Z to sheaves of right and left modules over the deformed algebra, respectively, gives rise to a Batalin-Vilkoviski algebra structure on the Tor-sheaf $\mathcal{T}or^{\bullet, \mathcal{O}^X}(\mathcal{O}_Y, \mathcal{O}_Z)$. The induced Gerstenhaber bracket on the Tor-sheaf turns out to be canonically defined; it is independent of the choices of deformations involved. There are similar results for Ext-sheaves as well.

Our construction is motivated by, and is closely related to, a result of Behrend-Fantechi [2], who studied intersections of Lagrangian submanifolds in a symplectic manifold.

1. Introduction

1.1. Main result. Let \mathbb{C} be a field of characteristic zero. We let $\mathbb{C}_\varepsilon := \mathbb{C}[\varepsilon]/(\varepsilon^2)$ denote the ring of dual numbers and let all unlabeled tensor products stand for $\otimes_{\mathbb{C}}$. Given an algebraic variety X , we write \mathcal{O}_X for the structure sheaf, resp. T_X for the tangent sheaf on X .

Fix a smooth algebraic variety X , over \mathbb{C} , and $P \in H^0(X, \Lambda^2 T_X)$, a Poisson bivector. Thus, there is a Poisson bracket $\{-, -\} : \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X$ given by the formula $\{f, g\} = \langle P, df \wedge dg \rangle$.

Let \mathcal{A} be a sheaf of (not necessarily commutative) \mathbb{C}_ε -algebras equipped with an algebra isomorphism $\mathcal{A}/\varepsilon\mathcal{A} \xrightarrow{\sim} \mathcal{O}_X$ so that \mathcal{A} gives a flat deformation of the structure sheaf \mathcal{O}_X . We require, in addition, that the Poisson bracket induced by the commutator in \mathcal{A} be equal to the bracket $\{-, -\}$. A particular example of such a deformation is the sheaf $\mathcal{A} := \mathbb{C}_\varepsilon \otimes \mathcal{O}_X = \mathcal{O}_X \oplus \varepsilon\mathcal{O}_X$, equipped with multiplication given by the well-known formula $f \times g \mapsto f * g = fg + \frac{\varepsilon}{2}\{f, g\}$, for any $f, g \in \mathcal{O}_X$.

Let $Z \subset X$ be a smooth subvariety. In this paper, we are interested in flat deformations of the sheaf \mathcal{O}_Z , viewed as an \mathcal{O}_X -module supported on Z , to either left or right \mathcal{A} -module \mathcal{C} set theoretically supported on Z . Associated with such a deformation \mathcal{C} to a left \mathcal{A} -module, there is a *transposed* deformation \mathcal{C}^t , which gives a right \mathcal{A} -module, see section 4.1 for the definition of \mathcal{C}^t .

Next, let $Y, Z \subset X$ be a pair of smooth subvarieties. Then, $Y \cap Z$, a *scheme theoretic* intersection, is a closed subscheme of X with structure sheaf $\mathcal{O}_{Y \cap Z} := \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Z$. More generally, we have $\mathcal{T}or^{\bullet, \mathcal{O}^X}(\mathcal{O}_Y, \mathcal{O}_Z)$, a coherent sheaf of supercommutative graded $\mathcal{O}_{Y \cap Z}$ -algebras. Similarly one has a sheaf $\mathcal{E}xt^{\bullet, \mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ that

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comes equipped with the natural structure of a graded $\mathcal{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ -module (the module structure is recalled in Section 3.2).

Recall that a graded commutative algebra $D = \bigoplus_{k \geq 0} D_k$ equipped with an operator $\delta : D_{\bullet} \rightarrow D_{\bullet-1}$ is called a Batalin-Vilkovisky (BV) algebra if δ is a differential operator of order ≤ 2 (with respect to multiplication in D) and one has $\delta^2 = 0$. In this case, the formula (for x, y homogeneous):

$$(1.1.1) \quad [x, y] := \delta(x \cdot y) - \delta(x) \cdot y - (-1)^{\deg x} x \cdot \delta(y),$$

provides D_{\bullet} with a structure of Gerstenhaber algebra (i.e., odd Poisson algebra). See e.g. [2] for more details on these definitions.

Similarly, given a graded D -module $M = \bigoplus_{k \geq 0} M_k$, a BV-module structure on M is the data of a linear operator $\delta' : M_{\bullet} \rightarrow M_{\bullet-1}$ such that $(\delta')^2 = 0$ and such that δ' has order ≤ 2 in the sense that for any homogeneous $x, y \in D$ and $m \in M$, the following equation holds

$$\begin{aligned} \delta(xy)m - (-1)^{\deg x} x\delta(y)m - \delta(x)ym &= \delta'(xym) - \\ &- (-1)^{\deg x} x\delta'(ym) - (-1)^{\deg x \deg y + \deg y} y\delta'(xm) + (-1)^{\deg x + \deg y} xy\delta'(m). \end{aligned}$$

In such a case, an analogue of formula (1.1.1) (for δ' instead of δ) gives a pairing $\{-, -\} : D \otimes M \rightarrow M$ that makes M a Gerstenhaber module over D .

The main result of this paper reads

Theorem 1.1.2. *Let X be a smooth Poisson variety with Poisson bivector P , and let \mathcal{A} be a flat \mathbb{C}_{ε} -deformation of \mathcal{O}_X such that the commutator in \mathcal{A} induces the Poisson bracket given by P . Let Y, Z be a pair of smooth coisotropic subvarieties in X . Then, we have*

(i) *Associated with the data of a flat \mathbb{C}_{ε} -deformation of the sheaf \mathcal{O}_Y to a right \mathcal{A} -module \mathcal{B} and of the sheaf \mathcal{O}_Z to a left \mathcal{A} -module \mathcal{C} , there is a second order differential operator $\delta : \mathcal{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) \rightarrow \mathcal{T}or_{\bullet-1}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ that squares to zero (i.e. makes $\mathcal{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ a BV algebra) provided the first order deformations locally admit extensions to second order deformations.*

(ii) *The induced bracket (1.1.1) on $\mathcal{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ is independent of the choice of deformations \mathcal{B} and \mathcal{C} .*

(iii) *Similarly, given an additional flat \mathbb{C}_{ε} -deformation of \mathcal{O}_Z to a right \mathcal{A} -module \mathcal{C}' , there is an associated second order differential operator $\delta' : \mathcal{E}xt_{\mathcal{O}_X}^{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z) \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^{\bullet-1}(\mathcal{O}_Y, \mathcal{O}_Z)$, such that $(\delta')^2 = 0$ if the first order deformations locally admit extensions to second order deformations. If $\mathcal{C}' = \mathcal{C}^t$ the corresponding operator δ' provides the sheaf $\mathcal{E}xt_{\mathcal{O}_X}^{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z)$ with a structure of BV-module over the BV algebra $\mathcal{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$. Moreover, the resulting pairing*

$$\{-, -\} : \mathcal{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) \times \mathcal{E}xt_{\mathcal{O}_X}^{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z) \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^{\bullet}(\mathcal{O}_Y, \mathcal{O}_Z)$$

is independent of the choice of deformations \mathcal{B} and \mathcal{C} .

Since flat deformations exist locally, Theorem 1.1.2 yields the following corollary, which is the second important result of the paper.

Corollary 1.1.3. *Let $Y, Z \subset X$ any pair of smooth coisotropic submanifolds of an arbitrary smooth Poisson algebraic variety X . Then, on $\mathcal{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ there is a canonical structure of Gerstenhaber algebra.*

Furthermore, the group $\mathcal{E}xt_{\mathcal{O}_X}^\bullet(\mathcal{O}_Y, \mathcal{O}_Z)$ has a canonical structure of Gerstenhaber module over the Gerstenhaber algebra $\mathcal{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$.

Several examples of such BV and Gerstenhaber structures are discussed in §5 below.

Our results above were, to a great extent, inspired by the work of K. Behrend and B. Fantechi [2]. Behrend and Fantechi consider a pair Y, Z , of Lagrangian submanifolds in a *holomorphic* symplectic manifold X . They show that one can equip the graded algebra $\mathcal{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ with a Gerstenhaber bracket, resp. the graded sheaf $\mathcal{E}xt_{\mathcal{O}_X}^\bullet(\mathcal{O}_Y, \mathcal{O}_Z)$ with a BV type differential. The approach in [2] is based on reducing the case of a general Lagrangian intersection to the special case where $X = T^*Y$ and $Z \subset T^*Y$ is the graph of a holomorphic function on Y (and Y is identified with the zero section of T^*Y).

Thus, the arguments in [2] rely in a crucial way on a version of Darboux theorem saying that any holomorphic symplectic manifold is locally isomorphic to a cotangent bundle. Such a result holds for holomorphic symplectic manifolds (equipped with the usual Hausdorff topology) but it is totally false in the algebraic setting. Indeed, an algebraic symplectic 2-form need not be locally exact, even in étale topology. The corresponding argument, kindly communicated to us by A. Beilinson, will be given in section 5.3.

1.2. Construction of the BV differential. Let \mathcal{A} be any flat \mathbb{C}_ε -deformation of the sheaf \mathcal{O}_X to a sheaf of associative \mathbb{C}_ε -algebras equipped with an algebra isomorphism $\mathcal{A}/\varepsilon\mathcal{A} \simeq \mathcal{O}_X$. Similarly, let \mathcal{B} be a flat deformation of \mathcal{O}_Y to a right \mathcal{A} -module and \mathcal{C} a flat deformation of \mathcal{O}_Z to a left \mathcal{A} -module. The flatness assumptions imply that multiplication by ε induces an isomorphism $\mathcal{O}_Y = \mathcal{B}/\varepsilon\mathcal{B} \xrightarrow{\sim} \varepsilon\mathcal{B}$, and similar isomorphism $\mathcal{O}_Z = \mathcal{C}/\varepsilon\mathcal{C} \xrightarrow{\sim} \varepsilon\mathcal{C}$.

The short exact sequence $0 \rightarrow \varepsilon\mathcal{C} \rightarrow \mathcal{C} \rightarrow \mathcal{C}/\varepsilon\mathcal{C} \rightarrow 0$ induces a long exact sequence

$$\dots \rightarrow \mathcal{T}or_{i+1}^{\mathcal{A}}(\mathcal{B}, \mathcal{C}/\varepsilon\mathcal{C}) \rightarrow \mathcal{T}or_i^{\mathcal{A}}(\mathcal{B}, \varepsilon\mathcal{C}) \rightarrow \mathcal{T}or_i^{\mathcal{A}}(\mathcal{B}, \mathcal{C}) \rightarrow \mathcal{T}or_i^{\mathcal{A}}(\mathcal{B}, \mathcal{C}/\varepsilon\mathcal{C}) \rightarrow \dots$$

Locally, we can choose a projective resolution P^\bullet of \mathcal{B} with \mathcal{A} -modules, such that $P^\bullet/\varepsilon P^\bullet$ is a resolution of \mathcal{O}_Y with projective \mathcal{O}_X -modules. Further, we have an isomorphism of functors $(\cdot) \otimes_{\mathcal{A}} \varepsilon\mathcal{C} \simeq (\cdot) \otimes_{\mathcal{A}} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ and similarly for $\mathcal{C}/\varepsilon\mathcal{C}$. We deduce canonical isomorphisms

$$\mathcal{T}or_i^{\mathcal{A}}(\mathcal{B}, \varepsilon\mathcal{C}) \simeq \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) \simeq \mathcal{T}or_i^{\mathcal{A}}(\mathcal{B}, \mathcal{C}/\varepsilon\mathcal{C})$$

Using these isomorphisms, the connecting morphism in the long exact sequence above yields a map

$$\delta : \mathcal{T}or_{i+1}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) \rightarrow \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z).$$

Similarly, suppose that we have a deformation \mathcal{C}' of \mathcal{O}_Z to a *right* \mathcal{A} -module. Then there is a long exact sequence

$$\dots \rightarrow \mathcal{E}xt_{\mathcal{A}}^{i-1}(\mathcal{B}, \mathcal{C}'/\varepsilon\mathcal{C}') \rightarrow \mathcal{E}xt_{\mathcal{A}}^i(\mathcal{B}, \varepsilon\mathcal{C}') \rightarrow \mathcal{E}xt_{\mathcal{A}}^i(\mathcal{B}, \mathcal{C}') \rightarrow \mathcal{E}xt_{\mathcal{A}}^i(\mathcal{B}, \mathcal{C}'/\varepsilon\mathcal{C}') \rightarrow \dots$$

In particular, one has a morphism

$$\delta' : \mathcal{E}xt_{\mathcal{A}}^i(\mathcal{B}, \varepsilon\mathcal{C}') \simeq \mathcal{E}xt_{\mathcal{O}_X}^i(\mathcal{O}_Y, \mathcal{O}_Z) \longrightarrow \mathcal{E}xt_{\mathcal{O}_X}^{i+1}(\mathcal{O}_Y, \mathcal{O}_Z) \simeq \mathcal{E}xt_{\mathcal{A}}^{i+1}(\mathcal{B}, \mathcal{C}'/\varepsilon\mathcal{C}')$$

When both \mathcal{C} and \mathcal{C}' are given we will assume that $\mathcal{C}' = \mathcal{C}^t$ or \mathcal{C}' is *transposed* to \mathcal{C} .

Notation. Given a vector bundle (a locally free sheaf) E , we write E^\vee for the dual vector bundle. Let Ω_X , resp. $T_X = \Omega_X^\vee$, denote the cotangent, resp. tangent, sheaf on a manifold X . Let $N_{X/Y}$ denote the normal sheaf for a submanifold $Y \subset X$.

We often abuse the notation and write $\mathrm{Tor}_\bullet^X(-, -)$ for $\mathrm{Tor}_\bullet^{\mathcal{O}^X}(-, -)$, and similarly for Ext 's.

1.3. A conjecture by physicists. Recall first that, for any (triangulated) category \mathcal{C} , one can define its Hochschild cohomology groups $HH^\bullet(\mathcal{C})$.

According to A. Kapustin and L. Rozansky one has the following

Conjecture 1.3.1. *To each pair $Y, Z \subset X$, of smooth Lagrangian submanifolds of a smooth algebraic symplectic variety X , one can associate a triangulated category $\mathcal{C}at_X(Y, Z)$, cf. [7], such that the Hochschild cohomology of the category $\mathcal{C}at_X(Y, Z)$ is given by*

$$HH^\bullet(\mathcal{C}at_X(Y, Z)) \cong \mathcal{T}or_\bullet^{\mathcal{O}^X}(\mathcal{O}_Y, \mathcal{O}_Z).$$

Moreover, the standard Gerstenhaber bracket on Hochschild cohomology goes, under the isomorphism above, to the canonical Gerstenhaber bracket on $\mathcal{T}or_\bullet^{\mathcal{O}^X}(\mathcal{O}_Y, \mathcal{O}_Z)$ provided by Corollary 1.1.3.

If $X = T^\vee Y$ is the cotangent bundle of Y and $Z = Y$ is the zero section, then one should have $\mathcal{C}at_X(Y, Z) = D^b(\mathrm{Coh} Y)$. In this case, we have

$$HH^\bullet(D^b(\mathrm{Coh} Y)) = H^\bullet(Y, \Lambda^\bullet T_Y) = \mathrm{Tor}_\bullet^{T^\vee Y}(\mathcal{O}_Y, \mathcal{O}_Y).$$

so that the Gerstenhaber bracket is induced by the Schouten bracket on $\Lambda^\bullet T_Y$.

More generally, let $X = T^\vee Y$ and Y be the zero section as above, and let $Z = \mathrm{Graph}(df)$ where $f \in \mathbb{C}[Y]$. Then $\mathcal{C}at_X(Y, Z)$ should be the category of matrix factorizations $(F \xrightarrow[d']{d} F', d \circ d' = f \cdot \mathrm{Id} = d' \circ d)$, associated with the function f , cf. [11].

Remark 1.3.2. Observe that the sheaves $\mathcal{T}or_\bullet^{\mathcal{O}^X}(\mathcal{O}_Y, \mathcal{O}_Z)$ are related to the hyper-Tor groups $\mathrm{Tor}_\bullet^X(\mathcal{O}_Y, \mathcal{O}_Z)$ via the local-to-global spectral sequence

$$H^\bullet(X, \mathcal{T}or_\bullet^{\mathcal{O}^X}(\mathcal{O}_Y, \mathcal{O}_Z)) \implies \mathrm{Tor}_\bullet^X(\mathcal{O}_Y, \mathcal{O}_Z).$$

At the same time, the global hyper-Tor may also be calculated by applying $R\Gamma$ to the sheaf of DG algebras T_\bullet described in Section 3 below. Thus, we expect that there exists a refined version of our results, in which Gerstenhaber or Batalin-Vilkovisky structures on the cohomology sheaves of T_\bullet , are replaced by their “strong homotopy” versions on T_\bullet itself. In fact, the lemmas of Section 3.2 point towards such a refinement. Similar remarks apply to the Ext groups (local and global), and the polydifferential version of the resolution E^\bullet in Section 3.

2. Existence of first and second order deformations.

2.1. Algebraic setup. Following Gerstenhaber, a \mathbb{C}_ε -flat deformation of an associative \mathbb{C} -algebra A is given by a \mathbb{C}_ε -bilinear associative product structure on the vector space $A_\varepsilon = A \oplus \varepsilon A$ defined by

$$(a_1 \oplus 0) * (a_2 \oplus 0) = a_1 a_2 \oplus \varepsilon \cdot \alpha_A(a_1, a_2), \quad a_1, a_2 \in A \subset A_\varepsilon$$

with $\alpha_A : A \otimes A \rightarrow A$ a \mathbb{C} -linear map. The associativity of the $*$ -product is equivalent to the equation

$$(2.1.1) \quad \alpha_A(a_1 a_2, a_3) - \alpha_A(a_1, a_2 a_3) + \alpha_A(a_1, a_2) a_3 - a_1 \alpha_A(a_2, a_3) = 0.$$

Fix a \mathbb{C}_ε -flat deformation of A as above. Given a right A -module B , one may consider \mathbb{C}_ε -flat extensions of the A -module structure on B to a right A_ε -module on $B_\varepsilon = B \oplus \varepsilon B$. Explicitly, such an A_ε -module structure on B_ε is determined by a bilinear map $\alpha_B : B \otimes A \rightarrow B$. The corresponding right A_ε -action is given by the formula

$$(b \oplus 0) * (a \oplus 0) = ba \oplus \varepsilon \cdot \alpha_B(b, a), \quad a \in A, b \in B.$$

The map $\alpha_B : B \otimes A \rightarrow B$ must satisfy the associativity equation

$$(2.1.2) \quad \alpha_B(b a_2, a_3) - \alpha_B(b, a_2 a_3) + \alpha_B(b, a_2) a_3 - b \alpha_A(a_2, a_3) = 0.$$

Further, any pair of \mathbb{C}_ε -linear automorphisms of the form

$$A_\varepsilon \rightarrow A_\varepsilon, \quad a \mapsto a \oplus \varepsilon \beta_A(a) \quad \text{and} \quad B_\varepsilon \rightarrow B_\varepsilon, \quad b \mapsto b \oplus \varepsilon \beta_B(b),$$

where $\beta_A : A \rightarrow A$ and $\beta_B : B \rightarrow B$ are \mathbb{C} -linear maps, induces equivalent deformations corresponding to cochains

$$(2.1.3) \quad (a_1, a_2) \mapsto \alpha_A(a_1, a_2) + \beta_A(a_1 a_2) - a_1 \beta_A(a_2) - \beta_A(a_1) a_2$$

$$(2.1.4) \quad (b, a_2) \mapsto \alpha_B(b, a_2) + \beta_B(b a_2) - b \beta_A(a_2) - \beta_B(b) a_2$$

A deformation as above extends to $\mathbb{C}[\varepsilon]/(\varepsilon^3)$ if and only if one has

$$(2.1.5) \quad \alpha_A(\alpha_A(a_1, a_2), a_3) - \alpha_A(a_1, \alpha_A(a_2, a_3)) = d\gamma_A(a_1, a_2, a_3)$$

$$(2.1.6) \quad \alpha_B(\alpha_B(b, a_1), a_2) - \alpha_B(b, \alpha_A(a_1, a_2)) = d\gamma_B(b, a_1, a_2),$$

where $\gamma_A : A \otimes A \rightarrow A$, $\gamma_B : B \otimes A \rightarrow B$ are some linear maps and $d\gamma_A$, $d\gamma_B$ are defined similarly to the LHS of (2.1.1) and (2.1.2), respectively.

2.2. Deformation complex. The identities of the previous subsection can be reformulated as follows. The A -module structure on B defines a homomorphism $g : A \rightarrow \text{End}_{\mathbb{C}} B$ of algebras over \mathbb{C} , and deforming the algebra/module structure amounts to deforming g to an algebra homomorphism $A_\varepsilon \rightarrow \text{End}_{\mathbb{C}_\varepsilon}(B_\varepsilon)$. Observe that $\text{End}_{\mathbb{C}_\varepsilon}(B_\varepsilon)$ is the trivial deformation of $\text{End}_{\mathbb{C}} B$. Thus, adjusting the definitions of [5], [4] (i.e. removing the term responsible for the deformation of the algebra $\text{End}_{\mathbb{C}} B$) we introduce a deformation complex $C_{A,B}^\bullet$ with terms

$$C_{A,B}^n = C^n(A, A) \oplus C^{n-1}(A, \text{End}_{\mathbb{C}} B) = \text{Hom}_{\mathbb{C}}(A^{\otimes n}, A) \oplus \text{Hom}_{\mathbb{C}}(A^{\otimes(n-1)}, \text{End}_{\mathbb{C}} B),$$

where $C^n(A, X)$ denotes the standard complex of Hochschild cochains of an A -bimodule X . The differential in the complex $C_{A,B}^\bullet$ is given by

$$(2.2.1) \quad d_{A,B}(\alpha_A \oplus \alpha_B) = d_{\text{Hoch}} \alpha_A \oplus (g \alpha_A - d_{\text{Hoch}} \alpha_B),$$

where d_{Hoch} stands for the standard Hochschild differential, cf. *loc. cit.* We put $H_{A,B}^n := H^n(C_{A,B}^\bullet)$.

Equations (2.1.1), (2.1.2) say that $\alpha = \alpha_A \oplus \alpha_B$ is a cocycle in $C_{A,B}^2$. Equations (2.1.3) and (2.1.4) say that the equivalence class of the deformation depends only on the image of α in $H_{A,B}^2$.

To reinterpret integrability conditions recall that by *loc. cit.* $H_{A,B}^{\bullet-1}$ has a structure of graded Lie algebra such that $H^{\bullet-1}(A, A)$, the Hochschild cohomology with Gerstenhaber's bracket, is a quotient Lie algebra of $H_{A,B}^{\bullet-1}$. Explicitly, up to a choice of signs, for $\alpha_A \oplus \alpha_B \in C_{A,B}^m$, $\alpha'_A \oplus \alpha'_B \in C_{A,B}^m$ one has

$$\begin{aligned} [\alpha_A \oplus \alpha_B, \alpha'_A \oplus \alpha'_B] &= (\alpha_A \circ \alpha'_A - (-1)^{(n-1)(m-1)} \alpha'_A \circ \alpha_A) \oplus \\ &\oplus (\alpha_B \circ \alpha'_A + \alpha_B \cup \alpha'_B - (-1)^{(n-1)(m-1)} (\alpha'_B \circ \alpha_A + \alpha'_B \cup \alpha_B)) \end{aligned}$$

where

$$\alpha_B \circ \alpha'_A := \sum_{s=1}^n (-1)^{(s-1)(m-1)} \alpha_B (1_A^{\otimes(s-1)} \otimes \alpha'_A \otimes 1_A^{\otimes(n-s)})$$

and similarly for the other terms. The cup product $\alpha_B \cup \alpha'_A : B \otimes A^{\otimes m+n-1} \rightarrow B$ is the composition of $\alpha_B \otimes \alpha'_A : B \otimes A^{\otimes m+n-1} \rightarrow B \otimes A$ and the action map $B \otimes A \rightarrow B$.

Then, equations (2.1.5), (2.1.6) say that $\frac{1}{2}[\alpha, \alpha] = d_{A,B}(\gamma)$, i.e. that $[\alpha, \alpha]$ represents the zero class in $H_{A,B}^3$.

Observe that $C^{\bullet-1}(A, \text{End}_{\mathbb{C}} B)$ is a subcomplex of $C_{A,B}^{\bullet}$, and $C^{\bullet}(A, A)$ is a quotient complex of $C_{A,B}^{\bullet}$. The corresponding long exact sequence of cohomology reads

$$(2.2.2) \quad \dots \rightarrow \text{Ext}_A^{n-1}(B, B) \rightarrow H_{A,B}^n \rightarrow H^n(A, A) \xrightarrow{g} \text{Ext}_A^n(B, B) \rightarrow \dots$$

We see that an n -cocycle $\alpha_A \in C^2(A, A)$ may be lifted to a class in $H_{A,B}^2$ if and only if the map $g \circ \alpha_A : A \times A \rightarrow \text{End}_{\mathbb{C}} B$, that represents the image of the class of α_A under the connecting homomorphism, gives the zero class in $\text{Ext}_A^2(B, B)$.

Similarly, given another algebra homomorphism $h : A \rightarrow (\text{End}_{\mathbb{C}} C)^{op}$, one can introduce a bigger deformation complex with terms

$$(2.2.3) \quad C_{A,B,C}^n = C^m(A, A) \oplus C^{m-1}(A, \text{End}_{\mathbb{C}} B) \oplus C^{n-1}(A, \text{End}_{\mathbb{C}} C).$$

The differential in the complex $C_{A,B,C}^{\bullet}$ is given by

$$d_{A,B,C}(\alpha_A \oplus \alpha_B \oplus \alpha_C) = d_{\text{Hoch}} \alpha_A \oplus (g\alpha_A - d_{\text{Hoch}} \alpha_B) \oplus (h\alpha_A - d_{\text{Hoch}} \alpha_C).$$

For the corresponding cohomology groups $H_{A,B,C}^{\bullet}$ there is a long exact sequence

$$\begin{aligned} \dots \rightarrow \text{Ext}_A^{n-1}(B, B) \oplus \text{Ext}_A^{n-1}(C, C) &\longrightarrow H_{A,B,C}^n \longrightarrow \\ &\longrightarrow H^n(A, A) \xrightarrow{g \oplus h} \text{Ext}_A^n(B, B) \oplus \text{Ext}_A^n(C, C) \rightarrow \dots \end{aligned}$$

2.3. Local deformations. Let now X be a smooth affine variety and $Y \subset X$ a smooth closed subvariety. Write $A := \mathbb{C}[X]$, resp. $B := \mathbb{C}[Y]$, for the corresponding coordinate rings.

A bivector $P \in H^0(X, \Lambda^2 T_X)$ with a vanishing Schouten bracket gives a Poisson structure on A . We will say that Y is *coisotropic* with respect to P if P projects to zero in $H^0(Y, \Lambda^2 N_{X/Y})$.

Proposition 2.3.1. *The map $H_{A,B}^\bullet \rightarrow H^\bullet(A, A)$, in (2.2.2), is injective. Furthermore, for any 2-cocycle $\alpha_A \in C^2(A, A)$ the following holds:*

(i) *Y is a coisotropic subvariety in X if and only if there exists $\alpha_B : B \otimes A \rightarrow B$ that gives a first order deformation of B , i.e., if and only if there exists α_B such that the pair (α_A, α_B) gives a 2-cocycle in the complex $C_{A,B}^\bullet$.*

(ii) *Assume that $\alpha_A(a_1, a_2) = \frac{1}{2}\langle P, da_1 \wedge da_2 \rangle$ with $P \in H^0(X, \Lambda^2 T_X)$. Then, in (i), one may choose $\alpha_B : B \otimes A \rightarrow B$ to be a sum of a bidifferential operator of bidegree $(1, 1)$ and a bidifferential operator of bidegree $(0, 2)$.*

(iii) *Assume, in addition, that the bivector P has a vanishing Schouten bracket with itself: $\{P, P\} = 0$. Then, there exists a symmetric bilinear map $\gamma_A : A \otimes A \rightarrow A$ such that equation (2.1.5) holds. If, moreover, the canonical class of Y is trivial, then there exist $\alpha_B : B \otimes A \rightarrow B$ and $\gamma_B : B \otimes A \rightarrow B$ such that equations (2.1.2) and (2.1.6) hold, i.e., the map $(b, a) \mapsto ba + \varepsilon \alpha_B(b, a) + \varepsilon^2 \gamma_B(b, a)$ gives a second order deformation of B .*

Proof. First, note that $g\alpha_A = d_{\text{Hoch}}\alpha_B$ in $C^2(A, \text{End}_C B)$ means that (2.1.2) holds by definition of the maps involved. This proves (i).

To prove (ii) let $I \subset A$ denote the defining ideal of the subvariety Y .

First, we are going to construct a map $\alpha_B^0 : B \otimes I \rightarrow B$, the restriction of the cocycle $\alpha_B : B \otimes A \rightarrow B$, that we are looking for, to $B \otimes I$. Observe that the cocycle equation (2.1.2) implies that, the map α_B^0 should satisfy the following two constraints:

$$\alpha_B^0(ba, x) - \alpha_B^0(b, ax) - b\alpha_A(a, x) = 0; \quad -\alpha_B^0(b, xa) + \alpha_B^0(b, x)a - b\alpha_A(x, a) = 0,$$

for any $b \in B$, $a \in A$, $x \in I$.

We will define α_B^0 to be a map of the following form:

$$\alpha_B^0(b, x) = \rho(db, x) + b\psi(x);$$

where $\rho : \Omega_B^1 \otimes (I/I^2) \rightarrow B$ is a B -bilinear map and $\psi : I/I^2 \rightarrow B$ is a first order algebraic differential operator. In terms of ρ and ψ , the two constraints above translate into the following pair of equations, for any $a \in A$, $x \in I$,

$$(2.3.2) \quad \psi(ax) - a\psi(x) = 1_B \alpha_X(a, x); \quad \rho(d(a|_Y), x) = 2 \cdot 1_B \alpha_X(a, x).$$

We remark that the second equation in (2.3.2) determines ρ uniquely, since every element of B is an image of some $a \in A$. Observe further that, for $a, x \in I$, we have $\alpha_A(a, x) \in I$ since Y is a coisotropic subvariety. Hence, in this case $1_B \alpha_A(a, x) = 0$. We see that we may (and will) use the second equation in (2.3.2) as a definition of ρ ; the resulting map ρ is well-defined.

Observe next that the first equation in (2.3.2) is a condition on the map $\sigma_\psi : I/I^2 \otimes \Omega_B^1 \rightarrow B$, the principal symbol of the first order differential operator ψ . Specifically, the equation says that $\sigma_\psi(a, x) = \frac{1}{2}\langle P|_Y, da \wedge dx \rangle$ for any $a \in I, x \in B$. Again, we may (and will) use the latter equation as the definition of σ_ψ . The resulting symbol is well-defined since P sends $I \otimes I$ to zero in B .

Recall next that, for any σ_ψ , one may find a differential operator ψ that has σ_ψ as its principal symbol. Indeed, let $\mathcal{D}^{\leq 1}(N_{X/Y}^\vee, \mathcal{O}_Y)$ denote the space of first order

algebraic differential operators $N_{X/Y}^\vee \rightarrow \mathcal{O}_Y$. The variety Y being smooth and affine, one has a short exact sequence, cf. [6],

$$0 \rightarrow \text{Hom}_{\mathcal{O}_Y}(N_{X/Y}^\vee, \mathcal{O}_Y) \rightarrow \mathcal{D}^{\leq 1}(N_{X/Y}^\vee, \mathcal{O}_Y) \rightarrow \text{Hom}_{\mathcal{O}_Y}(N_{X/Y}^\vee \otimes_{\mathcal{O}_Y} \Omega_Y^1, \mathcal{O}_Y) \rightarrow 0,$$

where the last arrow is the principal symbol map which is, therefore, surjective.

This completes the construction of the map $\alpha_B^0 : B \otimes I \rightarrow B$.

It remains to extend α_B^0 to construct a cocycle $\alpha_B : B \otimes A \rightarrow B$. To that end, note that since Y is smooth and affine we can choose a splitting of the short exact sequence

$$0 \rightarrow N_{X/Y}^\vee \rightarrow \Omega_X^1|_Y \rightarrow \Omega_Y^1 \rightarrow 0.$$

Such a splitting yields a B -linear map $p : \Omega_A^1 \otimes_A B \rightarrow I/I^2$. Similarly, a splitting of the projection $T_X^*|_Y \rightarrow T_Y^*$ yields a B -linear map $q : \Omega_B^1 \rightarrow \Omega_A^1 \otimes_A B$.

Using the splittings, we define

$$\alpha_B(b, a) = \rho(db, p(da)) - \frac{1}{2}P(q(da|_Y), q(db)) + b\psi(p(da)).$$

It is clear that the resulting map α_B is an extension of α_B^0 . Furthermore, an explicit calculation using identities (2.3.2) shows that the map α_B so defined satisfies the requirements of part (ii) of the proposition.

To prove part (iii) we need to recall an explicit version of the Hochschild-Kostant-Rosenberg isomorphism for Hochschild cohomology.

The Hochschild complex that we are interested in is the complex with terms $C^k(A, \text{End}_{\mathbb{C}} B) = \text{Hom}_{\mathbb{C}}(B \otimes A^{\otimes k}, B)$, equipped with the Hochschild differential d_{Hoch} . The Hochschild-Kostant-Rosenberg theorem says that, for $Y = \text{Spec } B$ smooth, one has an isomorphism $\text{Alt} : H^k(C^\bullet(A, \text{End}_{\mathbb{C}} B)) \xrightarrow{\sim} H^0(Y, \Lambda^k N_{X/Y})$. The isomorphism is constructed as follows. Given, $\gamma \in \text{Hom}_{\mathbb{C}}(B \otimes A^{\otimes k}, B)$, one obtains, by restriction to the ideal of Y , a polylinear map $B \otimes I^{\otimes k} \rightarrow B$. Let $\text{Alt}(\gamma)$ be the anti-symmetrization of this map with respect to the last k arguments. One shows, that if γ is a Hochschild *cocycle*, i.e. $d_{\text{Hoch}}\gamma = 0$, then $\text{Alt}(\gamma)(b, x_1, \dots, x_k) = 0$ whenever $x_i \in I^2$ for at least one $i = 1, \dots, k$. It follows that the map $\text{Alt}(\gamma)$ descends to a map $\text{Alt}(\gamma) : B \otimes \Lambda_{\mathbb{C}}^k(I/I^2) \rightarrow B$. Furthermore, the equation $d_{\text{Hoch}}\gamma = 0$ insures that the resulting map is B -polylinear, cf. [10, Proposition 1.3.12] for a similar result in the case of Hochschild homology. We conclude that the map $\text{Alt}(\gamma)$ descends to a well-defined B -linear map $\text{Alt}(\gamma) : \Lambda_B^k(I/I^2) \rightarrow B$. Giving such a map is the same as giving a section of $\Lambda^k N_{X/Y}$, and we are done.

We can now resume the proof of part (iii). First of all, we note that the existence of *some* γ_A is well-known, cf. e.g. [8]. By skew symmetry of $\alpha_A = \frac{1}{2}P$ it follows immediately that $\frac{1}{2}(\gamma_A(a, b) + \gamma_A(b, a))$ solves the same equation (2.1.5). Hence, from now on, we assume that the bilinear map γ_A is symmetric.

Assume now that the canonical bundle on Y is trivial and choose a trivialization, that is, a nowhere vanishing top degree differential form ω on Y . Let $\text{Lie}_\partial(\omega)$ denote the Lie derivative of ω with respect to a vector field ∂ on Y . The assignment $\partial \mapsto \text{Lie}_\partial(\omega) \cdot \omega^{-1}$ gives a first order differential operator $T_Y \rightarrow B$. We compose this differential operator with the B -linear map $I/I^2 \rightarrow T_Y$ given by the restriction of the bivector P to Y . This way, we obtain a first order differential operator $\psi : I/I^2 \rightarrow B$ that satisfies the identity $\psi(ax) - a\psi(x) = {}_B\alpha_X(a, x)$ (the first equation in (2.3.2)).

We use the above operator ψ to construct a cocycle $\alpha_B : B \otimes A \rightarrow B$ following the procedure explained in the proof of part (ii). The resulting operator $\alpha_B : B \otimes A \rightarrow B$ satisfies

$$(2.3.3) \quad \alpha_B(\alpha_B(b, x_1), x_2) - \alpha_B(\alpha_B(b, x_2), x_1) - 2\alpha_B(b, \alpha_A(x_1, x_2)) = 0, \quad x_1, x_2 \in I.$$

To complete the proof, we have to construct an operator $\gamma_B : B \otimes A \rightarrow B$ that satisfies the equation

$$(2.3.4) \quad d_{\text{Hoch}}\gamma_B(b, a, a') = \alpha_B(\alpha_B(b, a), a') - \alpha_B(b, \alpha_A(a, a')) + b\gamma_A(a, a'),$$

where the Hochschild differential $d_{\text{Hoch}} : \text{Hom}_{\mathbb{C}}(B \otimes A, B) \rightarrow \text{Hom}_{\mathbb{C}}(B \otimes A \otimes A, B)$ is given by the formula $d_{\text{Hoch}}\gamma_B(b, a, a') := \gamma_B(ba, a') - \gamma_B(b, aa') + \gamma_B(b, a)a'$.

Let $\eta(b, a, a')$ denote the RHS of (2.3.4). A straightforward computation shows that η is a Hochschild cocycle, explicitly, one has

$$\eta(ba, a', a'') - \eta(b, aa', a'') + \eta(b, a, a'a'') + \eta(b, a, a'')a'' = 0.$$

We claim further that η gives the zero class in Hochschild cohomology. To see this we use the Hochschild-Kostant-Rosenberg isomorphism. Thus, we must restrict η to $B \otimes I \otimes I$ and compute $\text{Alt}(\eta)$. But equation (2.3.3) says that the RHS of formula (2.3.4) is symmetric in the last two arguments. We conclude that $\text{Alt}(\eta) = 0$. Hence, η is a Hochschild coboundary, and part (iii) follows. \square

Remark 2.3.5. Here we sketch another proof of part (ii): Equation (2.1.2) says that $d\alpha_B = \alpha_A \cdot \text{Id}_B$ holds in $C^2(A, \text{End}_{\mathbb{C}}(B))$. Let $D^\bullet(A, \text{End}_{\mathbb{C}}(B))$ be a subcomplex of the Hochschild complex formed by cochains given by multidifferential operators. The arguments from [8], pp. 16-17 can be used to show that the cohomology groups of the complex $D^\bullet(A, \text{End}_{\mathbb{C}}(B))$ are $\wedge^\bullet N_{X/Y}$. Therefore, it follows from a version of the Hochschild-Kostant-Rosenberg result that the imbedding $j : D^\bullet(A, \text{End}_{\mathbb{C}}(B)) \hookrightarrow C^\bullet(A, \text{End}_{\mathbb{C}}(B))$ induces an isomorphism $H^\bullet(j)$, on cohomology.

Now, the Poisson bivector on X gives a class $\alpha_A \cdot \text{Id}_B \in D^2(A, \text{End}_{\mathbb{C}}(B))$. Since Y is coisotropic, $j(\alpha_A \cdot \text{Id}_B) \in C^2(A, \text{End}_{\mathbb{C}}(B))$ is a coboundary. By injectivity of $H^2(j)$ the class $\alpha_A \cdot \text{Id}_B \in D^2(A, \text{End}_{\mathbb{C}}(B))$ is itself a coboundary, i.e. $d\alpha_B = \alpha_A \cdot \text{Id}_B$ for some $\alpha_B \in D^2(A, \text{End}_{\mathbb{C}}(B))$. A separate easy calculation shows that the principal symbol of α_B is a linear combination of maps $\text{Sym}^2\Omega_{A/\mathbb{C}} \rightarrow B$ and $\Omega_{A/\mathbb{C}} \otimes \Omega_{B/\mathbb{C}} \rightarrow B$, i.e. its component $\text{Sym}^2\Omega_{B/\mathbb{C}} \rightarrow B$ is actually zero.

Remark 2.3.6. It can be shown using the arguments of the proof above and those of Section 4 below that the existence of a (not necessarily split) deformation for \mathcal{O}_Y with Y coisotropic, is equivalent to the vanishing of a certain class in $H^1(Y, N_{X/Y})$. This class is the cup product of the Atiyah class in $H^1(Y, \Omega_Y^1 \otimes_{\mathcal{O}_Y} \text{End}(N_{X/Y}))$ with the image of P in $H^0(Y, T_Y \otimes N_{X/Y})$. See Theorem 7 in [1] for more details.

3. An algebraic construction of BV operators

3.1. Complexes computing $\text{Tor}_\bullet^A(B, C)$ and $\text{Ext}_A^\bullet(B, C)$. In this subsection we fix a commutative algebra A and a pair of A -modules B, C . We have associated algebra homomorphisms $g : A \rightarrow \text{End}_{\mathbb{C}} B$, resp. $h : A \rightarrow \text{End}_{\mathbb{C}} C$. Write $T(A)$ for the tensor algebra of the vector space A .

Recall that the A -module B admits a free bar resolution $B \otimes T(A) \otimes A \rightarrow B$, cf. [12]. Therefore $\text{Tor}_A^i(B, C)$ and $\text{Ext}_A^\bullet(B, C)$ can be computed as the cohomology of complexes T_\bullet and E^\bullet , respectively, with terms

$$T_i = B \otimes A^{\otimes i} \otimes C, \quad E^i = \text{Hom}_k(B \otimes A^{\otimes i}, C)$$

The corresponding differentials, d_T and d_E , respectively, are given by

$$\begin{aligned} d_T(b \otimes a_1 \otimes \dots \otimes a_i \otimes c) &= ba_1 \otimes \dots \otimes a_i \otimes c + (-1)^i b \otimes a_1 \otimes \dots \otimes a_{i-1} \otimes a_i c + \\ &\quad + \sum_{s=1}^{i-1} (-1)^s b \otimes a_1 \otimes \dots \otimes a_s a_{s+1} \otimes \dots \otimes a_i \otimes c, \\ d_E \phi(b \otimes a_1 \otimes \dots \otimes a_{i+1}) &= -\phi(ba_1 \otimes \dots \otimes a_{i+1}) + (-1)^{i-1} \phi(b \otimes a_1 \otimes \dots \otimes a_i) a_{i+1} \\ &\quad + \sum_{s=1}^{i-1} (-1)^{s-1} \phi(b \otimes a_1 \otimes \dots \otimes a_s a_{s+1} \otimes \dots \otimes a_{i+1}) \end{aligned}$$

We consider deformations of the triple (A, B, C) . Such a deformation is determined by an element of the deformation complex $C_{A,B,C}^2$ given by a cocycle $(\alpha_A, \alpha_B, \alpha_C)$, see §2.2. Working with T_\bullet , we always assume that α_B gives a deformation of B to a right module and α_C a deformation of C to a left module.

The triple $(\alpha_A, \alpha_B, \alpha_C)$ induces an operation $\delta_\alpha : T_i \rightarrow T_{i-1}$ given essentially by the same formula as d_T where ba_1 is replaced by $\alpha_B(b, a_1)$, resp. $a_s a_{s+1}$ is replaced by $\alpha_A(a_s, a_{s+1})$ and $a_i c$ by $\alpha_C(a_i, c)$. If, in addition $\alpha'_C : C \otimes A \rightarrow A$ gives a deformation to a right module, then the triple $(\alpha_A, \alpha_B, \alpha'_C)$ induces an operation $\delta'_\alpha : E^i \rightarrow E^{i+1}$ given by a formula similar to d_E ; this time $\phi(X)a_i$ is replaced by $\alpha'_C(\phi(X), a_i)$.

The following result is proved by direct computation

Lemma 3.1.1. *Let δ_α be the operator on T_\bullet constructed from a triple $(\alpha_A, \alpha_B, \alpha_C)$.*

- (1) *If (α_A, α_B) is a cocycle in $C_{A,B}^2$ and (α_A, α_C) is a cocycle in C_h^2 then, we have*

$$\delta_\alpha d_T + d_T \delta_\alpha = 0.$$

- (2) *If $(\tilde{\alpha}_A, \tilde{\alpha}_B) - (\alpha_A, \alpha_B) = d(\beta_A, \beta_B)$ and $(\tilde{\alpha}_A, \tilde{\alpha}_C) - (\alpha_A, \alpha_C) = d(\beta_A, \beta_C)$ then, we have*

$$\delta_{\tilde{\alpha}} - \delta_\alpha = d_T \delta_\beta + \delta_\beta d_T,$$

where

$$\begin{aligned} \delta_\beta(b \otimes a_1 \otimes \dots \otimes a_n \otimes c) &= \beta_B(b) \otimes a_1 \otimes \dots \otimes a_n \otimes c + \\ &+ \sum_i (-1)^i b \otimes a_1 \otimes \dots \otimes \beta_A(a_i) \otimes \dots \otimes a_n \otimes c + (-1)^{n+1} b \otimes a_1 \otimes \dots \otimes a_n \otimes \beta_C(c). \end{aligned}$$

- (3) *If (2.1.5), (2.1.6) hold (with similar equation and notation assumed for α_C), then, we have*

$$\delta_\alpha^2 = d_T \delta_\gamma + \delta_\gamma d_T.$$

Similar identities hold for the map δ'_α constructed from $(\alpha_A, \alpha_B, \alpha'_C)$, with d_T replaced by d_E . \square

We now interpret δ_α in the context of the long exact sequence of Section 1.2. Since B_ε is flat over \mathbb{C}_ε we can construct a bar resolution using tensor products over \mathbb{C}_ε :

$$\dots \rightarrow B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon \rightarrow B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon \rightarrow B_\varepsilon \rightarrow 0$$

where the bar differential is defined using the deformed product $A_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon \rightarrow A_\varepsilon$ and the deformed action $B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon \rightarrow B_\varepsilon$. In particular, $\text{Tor}_i^{A_\varepsilon}(B_\varepsilon, C_\varepsilon)$ is the homology of the complex with the i -th term

$$T_i^\varepsilon = B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon^{\otimes_{\mathbb{C}_\varepsilon} i} \otimes_{\mathbb{C}_\varepsilon} C_\varepsilon \simeq [B \otimes A^{\otimes i} \otimes C] \oplus \varepsilon[B \otimes A^{\otimes i} \otimes C] = T_i \oplus \varepsilon T_i$$

It is easy to see that the differential of this complex is $d_\varepsilon = d + \varepsilon \delta_\alpha$. The spectral sequence of the filtered complex (with the two step filtration) $\varepsilon T_\bullet \subset T_\bullet^\varepsilon$ boils down to the long exact sequence

$$\dots \rightarrow H_i(T_\bullet, d) \rightarrow H_i(T_\bullet^\varepsilon, d_\varepsilon) \rightarrow H_i(T_\bullet, d) \rightarrow H_{i-1}(T_\bullet, d) \rightarrow \dots$$

By definition, we have $H_i(T_\bullet, d) = \text{Tor}_i^A(B, C)$. The connecting differential $\delta : H_i(T_\bullet, d) \rightarrow H_{i-1}(T_\bullet, d)$ is computed as usual: we take a representative $x \in T_i \subset T_i^\varepsilon$ and assume that $dx = 0$, hence $d_\varepsilon x \in \varepsilon T_i \subset T_i^\varepsilon$. Then, we let $\delta(x)$ be represented by the element $\frac{1}{\varepsilon} d_\varepsilon x$. Because of the definition of d_ε this is precisely δ_α .

In the case of $\text{Ext}_A^*(B, C)$, we assume that both B, C are deformed as right modules. Again, if $(\alpha_A, \alpha_B, \alpha'_C)$ satisfies the cocycle condition determining the first order deformation, the operation δ'_α descends to δ' on $\text{Ext}_A^*(B, C)$ defined in Section 1.2. In fact, $\text{Ext}_{A_\varepsilon}^i(B_\varepsilon, C_\varepsilon)$ may be computed as the cohomology of the complex

$$\begin{aligned} E_\varepsilon^i &= \text{Hom}_{A_\varepsilon}(B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon^{\otimes_{\mathbb{C}_\varepsilon} i} \otimes_{\mathbb{C}_\varepsilon} C_\varepsilon) = \text{Hom}_{\mathbb{C}_\varepsilon}(B_\varepsilon \otimes_{\mathbb{C}_\varepsilon} A_\varepsilon^{\otimes_{\mathbb{C}_\varepsilon} i}, C_\varepsilon) \\ &= \text{Hom}_{\mathbb{C}_\varepsilon}([B \otimes A^{\otimes i}] \oplus \varepsilon[B \otimes A^{\otimes i}], C \oplus \varepsilon C) \\ &= \text{Hom}_k(B \otimes A^{\otimes i}, C) \oplus \varepsilon \text{Hom}_k(B \otimes A^{\otimes i}, C) =: E^i \oplus \varepsilon E^i. \end{aligned}$$

The differential d_ε again splits into $d_E + \varepsilon \delta'_\alpha$. Hence, the connecting differential

$$\text{Ext}_A^{i-1}(B, C) \rightarrow \text{Ext}_A^i(B, C)$$

is induced by δ'_α .

Observe that for δ , resp. δ' , part (1) of the Lemma 3.1.1 implies that δ_α , resp. δ'_α , does descend to (co)homology. Part (2) says that the operator on cohomology does not change if $(\alpha_A, \alpha_B, \alpha_C)$ is replaced by $(\alpha_A, \alpha_B, \alpha_C) + d(\beta_A, \beta_B, \beta_C)$. Part (3) says that integrability conditions imply $\delta^2 = 0$, resp. $(\delta')^2 = 0$.

3.2. Multiplicative properties of T_\bullet and E^\bullet .

We begin with a general result. Let (D, d) be a differential associative $\mathbb{Z}/2\mathbb{Z}$ -graded algebra with an odd differential d and an odd linear map $\delta : D \rightarrow D$ such that $\delta(1) = 0$. We write $|x| \in \mathbb{Z}/2\mathbb{Z}$ for the parity of a homogeneous element $x \in D$, and introduce the notation $[d, \delta]_+ := d\delta + \delta d$. Define a bracket $[-, -] : D \times D \rightarrow D$ as follows

$$[x, y] = \delta(xy) - \delta(x)y - (-1)^{|x|}x\delta(y) \quad x, y \in D.$$

Also, for any homogeneous elements $x, y, z \in D$, put

$$\Xi(x, y, z) := \delta(xyz) - (-1)^{|x|}x\delta(yz) - \delta(xy)z - (-1)^{|y|(|x|-1)}y\delta(xz).$$

A straightforward computation yields the following result

Lemma 3.2.1. *The following identities hold:*

$$\begin{aligned}
 (1) \quad & d[x, y] - [dx, y] - (-1)^{|x|}[x, dy] = [d, \delta]_+(xy) - [d, \delta]_+(x)y - x[d, \delta]_+(y) \\
 (2) \quad & [x, yz] - [x, y]z - (-1)^{|y||x|}y[x, z] = \Xi(x, y, z) \\
 & \quad + (-1)^{|x|+|y|}xy\delta(z) + (-1)^{|x|}x\delta(y)z + \delta(x)yz. \\
 (3) \quad & [[x, y], z] + (-1)^{|x|(|y|+|z|)}[[y, z], x] + (-1)^{|z|(|x|+|y|)}[[z, x], y] \\
 & = \delta^2(xyz) - z\delta^2(xy) - x\delta^2(yz) - y\delta^2(xz) + yz\delta^2(x) + xz\delta^2(y) + zy\delta^2(z) \\
 & + \delta(\Xi(x, y, z)) - \Xi(\delta(x), y, z) - (-1)^{|x|}\Xi(x, \delta(y), z) - (-1)^{|x|+|y|}\Xi(x, y, \delta(z)).
 \end{aligned}$$

Recall next that the algebra structure on Tor-groups may be defined using the shuffle product. In more detail, according to [12], Exercise 8.6.5, Section 8.7.5 and Lemma 8.7.15 (as well as a similar statement for Ext groups), one has

Lemma 3.2.2. (i) *The algebra $\text{Tor}_\bullet^A(B, C)$ is isomorphic to the homology of the DG algebra T_\bullet , see §3.1, with the shuffle product $\bullet : T_i \otimes T_j \rightarrow T_{i+j}$ given by*

$$\begin{aligned}
 (b \otimes a_1 \otimes \dots \otimes a_i \otimes c) \bullet (b' \otimes a_{i+1} \otimes \dots \otimes a_{i+j} \otimes c') = \\
 \sum_{\sigma \in Sh(i, j)} (-1)^\sigma bb' \otimes a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(i+j)} \otimes cc'.
 \end{aligned}$$

(ii) *The $\text{Tor}_\bullet^A(B, C)$ -module $\text{Ext}_A^\bullet(B, C)$ is isomorphic to the cohomology of the T_\bullet -module E^\bullet with the action $\bullet' : T_i \otimes E^j \rightarrow E^{j-i}$ given by*

$$\begin{aligned}
 (b \otimes a_1 \otimes \dots \otimes a_i \otimes c) \bullet' \phi(b' \otimes a_{i+1} \otimes \dots \otimes a_j) = \\
 \sum_{\sigma \in Sh(i, j)} (-1)^\sigma \phi(bb' \otimes a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(i+j)})c.
 \end{aligned}$$

In order to be able to apply Lemma 3.2.1 in the situation we are interested in, we need the following result.

Lemma 3.2.3. *Let $(T = T_\bullet, d = d_T)$ be the DG algebra computing the Tor groups and let $\delta = \delta_\alpha$, be as in section 3.1. Then, the equation $\Xi(x, y, z) = 0$ holds for all x, y, z if and only if, for all $b_1, b_2 \in B$, $a \in A$, the cochain α_B satisfies*

$$(3.2.4) \quad \alpha_B(b_1 b_2, a) - b_1 \alpha_B(b_2, a) - b_2 \alpha_B(b_1, a) + b_1 b_2 \alpha_B(1, a) = 0$$

and the cochain α_C satisfies a similar identity.

Let α'_C be the transposed deformation defined by the formula $\alpha'_C(c, a) := -\alpha_C(a, c)$. Then, the above conditions also insure the following identity

$$(3.2.5) \quad \delta_\alpha(xy)m - x\delta_\alpha(y)m - y\delta_\alpha(x)m = \delta'_\alpha(xym) - x\delta'_\alpha(y)m - y\delta'_\alpha(xm) + xy\delta'_\alpha(m).$$

Proof. Let $x = b_x \otimes x_1 \dots x_{l_x} \otimes c_x$ and use the similar notation for y, z . If we plug in the formula for δ into the definition of $\Xi(x, y, z)$ we get three kinds of terms: those which involve α_B , α_A and α_C , respectively. For instance, the terms in $\delta(xyz)$ coming from α_A , will involved tensor factors of the type

$$\alpha_A(x_i, x_{i+1}), \alpha_A(y_j, y_{j+1}), \alpha_A(z_s, z_{s+1}), \\ \alpha_A(x_i, y_j), \alpha_A(y_j, x_i), \alpha_A(x_i, z_s), \alpha_A(z_s, x_i), \alpha_A(y_j, z_s), \alpha_A(z_s, y_j)$$

For $x\delta(yz)$ we need to include only those terms in which α_A is applied to y_j and z_s but not to x_i , and so on. Hence the terms in $\Xi(x, y, z)$ which depend on α_A cancel out by inclusion-exclusion formula. Looking at terms which involve α_B we get for $\delta(x, y, z)$:

$$\alpha_B(b_x b_y b_z, x_1), \alpha_B(b_x b_y b_z, y_1), \alpha_B(b_x b_y b_z, z_1)$$

For $x\delta(yz)$ we get

$$b_x \alpha_B(b_y b_z, y_1) + b_x \alpha_B(b_y b_z, z_1)$$

and similarly for other summands in $\Xi(x, y, z)$. Extracting the terms which only contain x_1 we get

$$\alpha_B(b_x b_y b_z, x_1) - b_y \alpha_B(b_x b_z, x_1) - b_z \alpha_B(b_x b_y, x_1) + b_y b_z \alpha_B(b_x, x_1) = 0$$

For $b_x = 1$ this gives (3.2.4). On the other hand, if (3.2.4) holds then

$$\begin{aligned} \alpha_B(b_x b_y b_z, x_1) &= b_y \alpha_B(b_x b_z, x_1) + b_x b_z \alpha_B(b_y, x_1) - b_x b_y b_z \alpha_B(1, x_1) \\ &= b_y \alpha_B(b_x b_z, x_1) + b_z [b_x \alpha_B(b_y, x_1) - b_x b_z \alpha_B(1, x_1)] \\ &= b_y \alpha_B(b_x b_z, x_1) + b_z [\alpha_B(b_y b_x, x_1) - b_y \alpha_B(b_x, x_1)] \end{aligned}$$

as required.

The calculation for (3.2.5) is similar: for terms involving $\alpha_B : B \otimes A \rightarrow B$ we get precisely (3.2.4). Comparing the terms involving $\alpha_C : A \otimes C \rightarrow C$ and $\alpha'_C : C \otimes A \rightarrow C$ we get the condition (3.2.4) for α'_C plus the equation

$$c_3 \alpha_C(a, c_1 c_2) - c_3 c_2 \alpha_C(a, c_1) = -\alpha'_C(c_1 c_2 c_3, a) + c_2 \alpha'_C(c_1 c_3, a)$$

Since α'_C is transposed to α_C this equation can also be reduced to (3.2.4) for α'_C . \square

4. Proofs of main results

4.1. The transposed deformation. Fix a flat \mathbb{C}_ε -algebra deformation \mathcal{A} , of \mathcal{O}_X . Associated with any deformation \mathcal{C} , of the sheaf \mathcal{O}_Z to a left \mathcal{A} -module, there is a transposed deformation \mathcal{C}^t , which gives a sheaf of right \mathcal{A} -modules.

To explain the definition of \mathcal{C}^t , recall first that any deformation \mathcal{C} admits local \mathbb{C} -linear splittings (in the Zariski topology) $\mathcal{C} \simeq \mathcal{O}_Z \oplus \varepsilon \mathcal{O}_Z$. So, locally, the deformed module structure can be written in the form $(a \oplus 0) * (c \oplus 0) = ac \oplus \varepsilon \alpha(a, c)$. Furthermore, by Proposition 2.3.1 the cochain $\alpha(a, c)$ can be chosen to be an algebraic differential operator in each of its arguments (which satisfies an associativity condition recalled in Section 2). Thus, X has a covering by affine open subsets U_i and on each of them there is a splitting as above. It follows that, on each double intersection $U_i \cap U_j$, the corresponding splittings differ by an automorphism

$$c_1 \oplus \varepsilon c_2 \mapsto c_1 \oplus \varepsilon (\psi_{ij}(c_1) + c_2),$$

where $\psi_{ij} : \mathcal{O}_Y \rightarrow \mathcal{O}_Y$ is an algebraic differential operator. The gluing condition for the locally defined cochains $\alpha_i(a, c)$ and $\alpha_j(a, c)$ reads:

$$(4.1.1) \quad \alpha_i(a, c) - \alpha_j(a, c) = a\psi_{ij}(c) - \psi_{ij}(ac).$$

Conversely, given a covering of X , a collection of α_i 's describing a deformation of $\mathcal{O}_Y|_{U_i}$ to a left module over $\mathcal{A}|_{U_i}$, and a set of operators ψ_{ij} such that (4.1.1) holds and, moreover, such that for each triple (i, j, k) , one has $\psi_{ij} + \psi_{jk} = \psi_{ik}$, gives rise to a deformation of \mathcal{O}_Z to a left \mathcal{A} -module \mathcal{C} if the above gluing conditions are satisfied. A similar statement holds for right \mathcal{A} -module deformations as well.

Now, given a left \mathcal{A} -module deformation \mathcal{C} , we define \mathcal{C}^t , the transposed right \mathcal{A} -module deformation, by gluing local deformations given, on each U_i , by the formula

$$(c \oplus 0) * (a \oplus 0) = ac \oplus -\varepsilon\alpha(a, c), \quad \text{i.e. we put } \alpha_i^t(c, a) := -\alpha_i(a, c).$$

Here, the minus signs appear since the opposite algebra \mathcal{A}^{op} may be viewed as a deformation coming from the bivector $-P$. The above defined local deformations are related, on double intersections, via the operators $\psi_{ij}^t := -\psi_{ij}$.

4.2. Proof of Theorem 1.1.2(i): BV differential. Given $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, a triple of deformations as in the theorem, we use the construction of the map $\delta : \mathcal{T}or_{\bullet}^{\mathcal{O}^X}(\mathcal{O}_Y, \mathcal{O}_Z) \rightarrow \mathcal{T}or_{\bullet-1}^{\mathcal{O}^X}(\mathcal{O}_Y, \mathcal{O}_Z)$ from section 1.2. The construction being local, the map δ restricts to a similar map on any open affine subset of X . Clearly, in order to verify the required properties of δ , it is sufficient to verify them for an open affine covering of X .

This way, we reduce the proof to the case where $X = \text{Spec } A$, resp. $Y = \text{Spec } B$ and $Z = \text{Spec } C$. For affine varieties all the deformations involved are automatically *split* over \mathbb{C} . It follows that these deformations may be written in terms of certain cocycles α_A, α_B , and α_C , respectively, as in Section 2.1. Now, we are in the setting of Sections 3.2-3.3. In particular, we may use the DG algebra (T_{\bullet}, d_T) and Lemma 3.2.2 for computing the algebra $Tor_{\bullet}^A(B, C)$ and we may interpret the map δ in terms of the Bar construction.

Observe next that, thanks to Lemma 3.1.1(2), replacing deformations by equivalent deformations doesn't affect the conclusion of the theorem. Therefore, we may adjust our cocycles using Proposition 2.3.1 to insure that: (1) the cocycle α_A be equal to the Poisson bracket, in particular, it can be extended to a second order deformation (α_A, γ_A) ; and (2) the cocycle α_B , resp. α_C , can also be extended to a second order deformation and, moreover, it is given by a bidifferential operator, as in Proposition 2.3.1(ii). Here, the existence of extensions to second order deformations means that there exist γ_B and γ_C such that for $\alpha := \alpha_A \oplus \alpha_B \oplus \alpha_C$, in the deformation complex $C_{A,B,C}^{\bullet}$, see (2.2.3), we have $[\alpha, \alpha] = d_{A,B,C}(\gamma_A \oplus \gamma_B \oplus \gamma_C)$; in particular, the second order deformations of B and C given by (α_B, γ_B) and (α_C, γ_C) , respectively, correspond to the *same* second order deformation of A given by a pair (α_A, γ_A) , as guaranteed by Proposition 2.3.1 (iii).

Recall that the existence of a second order extension of the deformation given by (α_A, α_B) is equivalent to equations (2.1.5)-(2.1.6). Hence, we deduce $[d_T, \delta]_+ = 0$, by Lemma 3.1.1(1). Also, equation $\delta^2 = 0$ follows from Lemma 3.1.1(3). Further, the constraints from Proposition 2.3.1(ii) on the order of bidifferential operators insure that (3.2.4) holds, hence $\Xi(x, y, z) = 0$ by Lemma 3.2.3. Thus, combining Lemma

3.2.3 with Lemma 3.2.1, we deduce that the Poisson and Jacobi identities hold already in the algebra T_\bullet . We conclude that these identities hold in the Tor algebra as well, and the theorem follows. \square

The proof of the properties of the differential δ' on the Ext-sheaves is completely similar and is left for the reader.

4.3. Proof of Theorem 1.1.2(ii)-(iii): Gerstenhaber structures. Let Y, Z be smooth coisotropic submanifolds of a smooth Poisson variety X . We must prove the following

Theorem 4.3.1. *The sheaf of graded algebras $\mathcal{T}or_\bullet^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ admits a canonical structure of a Gerstenhaber algebra, i.e. a graded symmetric bracket of degree (-1) which satisfies Poisson and Jacobi identities.*

Similarly, the sheaf $\mathcal{E}xt_{\mathcal{O}_X}^\bullet(\mathcal{O}_Y, \mathcal{O}_Z)$ has a canonical structure of a Gerstenhaber module, i.e. a bracket $[\cdot, \cdot]': \mathcal{T}or_i \otimes \mathcal{E}xt^j \rightarrow \mathcal{E}xt^{j-i+1}$ such that

$$\begin{aligned} [xy, m]' &= x[y, m]' + (-1)^{\deg y \deg x} y[x, m]'; \\ [x, ym]' &= [x, y]m + (-1)^{\deg y \deg x} y[x, m]'; \\ [[x, y], m]' &= [x, [y, m]']' + (-1)^{\deg y \deg x} [y, [x, m]']' \end{aligned}$$

Proof. Let \mathcal{A} be the standard (split) algebra deformation of the structure sheaf \mathcal{O}_X given by the formula $f * g = fg + \frac{\varepsilon}{2}\{f, g\}$.

On X , we choose an affine open covering $\{U_i\}$ such that each open subset U_i has trivial canonical class. By Proposition 2.3.1(iii), on each U_i , we can find deformations $\mathcal{B}_i, \mathcal{C}_i$ which extend to second order deformations. Writing A_i , resp. B_i, C_i , for the corresponding algebras of global sections and applying Theorem 1.1.2, we get a BV algebra structure on $\text{Tor}_\bullet^{A_i}(B_i, C_i)$. By Lemma 3.1.1(2), the corresponding BV differential δ_i on the Tor algebra is unaffected by a change of cocycles $\alpha_A, \alpha_B, \alpha_C$ provided neither the cohomology class of $\alpha_A \oplus \alpha_B$ nor the cohomology class of $\alpha_A \oplus \alpha_C$ is changed. In particular, we may insure that all the cocycles involved satisfy the conclusions of Proposition 2.3.1.

Next, fix a double intersection $U_i \cap U_j$. We must show that the restrictions to $U_i \cap U_j$ of the two BV differentials δ_i and δ_j , arising from the triples $(\alpha_A)_i, (\alpha_B)_i, (\alpha_C)_i$ and $(\alpha_A)_j, (\alpha_B)_j, (\alpha_C)_j$ respectively, induce the same Gerstenhaber bracket on the sheaf $\mathcal{T}or_\bullet^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)|_{U_i \cap U_j}$.

To this end, we observe that the deformation \mathcal{A} being globally defined and *globally split*, on $U_i \cap U_j$, we have $(\alpha_A)_i = (\alpha_A)_j$. Next, consider the complex $C_{A,B}^\bullet$, cf. (2.2.1), associated with the algebras $A_{ij} = \mathbb{C}[U_i \cap U_j]$ and $B_{ij} = \mathbb{C}[Y \cap U_i \cap U_j]$. The injectivity claim at the beginning of Proposition 2.3.1 implies that the cocycles $(\alpha_A)_i \oplus (\alpha_C)_i$ and $(\alpha_A)_j \oplus (\alpha_C)_j$ give equal cohomology classes in $H^2(C_{A,B}^\bullet)$. We conclude that there exist cochains $((\beta_A)_{ij}, (\beta_B)_{ij})$ such that, in $C_{A,B}^2$, we have

$$(4.3.2) \quad (0 \oplus (\alpha_C)_i - (\alpha_C)_j) = d_{A,B}((\beta_A)_{ij} \oplus (\beta_B)_{ij}).$$

We see that adjusting the restriction of the triple $(\alpha_A)_i, (\alpha_B)_i, (\alpha_C)_i$ to $U_i \cap U_j$ by a coboundary allows to achieve that $(\alpha_C)_i = (\alpha_C)_j$.

Remark 4.3.3. At this point, the reader should be alerted that, although an equation similar to (4.3.2) holds for the difference $0 \oplus (\alpha_B)_i - (\alpha_B)_j$ as well, that equation may require a completely different choice of the cocycle $(\beta_A)_{ij}$. Thus, on $U_i \cap U_j$, the cocycles in the complex $C_{A,B,C}^\bullet$, cf. (2.2.3), corresponding to $(\alpha_A)_i \oplus (\alpha_B)_i \oplus (\alpha_C)_i$ and $(\alpha_A)_j \oplus (\alpha_B)_j \oplus (\alpha_C)_j$ respectively, do not necessarily give equal cohomology classes, in general.

The arguments above show that we may assume, adjusting by coboundaries if necessary, that we have $(\alpha_A)_i = (\alpha_A)_j$ and $(\alpha_C)_i = (\alpha_C)_j$ (but not also $(\alpha_B)_i = (\alpha_B)_j$ at the same time). Further, let $C_{A,B}^\bullet$ be the deformation complex for the pair (A, B) . Taking the difference of the cocycles $(\alpha_A)_i \oplus (\alpha_B)_i$ and $(\alpha_A)_j \oplus (\alpha_B)_j$ yields a cocycle in $C_{A,B}^\bullet$ of the form $0 \oplus (\alpha_B)_i - (\alpha_B)_j$. Since the first component here is 0, the cocycle condition says, cf. (2.2.1), that $(\alpha_B)_i - (\alpha_B)_j : B \otimes A \rightarrow B$ is a Hochschild 1-cocycle. But any Hochschild 1-cocycle has the form $(b \otimes a) \mapsto b \cdot \xi(a)$ for a certain derivation $\xi : A_{ij} \rightarrow A_{ij}$. Thus, the difference of two BV differentials $\delta_i - \delta_j$ on $U_i \cap U_j$ is induced by the operator

$$\tilde{\delta}(b \otimes a_1 \otimes \dots \otimes a_i \otimes c) = b \cdot \xi(a_1) \otimes a_2 \otimes \dots \otimes a_i \otimes c$$

It is straightforward to verify that, since ξ is a first order differential operator, any operator $\tilde{\delta}$ as above induces the zero bracket on Tor . Hence the brackets induced by δ_i and δ_j agree on $U_i \cap U_j$. This means that the Gerstenhaber bracket is independent on the local choice of α_B and α_C , which finishes the proof of the first part of the theorem.

We turn to the second part of the theorem concerning Gerstenhaber modules. Observe that the module bracket can be defined via

$$[x, m]' = \delta'(xm) - \delta(x)m - (-1)^{\deg x} x \delta'(m).$$

Since arbitrary \mathbb{C} -linear maps $B \otimes A \rightarrow B$, etc., do not localize in general, we need to work with the subcomplex of E^\bullet given by polydifferential operators (by the last remark of Section 2 this subcomplex has the same cohomology). Observe that our deformations of Y and Z are indeed given locally by bidifferential operators.

Now, both versions of Poisson identity are equivalent to (3.2.5). The Jacobi identity on the module follows from $(\delta')^2 = 0$, once the Poisson identity is established, cf. Lemma 3.2.1(3). The remaining part of the proof is the same as for Tor . \square

5. Examples.

The proof of the previous section shows that sometimes the BV structure on Tor or Ext is well-defined globally. This is the case, for instance, whenever in the setting of the proof of Theorem 4.3.1, the cocycles α_C and α_B , are defined globally.

Here is one such example.

5.1. Koszul bracket. Take $X = Y \times Y$ and view $Y \subset X$ as the diagonal. Then, one easily finds

$$\mathcal{T}or_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \Omega_Y^\bullet; \quad \mathcal{E}xt_{\mathcal{O}_X}^\bullet(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \Lambda^\bullet \Omega_Y^\vee.$$

Proposition 5.1.1. *For any Poisson bivector field $P \in H^0(Y, \Lambda^2 T_Y)$, the induced BV differential δ on $\Omega_{B/k}^\bullet$ is given by the formula $\delta = i_P d_{DR} + d_{DR} i_P$, where the de Rham differential d_{DR} has degree +1, and where i_P is a degree (-2) contraction operation (with the bivector P). The BV differential δ' on $\Lambda^\bullet \Omega_{B/k}^\vee$ is given by the Schouten bracket $[P, -]$.*

Proof. To simplify notation we will work in the affine case although all formulas make sense globally. Thus we consider $A = B \otimes B$ with the quotient map $m : B \otimes B \rightarrow B$ given by multiplication. Also, take $C = B$. By standard results, e.g. [12], we have a Hochschild cocycle on A :

$$\alpha_A(x \otimes x', y \otimes y') = \frac{1}{2}P(dx \wedge dy) \otimes x'y' - xy \otimes \frac{1}{2}P(dx' \wedge dy')$$

and the diagonal $Y_\Delta \subset X = Y \times Y$ is coisotropic with respect to the corresponding Poisson structure. We also have a right deformation of B induced by the cocycle

$$\alpha_B(b, x \otimes y) = -\frac{1}{2}P(d(bx) \wedge dy) + \frac{1}{2}P(db \wedge dx)y : B \otimes A \rightarrow B$$

For the second argument of $\text{Tor}_\bullet^A(\cdot, \cdot)$ we use the transposed map $\alpha_B^t : A \otimes B \rightarrow B$. Our goal is to compute the induced BV differential on $\Omega_{B/k}^\bullet$. To that end, we need explicit quasi-isomorphisms between $\Omega_{B/k}^\bullet$ and T_\bullet .

Observe that usually $\Omega_{B/k}^\bullet$ is identified with the cohomology of $C_\bullet(A, A) = A^{\otimes(\bullet+1)}$ and the standard Hochschild differential. Our complex T_\bullet is slightly different although quasi-isomorphic to $C_\bullet(A, A)$ by the map:

$$b \otimes a_1 \otimes \dots \otimes a_n \otimes b' \mapsto (b'b) \otimes m(a_1) \otimes \dots \otimes m(a_n)$$

Combining with the Hochschild-Kostant-Rosenberg isomorphism we have a pair of mutually inverse quasi-isomorphisms:

$$b \otimes a_1 \otimes \dots \otimes a_n \otimes b' \mapsto \frac{1}{n!}b'b \cdot dm(a_1) \wedge \dots \wedge dm(a_n) : T_\bullet \rightarrow \Omega_{B/k}^\bullet$$

respectively the map

$$b_0 db_1 \mapsto b_0 \otimes (b_1 \otimes 1) \otimes 1 : \Omega_{B/k}^1 \rightarrow T_1$$

extended multiplicatively. For example, $b_0 db_1 \wedge db_2 \wedge db_3$ maps to the antisymmetrization of

$$b_0 \otimes (b_1 \otimes 1) \otimes (b_2 \otimes 1) \otimes (b_3 \otimes 1) \otimes 1 \in T_3$$

in the three middle terms. The assertion follows from the above definitions of α_A and α_B and a straightforward computation.

The case of δ' is entirely similar. □

We observe here that the differential δ of the above proposition was first constructed from a Poisson bivector P by Koszul in [9]. Also, the differential δ (and not just the induced Gerstenhaber bracket) is canonically defined since the two arguments of $\text{Tor}_\bullet^A(B, B)$ are taken with their conjugate deformations. We also remark that the (co)homology of the differentials δ' and δ in this case are called Poisson cohomology and homology, respectively.

5.2. Self-intersection of a coisotropic submanifold. Let X be an arbitrary Poisson variety, and let $Y = Z$, a coisotropic subvariety. In this case, we have $\mathcal{E}xt_{\mathcal{O}_X}^{\bullet}(\mathcal{O}_Y, \mathcal{O}_Y) = \Lambda^{\bullet}(N_{X/Y})$.

The proof of Theorem 4.3.1 shows that the differential δ' is well-defined globally: although $\alpha_B(b, a)$ exist in general only locally, on double intersections of an affine cover the difference between two choices of α_B is a coboundary (since $B = C$, where B is the affine counterpart of \mathcal{O}_Y and C is the affine counterpart of \mathcal{O}_Z , as usual). Hence these two choices give the same δ' on cohomology.

The subsheaf $\mathcal{K} \subset \Lambda^{\bullet}(T_X)$ formed by vector fields which project to zero in $\Lambda^{\bullet}(N_{X/Y})$, is a Lie subalgebra with respect to the Schouten bracket. Then \mathcal{K} acts on $\Lambda^{\bullet}(N_{X/Y})$ since any Lie subalgebra acts on a quotient by itself. Note that P is a section of \mathcal{K} .

Proposition 5.2.1. *The BV differential on $\Lambda^{\bullet}(N_{X/Y})$ is given by the action of the Poisson bivector.*

Proof. It suffices to prove the statement on an affine open subset. Observe that $\text{Ext}_A^{\bullet}(B, B)$ is computed by the subcomplex

$$C^{\bullet-1}(A, \text{End}_{\mathbb{C}} B) \subset C_{A,B}^{\bullet}.$$

The differential δ' of $\gamma_B : A^{\otimes(n-1)} \rightarrow \text{End}_{\mathbb{C}} B$ is explicitly given by

$$\delta'(\gamma_B) = [\alpha_A \oplus \alpha_B, 0 \oplus \gamma_B] : A^{\otimes n} \rightarrow \text{End}_{\mathbb{C}} B$$

where $[\cdot, \cdot]$ is the Lie bracket on $C_{A,B}^n$ introduced in Section 2.2. Now a straightforward calculation finishes the proof. \square

5.3. Symplectic 2-form need not be locally exact. The following construction of a large class of examples of algebraic closed differential forms which are not locally exact in étale topology was explained to us by A. Beilinson.

Let X be a smooth algebraic variety, and ω a non-zero holomorphic i -form on X . We assume, in addition, that for some compactification of X the form ω has logarithmic singularities at infinity (it is then automatically closed thanks to Deligne results [3]). In any case, one can merely assume that X is itself a projective variety.

Claim 5.3.1. *For any étale morphism $\pi : U \rightarrow X$, the i -form $\omega_U = \pi^*\omega$, on U , is not exact.*

Proof. Let $X \hookrightarrow Z$ be a compactification such that $Z \setminus X$ is a divisor with normal crossings. By Hironaka's resolution of singularities, one can find a similar compactification $U \hookrightarrow T$ such that the map $\pi : U \rightarrow X$ extends to a morphism $T \rightarrow Z$. It is known that in such a case, the i -form ω_U has logarithmic singularities at $T \setminus U$. Furthermore, this form is clearly non-zero since its restriction to U is non-zero. The map sending a differential form β on T with logarithmic singularities at $T \setminus U$ to $[\beta] \in H^i(U)$, the corresponding de Rham cohomology class, is known to be *injective*, by [De]. So, the class $[\omega_U] \in H^i(U)$ is non-zero. We conclude that ω_U can not be an exact form, as claimed. \square

The above result produces examples of smooth symplectic varieties (e.g. X an abelian surface) such that the corresponding symplectic 2-form cannot be made exact by passing to any étale open subset.

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