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ON DIRECT SUMMANDS OF MODULES OF FINITE PHANTOM PROJECTIVE DIMENSION

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Abstract. The main result of this paper is the construction of an example of a local ring R of finite characteristic and cyclic R-modules M and N such that $M \oplus N$ has a finite phantom projective dimension, however neither M nor N has finite phantom projective resolution.

1. Introduction

Throughout this paper by a ring we mean commutative ring with an identity element. By a local ring $(R, \mathfrak{m})$ we mean a Noetherian ring R with a unique maximal ideal $\mathfrak{m}$.

Hochster and Huneke in [7] have introduced a tight closure operation on ideals and submodules of modules. This operation has turned out to be extremely useful when dealing with rings containing a field and has led to a large number of strong results (see, for instance, [9] for an overview of the field and references therein).

In its simplest form, tight closure can be defined as follows. Let R be a Noetherian ring of positive prime characteristic p and let $N \subseteq M$ be R-modules. For every integer $e \geq 0$ the eth power of Frobenius endomorphism maps R to itself by $x \mapsto x^q$ where $q = p^e$. Let S_e denote R considered as an R-algebra via the eth power of Frobenius. The Peskine-Szpiro functor, F^e is defined as $S_e \otimes_R -$, a covariant functor from R-modules to S_e-modules. Since $S_e = R$ this is actually a functor from R-modules to R-modules. We say that $x \in M$ is in the *tight closure* of N (denoted by N^*_M), if there exists $c \in R$ that does not belong to any minimal prime of R and such that in $F^e(M)$ we have $cx^q \in \operatorname{im}(F^e(N) \to F^e(M))$ for all $q \gg 0$. Here x^q stands for the image of x under the natural map $M \to F^e(M)$ that takes $m \mapsto 1 \otimes m$.

One of the beautiful notions emerging from tight closure theory is that of phantom homology, also introduced by Hochster and Huneke in [7]. Roughly speaking, the idea is to consider projective resolutions that are "almost acyclic": instead of requiring that the boundaries are the same as cycles, as is the case for the usual acyclic resolution, it is assumed that they are the same "up to tight closure".

Specifically, let

$$P_\bullet : \ldots \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to 0$$

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be a complex of finitely generated projective R-modules, where R is a Noetherian ring of positive prime characteristic p. We say that $P_\bullet$ has *phantom homology* at the ith spot if the cycles at that spot lie inside the tight closure of the boundaries within the ambient module P_i: that is, $Z_i \subseteq (B_i)^*_{P_i}$. If $P_\bullet$ has a phantom homology at the ith spot for all $i \geq 1$ then we say that $P_\bullet$ is *phantom acyclic*. If not only $P_\bullet$, but also all of its Frobenius iterates $F^e(P_\bullet)$ are phantom acyclic for all $e \geq 0$, then $P_\bullet$ is called *stably phantom acyclic*. In this last case $P_\bullet$ is called a *phantom resolution* of the augmentation module $H^0(P_\bullet) = M$ (assuming that the augmentation module M is nonzero). The length of the shortest finite stably phantom projective resolution of M, if one exists, is called the *phantom projective dimension* of M over R and is denoted by $\mathrm{ppd}_R(M)$. For the zero module $M = 0$ we define $\mathrm{ppd}_R(M) = -1$.

A module M that has at least one phantom projective resolution will necessarily have a finite one as well (see Theorem 2.1.7 in [1]). Thus $\mathrm{ppd}_R(M)$ is a well-defined natural number when M has any phantom projective resolution; otherwise, we define $\mathrm{ppd}_R(M) = \infty$.

Phantom homology turned out to be a very fruitful idea and was developed in a number of papers (see, for instance, [1], [2], [3], [8], [9]). The class of modules of finite phantom projective dimension over a fixed ring R includes those of finite projective dimension but is usually larger (the two classes coincide for Cohen-Macaulay rings). Phantom resolutions are not at all exotic and arise very naturally. An important source of phantom resolutions is base change: if M is a module of finite projective dimension, then any of its finite projective resolutions tensored with a module-finite extension S of R produces a phantom resolution of $S \otimes_R M$ (under mild conditions on the ring R; e.g., it is enough to assume that R is excellent, equidimensional and reduced; see, for instance, discussion of Phantom Acyclicity Criterion in [7] and [8]).

There are lots of results on modules of finite phantom projective dimension that parallel the results for the usual notion of projective dimension. Just to name a few, the Buchsbaum-Eisenbud Acyclicity Criterion for finite complexes of free modules (see [5]) has a phantom analogue with depth conditions replaced by the height conditions (see, for instance, Theorem 9.8 in [7]), Auslander-Buchsbaum's formula that expresses the projective dimension via depths has a very similar version for phantom projective dimension and phantom depth (see Theorem 3.2.7 in [1]), an extension of the regularity theorem by Auslander-Buchsbaum-Serre for a local ring holds with finite phantom projective dimension (see Theorem 2.4.1 in [1]), etc. These phantom homology ideas were used to give extremely powerful results on vanishing of maps of Tors (see Section 4 in [8]). Another reason to be interested in phantom projective resolutions stems from a result in [2], which gives an instance when tight closure commutes with localization: that is, under mild conditions on the ring R, for a pair of finitely generated modules $N \subseteq M$ for which $\mathrm{ppd}_R(M/N) < \infty$ and for any multiplicative system W in R we have $W^{-1}(N^*_M) = (W^{-1}N)^*_{W^{-1}M}$ (see section 5 in [2]) Tight closure does not commute with localization in general, as shown recently by Brenner and Monsky (see [4]).

Nevertheless, the notions of phantom projective dimension and phantom depth do not enjoy all the good properties of the usual ones. One of the most important reasons for this is that when one is given a module of finite phantom projective dimension, there is no canonical "constructive" way to build up a phantom resolution. For a module of finite phantom projective dimension different resolutions can even fail to be chain-isomorphic. Also, the behavior of such modules in short exact sequences is more complicated (e.g., it can happen that for R-modules $N \subseteq M$ we have $\mathrm{ppd}_R(M) < \infty$, $\mathrm{ppd}_R(M/N) < \infty$ but N has no finite phantom resolution).

The main result of this paper is a demonstration of further instances of "bad" behavior for modules of finite phantom projective dimension that are not parallel to what happens with modules of finite projective dimension. In [1], the following two natural conjectures were posed:

Conjecture 1. *Let R be a Noetherian ring of prime characteristic p and let M be a finitely generated R-module. Then $\mathrm{ppd}_R M < \infty$ if and only if $\mathrm{ppd}_{R_\mathfrak{m}} M_\mathfrak{m} < \infty$ for all maximal ideals $\mathfrak{m}$ of R.*

Conjecture 2. *Let R be a Noetherian ring of characteristic p and let M and N be finitely generated R-modules. Then $\mathrm{ppd}_R(M \oplus N) < \infty$ if and only if $\mathrm{ppd}_R(M) < \infty$ and $\mathrm{ppd}_R(N) < \infty$.*

Conjecture 1 was settled negatively in [3]. Techniques developed in [3] that were used to produce a counterexample for Conjecture 1 actually enable one to construct a *non-local* counterexample for Conjecture 2 as well. Specifically, in [3] an example of a ring R and an R-module M is given such that $\mathrm{ppd}_R M \oplus M < \infty$ but $\mathrm{ppd}_R M = \infty$. However, that example is "very far from being local", so to speak: none of its localizations provides a counterexample to the Conjecture 2. In fact, in [3] it is shown that $\mathrm{ppd}_{R_\mathfrak{m}} M_\mathfrak{m} < \infty$ for all maximal ideals of R. The goal of this paper is to give a *local* counterexample for Conjecture 2; this requires new ideas and techniques quite different from those in [3].

2. Notation, Main Result and Sketch of Proof

The direct sum of phantom resolutions is still a phantom resolution, so the "if" part of the Conjecture 2 is obviously true. We are going to construct a counterexample for the "only if" part: namely, we want to construct an example of a local domain R, essentially of finite type over a field k of finite characteristic, and two finitely generated R-modules M and N for which $\mathrm{ppd}_R(M \oplus N) < \infty$ but $\mathrm{ppd}_R(M) = \infty$ and $\mathrm{ppd}_R(N) = \infty$.

The notation we introduce in this section will be fixed throughout the rest of this paper.

Let k be an arbitrary field of finite characteristic $p \neq 3$, and let

$$x, y, z, x', y', z', u, v, w, a_{ij}, \quad 1 \leq i, j \leq 6$$

be indeterminates over k. Let $(b_{ij})_{6\times 6} = (a_{ij})^{-1}_{6\times 6}$. We will often omit the conditions $1 \le i, j \le 6$ and just write a_{ij} to indicate the collection of all 36 indeterminates.

For an arbitrary matrix A of size $m \times n$ and any $1 \le l \le \min\{m, n\}$ we will denote by $I_l(A)$ the set of $l \times l$ minors of A. We will often refer to the following matrices:

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, X' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, T = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}, T' = \begin{pmatrix} 0 & -z' & y' \\ z' & 0 & -x' \\ -y' & x' & 0 \end{pmatrix},$$

as well as the block matrices made of the ones above:

$$\Phi = \begin{bmatrix} T & 0_{3\times 3} \\ 0_{3\times 3} & T' \end{bmatrix} (a_{ij})_{6\times 6},$$

$$\Psi = (b_{ij})_{6\times 6} \begin{bmatrix} X & 0_{3\times 1} \\ 0_{3\times 1} & X' \end{bmatrix}$$

Consider the localized polynomial ring

$$S = k[u, v, w, x, y, z, x', y', z', a_{ij}]_{(u,v,w,x,y,z,x',y',z')}$$

and the sequence of its subrings:

$$P_n = k \begin{bmatrix} u, v, w, & ux, uy, uz, & x^{p^n}, y^{p^n}, z^{p^n}, \\ & vx', vy', vz', & x'^{p^n}, y'^{p^n}, z'^{p^n} \\ I_1(\Phi), & I_1(\Psi), & \\ wa_{ij}, & a_{ij}^{p^n}, & \frac{1}{(\det(a_{ij}))^{p^n}} \end{bmatrix}$$

where $i, j = 1, \ldots, 6$ and n is an arbitrary positive integer.

For every n we can localize P_n at $\mathfrak{p}_n = (u, v, w, x, y, z, x', y', z')S \cap P_n$. We get a sequence of *local* subrings of S

$$R_n = (P_n)_{\mathfrak{p}_n}$$

Define the cyclic R_n-modules M and N to be

$$M_n = R_n/(ux, uy, uz)$$

and

$$N_n = R_n/(vx', vy', vz')$$

We aim to prove the following result:

Main Theorem. *If the rings R_n and R_n-modules M_n and N_n are defined as above, we have $ppd_{R_n}(M_n \oplus N_n) < \infty$ for every n, but there is a positive integer N such that for all $n > N$ we have $ppd_{R_n}(M_n) = \infty$ and $ppd_{R_n} N_n = \infty$.*

The fact that $M_n \oplus N_n$ has a finite phantom projective dimension over R_n is relatively easy to verify. We have the following resolution of $M_n \oplus N_n$:

$$0 \to R_n^2 \xrightarrow{\Psi} R_n^6 \xrightarrow{\Phi} R_n^6 \xrightarrow{A_1} R_n^2 \to 0 \tag{1}$$

where

$$A_1 = \begin{pmatrix} ux & uy & uz & 0 & 0 & 0 \\ 0 & 0 & 0 & vx' & vy' & vz' \end{pmatrix}.$$

All these matrices clearly have entries in the ring R_n. In order to verify that the resolution is phantom, one can apply the Phantom Acyclicity Criterion (see e.g., (3.21) of [8]). Observe that, over the module-finite extension of $R_n[a_{ij}]$ of R_n, the resolution (1) is isomorphic to the standard Koszul resolution (which is known to be phantom). The rank condition is of course preserved under base change. The fact that the height condition is also preserved follows from the following Lemma.

Lemma 3. *Let R be any universally catenary Noetherian integral domain and let S be an integral domain extension of R that is integral over R. Then for any ideal I of R we have $ht_R(I) = ht_S(IS)$.*

Proof. This immediately follows from Proposition 4.8.6 in [10] and the fact that all rings under consideration are universally catenary. □

Thus $\mathrm{ppd}_{R_n}(M_n \oplus N_n) = 3$ for all n.

Showing that M_n and N_n do not have finite phantom projective dimension is more difficult and takes the rest of this paper. The idea is that we have included in the ring R_n only those elements that are absolutely necessary to turn (1) into a phantom resolution; however, there are not enough elements in the ring R_n to enable one to decompose the resolution of $M_n \oplus N_n$ into a direct sum of phantom resolutions for M_n and N_n.

By symmetry, it is enough to show the result only for M_n. Briefly, the proof goes as follows: assume that there is a phantom resolution of M_n over R_n. First, we reduce to the case when the resolution has the form

$$P_\bullet : \; 0 \to R_n \xrightarrow{F} R_n^3 \xrightarrow{G} R_n^3 \xrightarrow{(ux,uy,uz)} R_n \to 0$$

Then we show that, after tensoring $P_\bullet$ with S over R_n, we will get an *honest* resolution of $M_n \otimes_{R_n} S = S/(ux, uy, uz)S$ over S. The latter has to be chain isomorphic to the standard "Koszul-type" resolution:

$$0 \to S \xrightarrow{X} S^3 \xrightarrow{T} S^3 \xrightarrow{(ux,uy,uz)} S \to 0$$

i.e. there should exist an invertible matrix Θ with elements in S and invertible element λ of S such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S & \xrightarrow{F\otimes id_S} & S^3 & \xrightarrow{G\otimes id_S} & S^3 & \xrightarrow{(ux,uy,uz)} & S & \longrightarrow & 0 \\
 & & \lambda\downarrow & & \Theta\downarrow & & id_{S^3}\downarrow & & id_S\downarrow & & \\
0 & \longrightarrow & S & \xrightarrow{X} & S^3 & \xrightarrow{T} & S^3 & \xrightarrow{(ux,uy,uz)} & S & \longrightarrow & 0
\end{array}$$

Tensoring the resolution $P_\bullet$ over R_n with its overring S does not actually change the matrices F and G. Thus the commutativity of the diagram above implies that the matrices $F = (\lambda\Theta^{-1}) \cdot X$ and $G = T \cdot \Theta$ should have all their entries in R_n. Since R_n contains relatively few elements of S, the latter requirements on matrices F and G impose quite rigid conditions on the rows of the matrix $\lambda\Theta^{-1}$ and columns of the matrix Θ. However, deriving these conditions takes a great deal of work. The final part of the proof is showing that these conditions on rows of $\lambda\Theta^{-1}$ and columns Θ

cannot be simultaneously met to give the matrix Θ needed for the resolution $P_\bullet$.

3. Reducing the phantom resolution to the form $0 \to R_n \to R_n^3 \to R_n^3 \xrightarrow{(ux,uy,uz)} R_n \to 0$

First note that the localized polynomial ring

$$S = k[u, v, w, x, y, z, x', y', z', a_{ij}]_{(u,v,w,x,y,z,x',y',z')}$$

is the normalization of each of its subrings R_n. In order to see this, first note that the quotient fields of R_n and S are both $k(u, v, w, x, y, z, x', y', z', a_{ij})$. Since p^n-th power of every element of S is in R_n, S is integral over R_n. Since S is also integrally closed (being localization of a polynomial ring over a field), it is then the integral closure of R_n within its field of fractions. As such, it is module-finite over R_n (see e.g., Theorem 4.14 in [6]).

Now we start with the proof of the fact that $\mathrm{ppd}_{R_n}(M_n) = \infty$ for all n large enough. Fix any positive integer n and assume that M_n has finite phantom projective dimension over R_n. The ring R_n is local so we can take a minimal free phantom resolution of M_n over R_n:

$$0 \to R_n^{b_m} \xrightarrow{d_m} \dots \xrightarrow{d_2} R_n^{b_1} \xrightarrow{d_1} R_n^{b_0} \to 0 \tag{2}$$

The Phantom Acyclicity Criterion shows that the rank and height conditions are satisfied for (2): $\mathrm{rk}(d_{i+1}) + \mathrm{rk}(d_i) = b_i$ and $\mathrm{ht}(I_{rk(d_i)}(d_i)) \geq i$ for $i = 1, \dots, m$.

Next, we tensor the resolution (2) with over-ring S of R_n; we get the resolution

$$0 \to S^{b_m} \xrightarrow{d_m \otimes id_S} \dots \xrightarrow{d_2 \otimes id_S} S^{b_1} \xrightarrow{d_1 \otimes id_S} S^{b_0} \to 0 \tag{3}$$

Note that the matrices in resolutions (2) and (3) are the same, so that the rank conditions are still satisfied. Height conditions $ht(I_{rk(d_i)}(d_i)) \geq i$ are also satisfied in (3) by Lemma 3. Therefore, again by the Phantom Acyclicity Criterion, (3) is a phantom resolution of $M_n \otimes S = S/(ux, uy, uz)S$ over S. However, the localized polynomial ring $S = k[u, v, w, x, y, z, x', y', z', a_{ij}]_{(u,v,w,x,y,z,x',y',z')}$ is regular, so the phantom acyclicity coincides with the usual acyclicity in this case. Thus (3) is an honest (clearly, still minimal) resolution of $S/(ux, uy, uz)$ over S.

We also have the usual "Koszul-type" minimal resolution for $S/(ux, uy, uz)S$ over S:

$$0 \to S \xrightarrow{X} S^3 \xrightarrow{T} S^3 \xrightarrow{(ux,uy,uz)} S \to 0$$

Any two minimal projective resolutions should be chain isomorphic. Since the Betti numbers b_i are not changing when tensoring with an overring, we get $m = 3$, $b_0 = 1$, $b_1 = b_2 = 3$ and $b_3 = 1$. Thus the original phantom resolution has the form:

$$0 \to R_n \xrightarrow{F} R_n^3 \xrightarrow{G} R_n^3 \xrightarrow{(r_1\ r_2\ r_3)} R_n \to 0$$

where the matrices F, G and $(r_1\ r_2\ r_3)$ have entries in the maximal ideal of R_n.

The augmentation module is $R_n/(ux, uy, uz)R_n$ so we should necessarily have $(r_1, r_2, r_3)R_n = (ux, uy, uz)R_n$. This means that there is a 3×3 matrix A over R_n such that

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = A \cdot \begin{bmatrix} ux \\ uy \\ uz \end{bmatrix}$$

We will need the following simple result here:

Lemma 4. *Let $(R, \mathfrak{m}, K)$ be a local ring and let $m_1, \dots, m_n$ be minimal generators of an R-module M. Assume also that the $n \times n$ matrix A with elements in R is such that the n elements of*

$$A \cdot \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}$$

are also minimal generators for the module M. Then A is invertible.

Proof. We have a commutative diagram

$$\begin{array}{ccc} R^n & \longrightarrow & M \\ {\scriptstyle A}\big\downarrow & & {\scriptstyle id_M}\big\downarrow \\ R^n & \longrightarrow & M \end{array}$$

where the horizontal maps are natural surjections that map generators of R^n on elements $m_1, \dots, m_n$ and $A \cdot \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}$ respectively. Both these sets are minimal generators of M over R, so by Nakayama's lemma after tensoring with K the horizontal maps will become isomorphisms:

$$\begin{array}{ccc} K^n & \xrightarrow{\cong} & K \otimes_R M \\ {\scriptstyle A mod\, \mathfrak{m}}\big\downarrow & & {\scriptstyle id}\big\downarrow \\ K^n & \xrightarrow{\cong} & K \otimes_R M \end{array}$$

Therefore $A \operatorname{mod} \mathfrak{m}$ is an invertible matrix over $R/\mathfrak{m}$, so that A is also invertible in R. ☐

By this lemma, the matrix A is invertible, so we can change basis in the rightmost R_n^3 via this matrix. Then, without loss of generality the last map in the phantom resolution is given by the matrix $(ux\ uy\ uz)$.

Thus we have reduced to the case when the phantom resolution of M_n over R_n has the form:

$$P_\bullet : \ 0 \to R_n \xrightarrow{F} R_n^3 \xrightarrow{G} R_n^3 \xrightarrow{(ux,uy,uz)} R_n \to 0$$

4. Necessary conditions on matrices defining maps in the phantom resolution

In the previous section it was shown that after tensoring $P_\bullet$ with S we obtain an honest resolution of $S/(ux, uy, uz)S$ over S. Note that the matrices giving the maps in $P_\bullet$ remain unchanged after tensoring with overring S:

$$P_\bullet \otimes_{R_n} S: \ 0 \to S \xrightarrow{F} S^3 \xrightarrow{G} S^3 \xrightarrow{(ux,uy,uz)} S \to 0$$

We also have the "standard" Koszul resolution of $S/(ux, uy, uz)S$ over S:

$$0 \to S \xrightarrow{X} S^3 \xrightarrow{T} S^3 \xrightarrow{(ux,uy,uz)} S \to 0$$

Any two minimal free resolutions have to be chain isomorphic, and we can specify several first isomorphisms as long as the corresponding squares commute. In particular, this means that there should exist an invertible matrix Θ with elements in S and invertible element λ of S such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \xrightarrow{F} & S^3 & \xrightarrow{G} & S^3 & \xrightarrow{(ux,uy,uz)} & S & \longrightarrow & 0 \\ & & \lambda\big\downarrow & & \Theta\big\downarrow & & id_{S^3}\big\downarrow & & id_S\big\downarrow & & \\ 0 & \longrightarrow & S & \xrightarrow{X} & S^3 & \xrightarrow{T} & S^3 & \xrightarrow{(ux,uy,uz)} & S & \longrightarrow & 0 \end{array}$$

The commutativity of two leftmost squares of this diagram means that $\Theta F = \lambda X$ and $id_{S^3} G = T\Theta$, i.e. $F = (\lambda\Theta^{-1}) \cdot X$ and $G = T\Theta$. These two matrices were defining maps in the phantom resolution $P_\bullet$, so they have to have entries in R_n.

Therefore, in order to show that M_n does *not* have a finite phantom projective dimension and thus finish off the proof, it is enough to show that for all sufficiently large n, $n > N$, there does *not* exist a matrix $\Theta = (\Theta_{ij})_{3\times 3}$ with entries in S and an invertible element λ of S such that the matrices $T \cdot \Theta$ and $(\lambda\Theta^{-1}) \cdot X$ have entries in R_n.

5. Conditions for $(\lambda\Theta^{-1}) \cdot X$ to have entries in R_n

We want to find a general form for the triples (a, b, c) that can form the *rows* of the matrix $\lambda\Theta^{-1}$. Specifically, we want to prove the following result:

Lemma 5. *Every triple $(a, b, c) \in S^3$ such that $ax + by + cz \in R_n$ can be written as*

$$\begin{pmatrix} a & b & c \end{pmatrix} = r_1 \cdot \begin{pmatrix} u & 0 & 0 \end{pmatrix} + r_2 \cdot \begin{pmatrix} 0 & u & 0 \end{pmatrix} + r_3 \cdot \begin{pmatrix} 0 & 0 & u \end{pmatrix} +$$

$$r_4 \cdot \begin{pmatrix} x^{p^{n-1}} & 0 & 0 \end{pmatrix} + r_5 \cdot \begin{pmatrix} 0 & y^{p^{n-1}} & 0 \end{pmatrix} + r_6 \cdot \begin{pmatrix} 0 & 0 & z^{p^{n-1}} \end{pmatrix} +$$

$$\sum_{k=1}^{6} \left[r_7^{(k)} \cdot \begin{pmatrix} 0 & a_{3k} & -a_{2k} \end{pmatrix} + r_8^{(k)} \cdot \begin{pmatrix} -a_{3k} & 0 & a_{1k} \end{pmatrix} + r_9^{(k)} \cdot \begin{pmatrix} a_{2k} & -a_{1k} & 0 \end{pmatrix} \right] +$$

$$\sum_{k=1}^{6} \left[r_{10}^{(k)} \cdot \begin{pmatrix} b_{k1} & b_{k2} & b_{k3} \end{pmatrix} \right] + s_1 \cdot \begin{pmatrix} 0 & -z & y \end{pmatrix} + s_2 \cdot \begin{pmatrix} z & 0 & -x \end{pmatrix} +$$

$$+ s_3 \cdot \begin{pmatrix} -y & x & 0 \end{pmatrix}$$

for some $r_1, \ldots, r_6, r_7^{(k)}, \ldots, r_{10}^{(k)} \in R_n$ *and* $s_1, s_2, s_3 \in S$.

In particular, every row of the matrix $\lambda\Theta^{-1}$ *can be written in this form.*

Proof. Assume that $(a, b, c) \in S^3$ is such that $r := ax + by + cz \in R_n$. Clearly, we have $r \in (x, y, z)S \cap R_n$. It is easy to write down the R_n-generators of this ideal:

$$(x, y, z)S \cap R_n = (ux, uy, uz, x^{p^n}, y^{p^n}, z^{p^n}, I_1(T \cdot (a_{ij})_{3\times 6}), I_1((b_{ij})_{6\times 3} \cdot X)R_n$$

If $r = a'x + b'y + c'z$ is another presentation of r as an element of $(x, y, z)S$ then we have $(a - a')x + (b - b')y + (c - c')z = 0$ in S. As x, y, z is a regular sequence in S, the relation $(a - a', b - b', c - c')$ on x, y, z is S-linear combination of Koszul relations.

Therefore, the triple $(a, b, c) \in S^3$ can presented as a sum of an R_n-linear combination of triples corresponding to R_n-generators of the ideal $(x, y, z)S \cap R_n$ (that were written above) and an S-linear combination of Koszul relations on x, y, z. This is exactly the statement that we want.

□

6. Conditions for $T \cdot \Theta$ to have entries in R_n

Now we want to find a general form for the triples $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ that can form the *columns* of the matrix Θ. Specifically, we want to show the following:

Lemma 6. *Every triple* $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in S^3$ *such that* $T \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ *has entries in* R_n *can be written as*

$$\sum_{l=1}^{6} \left[r_l \begin{pmatrix} a_{1l} \\ a_{2l} \\ a_{3l} \end{pmatrix} \right] + \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix}$$

where $r_l \in R_n$ *and* A', B', C' *are elements of the maximal ideal*

$$(x, y, z, x', y', z', u, v, w)S$$

of S.

In particular, every column of the matrix Θ *can be written in this form.*

Proof. Assume that $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in S^3$ is such that $T \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -bz + cy \\ az - cx \\ -ay + bx \end{pmatrix}$ has entries in R_n. We can rewrite this condition as

$$I_2 \begin{pmatrix} x & y & z \\ a & b & c \end{pmatrix} \in R_n$$

Therefore the triple $(-bz + cy, az - cx, -ay + bx)$ gives an R_n-relation on x, y, z.

The converse is also true: every R_n-relation on x, y, z has the form

$$(-bz + cy, az - cx, -ay + bx)$$

for some choice of $a, b, c \in S$. Indeed, since x, y, z form a regular sequence in S, every R_n-relation (in fact, every relation over S) is an S-linear combination of the Koszul relations, i.e. has the form

$$s_1(0, -z, y) + s_2(z, 0, -x) + s_3(-y, x, 0) = (s_2 z - s_3 y, -s_1 z + s_3 x, s_1 y - s_2 x)$$

This has the form we want with $(a, b, c) = (-s_1, -s_2, -s_3)$.

These remarks show that it is enough to describe all the relations on x, y, z over R_n. Since we are interested in the triples modulo the maximal ideal of S, without loss of generality we can kill the elements x', y', z', u, v, w. Also, since the relations over a localization of a ring all come from a relation over a ring itself, it is enough to describe the relations on x, y, z over the ring

$$\bar{R}_n = k\left[x^{p^n}, y^{p^n}, z^{p^n}, I_1\left(\begin{pmatrix} T & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3}\end{pmatrix} \cdot (a_{ij})_{6\times 6}\right),\right.$$
$$\left. I_1\left((A_{ij})_{6\times 6} \cdot \begin{pmatrix} X & 0_{3\times 1} \\ 0_{3\times 1} & 0_{3\times 1}\end{pmatrix}\right), a_{ij}^{p^n}\right]$$

where $(A_{ij})_{6\times 6}$ is the classical adjoint of the matrix $(a_{ij})_{6\times 6}$, so that each its entry is a homogeneous polynomial of degree 5 in a_{ij}. We want to show that these relations are necessarily of the form

$$\sum_{l=1}^{6} r_l\,(a_{2l} z - a_{3l} y,\ a_{1l} z - a_{3l} x,\ a_{1l} y - a_{2l} x) + (A'',\ B'',\ C'')$$

where $r_l \in \bar{R}_n$ and the elements A'', B'', C'' are in the ideal $(x, y, z)^2$ of the ring $\bar{S} = k[x, y, z, a_{ij}^{p^n}]$. The latter condition comes from the following observation: if the elements A', B', C' are in the maximal ideal $(x, y, z, x', y', z', u, v, w)S$ of S, then the entries of $T \cdot \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix}$ will be in the ideal $(x, y, z)^2 S + (x', y', z', u, v, w)S$ of S.

Introduce an $\mathbb{N}^2$-grading on $\bar{S} = k[x, y, z, a_{ij}^{p^n}]$ given by

$$(\text{degree in } x, y, z, \quad \text{degree in } a_{ij})$$

Then $\bar{R}_n$ is a graded subring of $\bar{S}$ and x, y, z are homogeneous elements, so it is enough to find the *bi-homogeneous* $\bar{R}_n$-relations under this grading.

The elements of bi-degree (i, j) for $i \geq 2$ are already in $(x, y, z)^2 \bar{S}$ so we need to consider only the elements in bi-degrees $(0, j)$ and $(1, j)$. The elements of bi-degree $(0, j)$ cannot give rise to a relation on x, y, z, since x, y, z are clearly algebraically independent over $k(a_{ij})$. So assume that the triple has the form

$$(\gamma_{11} x + \gamma_{12} y + \gamma_{13} z, \gamma_{21} x + \gamma_{22} y + \gamma_{23} z, \gamma_{31} x + \gamma_{32} y + \gamma_{33} z)$$

where γ_{ij} are homogeneous polynomials in a_{ij} of the same degree, is an $\bar{R}_n$-relation on x, y, z, i.e.

$$(\gamma_{11}x + \gamma_{12}y + \gamma_{13}z)x + (\gamma_{21}x + \gamma_{22}y + (\gamma_{23}z)y + \gamma_{31}x + \gamma_{32}y + \gamma_{33})z = 0$$

By looking at the homogeneous polynomials

$$x^{p^n}, y^{p^n}, z^{p^n}, a_{ij}^{p^n},$$

and elements of the matrices

$$\begin{pmatrix} T & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{pmatrix} \cdot (a_{ij})_{6\times 6}, (A_{ij})_{6\times 6} \cdot \begin{pmatrix} X & 0_{3\times 1} \\ 0_{3\times 1} & 0_{3\times 1} \end{pmatrix}$$

that generate $\bar{R}_n$ over k, one notices that the only way to get bi-degree $(1, j)$ elements

$$\gamma_{i1}x + \gamma_{i2}y + \gamma_{i3}z, \quad i = 1, 2, 3,$$

is either to use $k[a_{ij}^{p^n}]$ -linear combinations of the elements

$$I_1\left(\begin{pmatrix} T & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{pmatrix} \cdot (a_{ij})_{6\times 6}\right)$$

or to use a $k[a_{ij}^{p^n}]$ -linear combinations of the elements

$$I_1\left((A_{ij})_{6\times 6} \cdot \begin{pmatrix} X & 0_{3\times 1} \\ 0_{3\times 1} & 0_{3\times 1} \end{pmatrix}\right)$$

In the first case $j = 1 + h \cdot p^n$ and in the second case $j = 5 + h \cdot p^n$ for some $h \in \mathbb{N}$. Note that $1 + h_1 \cdot p^n = 1 + h_2 \cdot p^n$ implies $(h_1 - h_2) \cdot p^n = 4$ which does not happen for $n \geq 3$. Therefore these two cannot be used simultaneously. In order to get a homogeneous relation we need to use only one of this forms for all three elements $\gamma_{i1}x + \gamma_{i2}y + \gamma_{i3}z$. We consider these two cases separately.

Case 1

Assume that each $\gamma_{i1}x + \gamma_{i2}y + \gamma_{i3}z$ is a $k[a_{ij}^{p^n}]$ -linear combination of the elements

$$I_1\left(\begin{pmatrix} T & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{pmatrix} \cdot (a_{ij})_{6\times 6}\right)$$

i.e of the elements

$$-a_{2k}z + a_{3k}y, \quad a_{1k}z - a_{3k}x, \quad -a_{1k}y + a_{2k}x$$

where $k = 1, \ldots, 6$. Specifically, for every $i = 1, 2, 3$ let

$$\gamma_{i1}x + \gamma_{i2}y + \gamma_{i3}z = \sum_{k=1}^{6} A_k^i(-a_{2k}z + a_{3k}y) + \sum_{k=1}^{6} B_k^i(a_{1k}z - a_{3k}x) + \sum_{k=1}^{6} C_k^i(-a_{1k}y + a_{2k}x)$$

where $A_k^i, B_k^i, C_k^i \in k[a_{ij}^{p^n}]$.

As these form a relation on x, y, z, we have

$$(\gamma_{11}x + \gamma_{12}y + \gamma_{13}z)x + (\gamma_{21}x + \gamma_{22}y + \gamma_{23}z)y +$$

$$(\gamma_{31}x + \gamma_{32}y + \gamma_{33}z)z = 0 \tag{4}$$

Setting the coefficient of x^2 in (4) to 0 we get

$$\sum_{k=1}^{6}(-B_k^1 a_{3k}) + \sum_{k=1}^{6} C_k^1 a_{2k} = 0$$

Note that a_{3k} appears in degrees of the form $1 + h \cdot p^n$ in the first sum and in degrees of the form $h \cdot p^n$ in the second sum, so they cannot cancel one another. This shows that we have to have $B_k^1 = C_k^1 = 0$ for all $k = 1, \ldots, 6$ so that

$$\gamma_{11}x + \gamma_{12}y + \gamma_{13}z = \sum_{k=1}^{6} A_k^1(-a_{2k}z + a_{3k}y)$$

Similarly, we get

$$\gamma_{21}x + \gamma_{22}y + \gamma_{23}z = \sum_{k=1}^{6} B_k^2(a_{1k}z - a_{3k}x)$$

and

$$\gamma_{31}x + \gamma_{32}y + \gamma_{33}z = \sum_{k=1}^{6} C_k^3(-a_{1k}y + a_{2k}x)$$

Now, setting the coefficient of xy in (4) to 0 we get

$$\sum_{k=1}^{6}(A_k^1 - B_k^2)a_{3k} = 0 \tag{5}$$

For each $k = 1, \ldots, 6$ the coefficient of a_{3k} appears in degrees of the form $1 + h \cdot p^n$ in the k-th summand in (5) and in degrees of the form $h \cdot p^n$ in all the other summands. Thus they cannot cancel one another, which implies that $A_k^1 = B_k^2$ for all $k = 1, \ldots, 6$. Similarly, by considering the coefficient of yz in (4), we obtain $B_k^2 = C_k^3$. Combining these results we see that the relation on x, y, z is of the form we want:

$$\begin{pmatrix} \gamma_{11}x + \gamma_{12}y + \gamma_{13}z \\ \gamma_{21}x + \gamma_{22}y + \gamma_{23}z \\ \gamma_{31}x + \gamma_{32}y + \gamma_{33}z \end{pmatrix} = \sum_{k=1}^{6} A_k^1 \begin{pmatrix} -a_{2k}z + a_{3k}y \\ a_{1k}z - a_{3k}x \\ -a_{1k}y + a_{2k}x \end{pmatrix}$$

where $A_k^1 \in k[a_{ij}^{p^n}]$.

Case 2

Now assume that each $\gamma_{i1}x + \gamma_{i2}y + \gamma_{i3}z$ is a $k[a_{ij}^{p^n}]$ -linear combination of the elements

$$I_1\left((A_{ij})_{6\times 6} \cdot \begin{pmatrix} X & 0_{3\times 1} \\ 0_{3\times 1} & 0_{3\times 1} \end{pmatrix}\right)$$

i.e. of the elements

$$A_{k1}x + A_{k2}y + A_{k3}z$$

where $k = 1, \ldots, 6$. Specifically, for every $i = 1, 2, 3$ let

$$\gamma_{i1}x + \gamma_{i2}y + \gamma_{i3}z = \sum_{k=1}^{6} \beta_k^i(A_{k1}x + A_{k2}y + A_{k3}z)$$

where $\beta_k^i \in k[a_{ij}^{p^n}]$. (Recall that $(A_{ij})_{6\times 6}$ is the classical adjoint of $(a_{ij})_{6\times 6}$.)

As in previous case, these form a relation on x, y, z so that (4) holds. Setting the coefficient of x^2 to 0 gives

$$\sum_{k=1}^{6} \beta_k^1 A_{k1} = 0$$

which is a linear dependence relation on A_{k1}, $k = 1, \dots, 6$ over $k(a_{ij}^{p^n})$.

If these elements are linearly independent over some $k(a_{ij}^{p^{n_0}})$ (and therefore also over $k(a_{ij}^{p^n})$ for all $n > n_0$), then we will necessarily have $\beta_k^1 = 0$ for all $n > n_0$, and similarly $\beta_k^2 = \beta_k^3 = 0$ for all n large enough, so that we do not have any non-trivial relations in this case.

Assume that this is not the case: i.e. for every n we have a non-trivial linear dependence relation on some of A_{ij}s over $k(a_{ij}^{p^n})$.

We will need the following linear algebra result:

Proposition 7. *Let*

$$K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots \supseteq K_n \supseteq \dots$$

be a non-increasing sequence of fields and let $v_1, \dots, v_r$ be elements of a vector space V over K_0 such that for every positive integer n there is a non-trivial linear dependence relation on $v_1, \dots, v_r$ over K_n. Then there exists a non-trivial linear dependence relation on $v_1, \dots, v_r$ over $\cap_{n=0}^{\infty} K_n$.

Proof. For every n, let V_n be the vector space of relations on $v_1, \dots, v_r$ over K_n. This is a vector subspace of $K_n^{\oplus r}$. Choose a finite basis for V_n in $K_n^{\oplus r}$ and make it into rows of a matrix, then put this matrix into reduced row echelon form. The rows of the resulting matrix M_n would be relations on $v_1, \dots, v_r$ over K_n and every relation over K_n on these elements would be a K_n-linear combination of these rows.

Note that every relation on $v_1, \dots, v_r$ over K_n is also a relation over K_m for $m < n$. Thus the dimension of K_n-span of $v_1, \dots, v_r$ can only increase as n increases. It is bounded by r, so this dimension will be constant after some point on, say for all $n > N$. For such n the dimensions of the matrix M_n would also be all the same: $\dim(V_n) \times r$.

For any $n_1 > n_2 > N$ the rows of the matrix M_{n_2} are linearly independent relations on $v_1, \dots, v_r$ over K_{n_2}, so also over K_{n_1} as well. By the uniqueness of the reduced row echelon form the corresponding matrices are also the same: $M_{n_1} = M_{n_2}$. So we obtain that $M_{N+1} = M_{N+2} = \dots, M_n, \dots$ for all $n > N$. The entries of this common matrix are in $\cap_{n=N}^{\infty} K_n = \cap_{n=0}^{\infty} K_n$. Therefore any row of this matrix will give a non-trivial linear dependence relation on $v_1, \dots, v_r$ over $\cap_{n=0}^{\infty} K_n$. □

Applying Proposition 7 to the elements A_{ij} and the sequence of fields $k(a_{ij}^{p^n})$ we see that in the remaining case we should necessarily have a linear dependence relation on A_{ij} over k. This is however not true: in fact, A_{ij} are not only linearly but even algebraically independent over k:

Lemma 8. *Let k be any field and let a_{ij} be a $n\times n$ matrix of indeterminates over k. Let $(A_{ij})_{n\times n}$ be the classical adjoint matrix of $(a_{ij})_{n\times n}$. Then n^2 elements A_{ij} are algebraically independent over k*

Proof. Denote $\delta := det((a_{ij})_{n\times n})$. Note that $\delta^{n-1} = det((A_{ij})_{n\times n})$ so that $k(A_{ij})(\delta)$ is an algebraic extension of $k(A_{ij})$. But we have $k(A_{ij})(\delta) = k(a_{ij})$ (since $(\frac{A_{ij}}{\delta})_{n\times n} = (a_{ij})^{-1}_{n\times n}$). Therefore $k(A_{ij})$ should have the same transcendence degree over k as $k(A_{ij})(\delta) = k(a_{ij})$, namely n^2 and is generated over k by n^2 elements A_{ij}. So these elements are algebraically independent over k. □

Thus Case 2 does not actually ever hold. □

7. Proof of the non-existence of the matrix Θ

First, we will reduce to the field case. Let us kill the maximal ideal

$$(u, v, w, x, y, z, x', y', z')$$

of S and the maximal ideal

$$(u, v, w, x, y, z, x', y', z')S \cap R_n$$

of R_n. After this reduction the ring R_n is replaced by the field $L_n = k(a_{ij}^{p^n})$ and S is replaced by the field $L = k(a_{ij})$. Consider the images of the matrix Θ and the element λ; both are still invertible in this new ring L.

Lemmas 5 and 6 show that, after killing the maximal ideals of R_n and S, every row of the matrix $\lambda\cdot\Theta^{-1}$ is an L_n-linear combination of the vectors

$$\begin{pmatrix} 0 & a_{3k} & -a_{2k} \end{pmatrix}, \begin{pmatrix} -a_{3k} & 0 & a_{1k} \end{pmatrix}, \begin{pmatrix} a_{2k} & -a_{1k} & 0 \end{pmatrix}, \begin{pmatrix} b_{k1} & b_{k2} & b_{k3} \end{pmatrix}$$

where $k = 1,\ldots,6$, and every column of the matrix Θ is an L_n-linear combination of the vectors

$$\begin{pmatrix} a_{1l} \\ a_{2l} \\ a_{3l} \end{pmatrix} \qquad l = 1,\ldots,6$$

The entries off the diagonal for the matrix $(\lambda\cdot\Theta^{-1})\cdot\Theta$ are all 0 and the diagonal entries are all equal to λ which is a non-zero element of S. This shows that the following system of equations should have a solution in L_n:

$$\sum_{k=1}^{6}\big(X_k^i \begin{pmatrix} 0 & a_{3k} & -a_{2k} \end{pmatrix} + Y_k^i \begin{pmatrix} -a_{3k} & 0 & a_{1k} \end{pmatrix} + Z_k^i \begin{pmatrix} a_{2k} & -a_{1k} & 0 \end{pmatrix} +$$

$$+U_k^i \begin{pmatrix} b_{k1} & b_{k2} & b_{k3} \end{pmatrix}\big)\cdot\sum_{l=1}^{6}\left[V_l^j \begin{pmatrix} a_{1l} \\ a_{2l} \\ a_{3l} \end{pmatrix}\right] =$$

$$= \begin{cases} 0 \quad \text{for} \quad i \neq j, 1 \leq i, j \leq 3 \\ \text{equal to each other and non-zero for } 1 \leq i = j \leq 3 \end{cases}$$

for some $X_k^i, Y_k^i, Z_k^i, U_k^i, V_l^j \in L_n$.

We can rewrite this system as

$$\sum_{k=1}^{6}\sum_{l=1}^{6} U_k^i V_l^j \left(b_{k1}a_{1l} + b_{k2}a_{2l} + b_{k3}a_{3l}\right) =$$

$$(\dagger) \qquad = \begin{cases} 0 \quad \text{for} \quad i \neq j, 1 \leq i, j \leq 3 \\ \text{equal and non-zero for } 1 \leq i = j \leq 3 \end{cases}$$

As $(b_{ij})_{6\times 6} = (a_{ij})^{-1}_{6\times 6}$, after multiplying throughout by $\det(a_{ij})$, we can assume that each $(b_{ij})_{6\times 6}$ is the classical adjoint of the matrix $(a_{ij})_{6\times 6}$ and so each b_{ij} is a homogeneous polynomial of degree 5 in elements a_{ij}.

If we show that this system of equations does not have any solution in L_N for some N (and so of course it would also have no solutions in L_n for all $n > N$), then we are done: the ring R_N will then produce the counterexample that we want.

Assume conversely that this system has a solution in L_n for every n. In order to arrive to a contradiction we will need the following result:

Lemma 9. *Every solution of the system* (†) *in* L_n *gives rise to a* non-trivial *linear dependence relation on* 36 *elements of* $L = k(a_{ij})$

$$\alpha_{kl} = b_{k1}a_{1l} + b_{k2}a_{2l} + b_{k3}a_{3l} \quad 1 \leq k, l \leq 6$$

over $L_n = k(a_{ij}^{p^n})$.

Proof. Any of the equations of the system (†) is clearly a linear dependence relation on α_{kl}, with coefficients equal to $U_k^i V_l^j$ where $1 \leq i \neq j \leq 3$, $1 \leq k, l \leq 6$. Therefore it is enough to show that there is a non-trivial linear dependence relation among these.

Assume conversely that all these coefficients $U_k^i V_l^j = 0$. Note that

$$\sum_{l=1}^{6} \left[V_l^j \begin{pmatrix} a_{1l} \\ a_{2l} \\ a_{3l} \end{pmatrix} \right]$$

gives j-th column of an invertible matrix $\Theta \in GL_3(L)$, so for every $j = 1, 2, 3$ there is at least one value of l for which $V_l^j \neq 0$.

Fix arbitrary i and k. Choose some $j \neq i$, and then for that value of j choose l such that $V_l^j \neq 0$. Then $U_k^i V_l^j = 0$ would imply that $U_k^i = 0$ for all i and k. For $i = j$ the system (†) however requires that

$$\sum_{k=1}^{6}\sum_{l=1}^{6} U_k^i V_l^i \left(b_{k1}a_{1l} + b_{k2}a_{2l} + b_{k3}a_{3l}\right) = \lambda \neq 0$$

which is a contradiction. □

By Lemma 9 we have a linear dependence relation on α_{kl}, $1 \leq k, l \leq 6$ over $L_n = k(a_{ij}^{p^n})$ for every n. Proposition 7 applied to the sequence of fields

$$L_1 \supseteq L_2 \supseteq \ldots \supseteq L_n \supseteq \ldots$$

shows that there is a non-trivial linear dependence relation on α_{kl}, $1 \leq k, l \leq 6$ over

$$\bigcap_{n=1}^{\infty} L_n = \bigcap_{n=1}^{\infty} k(a_{ij}^{p^n}) = k$$

This means that the elements

$$\alpha_{kl} = b_{k1}a_{1l} + b_{k2}a_{2l} + b_{k3}a_{3l} \quad 1 \leq k, l \leq 6$$

are linearly dependent over k. However this is not the case; the following lemma will thus finish off the proof of non-existence of the matrix Θ.

Lemma 10. *Let k be a field with $char(k) \neq 3$, let $(a_{ij})_{6\times 6}$ be a matrix of indeterminates over k and let $(b_{ij})_{6\times 6}$ be its classical adjoint matrix. Then the elements*

$$\alpha_{kl} = b_{k1}a_{1l} + b_{k2}a_{2l} + b_{k3}a_{3l} \quad 1 \leq k, l \leq 6$$

are linearly independent over k.

Proof. First note that each element α_{kl} can be written as a determinant of a matrix

$$\begin{pmatrix} a_{1l} & a_{11} & a_{12} & \dots & a_{1,k-1} & a_{1,k+1} & \dots & a_{16} \\ a_{2l} & a_{21} & a_{22} & \dots & a_{2,k-1} & a_{2,k+1} & \dots & a_{26} \\ a_{3l} & a_{31} & a_{32} & \dots & a_{3,k-1} & a_{3,k+1} & \dots & a_{36} \\ 0 & a_{41} & a_{42} & \dots & a_{4,k-1} & a_{4,k+1} & \dots & a_{46} \\ 0 & a_{51} & a_{52} & \dots & a_{5,k-1} & a_{5,k+1} & \dots & a_{56} \\ 0 & a_{61} & a_{62} & \dots & a_{6,k-1} & a_{6,k+1} & \dots & a_{66} \end{pmatrix}$$

which is obtained from $(a_{ij})_{6\times 6}$ by removing the k-th column, adding the l-th column from the left and replacing last 3 entries of the first column of the resulting matrix by zeroes. We want to see that these 36 elements α_{kl} are linearly independent over k.

Assume there is a linear dependence relation on elements α_{kl} over k:

$$\sum_{k=1}^{6}\sum_{l=1}^{6} \lambda_{kl}\alpha_{kl} = 0 \quad \lambda_{kl} \in k$$

Consider $\mathbb{N}^6$-grading on the ring $k[a_{ij}]$ where the element a_{ij} has multi-degree

$$(0, \dots, 0, \underset{\substack{\uparrow \\ j\text{-th spot}}}{1,} \quad 0, \dots, 0)$$

With this grading, α_{kl} for $k \neq l$ has multi-degree

$$(1, \dots, 1, \underset{\substack{\uparrow \\ k\text{-th spot}}}{0,} \quad 1, \dots, 1, \underset{\substack{\uparrow \\ l\text{-th spot}}}{2,} \quad 1, \dots, 1)$$

whereas for $k = l$ it has multi-degree $(1, \dots, 1)$. This implies that $\lambda_{kl} = 0$ for $k \neq l$. So it remains to see only that the elements $\alpha_{11}, \dots, \alpha_{66}$ are linearly independent over k.

These elements, up to the sign, are determinants of the matrix obtained from $(a_{ij})_{6\times 6}$ by replacing last 3 elements of one of the columns by zeroes. Let ϕ be any permutation of $\{1,\dots,6\}$. Note that every monomial

$$a_{1\phi(1)}a_{2\phi(2)}a_{3\phi(3)}a_{4\phi(4)}a_{5\phi(5)}a_{6\phi(6)}$$

appears in exactly 3 of the determinants α_{ii}, namely in $\alpha_{\phi(1),\phi(1)}$, $\alpha_{\phi(2),\phi(2)}$ and $\alpha_{\phi(3),\phi(3)}$, so the sum of the corresponding coefficients λ_{ii} is zero. By varying ϕ through all possible permutations, we obtain that the sum of any 3 of λ_{ii} is zero, so that they are all equal to each other. But the sum of 3 equal non-zero numbers is not zero in a field of characteristic $\neq 3$ (and it *is* zero in characteristic 3, so that assumption $\mathrm{char}(k) \neq 3$ is quite necessary to make this proof work). □

The proof of the Main Theorem heavily relied on the fact that rings R_n are not normal. The same remark applies to the counterexample to Conjecture 1 given in [3]. A natural question to ask is whether these conjectures hold under assumption of normality of the ring R.

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