

VIRTUALLY GEOMETRIC WORDS AND WHITEHEAD’S ALGORITHM

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ABSTRACT. Motivated by a question of Gordon and Wilton, we consider the question of which collections of words are “virtually geometric”. In particular, we prove that some words (e.g. *bbaccabc*) are not virtually geometric.

1. Definitions and introduction

Let G be a group, $\underline{g} = \{[g_i]\}_{i \in I}$ a collection of conjugacy classes in G , and X some space. Any homomorphism $\phi: G \rightarrow \pi_1(X)$ induces a map,

$$\phi_{\underline{g}}: \bigsqcup_{i \in I} S^1_i \rightarrow X,$$

well-defined up to homotopy, which sends the circle S^1_i to a loop in X freely homotopic to the element $\phi(g_i)$.

Definition 1.1. Let $F_n = \langle x_1, \dots, x_n \rangle$ be the free group of rank n , and let H be a 3-dimensional handlebody of genus n . We say a collection of conjugacy classes $\underline{w} = \{[w_1], \dots, [w_r]\}$ in F is *geometric* if for some isomorphism

$$\phi: F_n \rightarrow \pi_1(H)$$

the induced map $\phi_{\underline{w}}$ is homotopic in H to an embedding

$$\phi'_{\underline{w}}: \bigsqcup_{i=1}^r S^1_i \hookrightarrow \partial H.$$

In this case we also say that the finite presentation

$$\langle x_1, \dots, x_n \mid w_1, \dots, w_r \rangle$$

is *geometric*.

Remark 1.2. Let $\phi: F_n \rightarrow \pi_1(H)$ be an isomorphism as in Definition 1.1. Since any automorphism of a free group F_n can be realized as a homeomorphism of the handlebody H , the phrase “for some isomorphism” can be replaced by “for any isomorphism”. (To see that any automorphism can be realized, first realize elementary Nielsen transformations

$$\alpha_i: \quad x_i \mapsto x_i^{-1}, \quad x_j \mapsto x_j, \quad \forall j \neq i,$$

and

$$\beta_{ij}: \quad x_i \mapsto x_i x_j, \quad x_k \mapsto x_k, \quad \forall k \neq i,$$

and then use the fact that these transformations generate $\text{Aut}(F_n)$, [13, I.4].)

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Remark 1.3. A geometric presentation $\langle x_1, \dots, x_n \mid w_1, \dots, w_r \rangle$ gives a (not necessarily unique) way to build a 3-manifold M with $\pi_1(M) = \langle x_1, \dots, x_n \mid w_1, \dots, w_r \rangle$, by attaching 2-handles to the boundary of the handlebody H , along attaching curves $\phi'_w(S^1_i)$ for $1 \leq i \leq r$. Thus every geometric presentation is a presentation of the fundamental group of some 3-manifold.

Moreover, the double $D_n(\underline{w})$

$$\left\langle \begin{array}{l} x_1, \dots, x_n, \\ x'_1, \dots, x'_n, \\ t_2, \dots, t_r \end{array} \mid \begin{array}{l} w_1(x_1, \dots, x_n) = w_1(x'_1, \dots, x'_n), \\ t_i(w_i(x_1, \dots, x_n))t_i^{-1} = w_i(x'_1, \dots, x'_n), i \geq 2 \end{array} \right\rangle$$

is the fundamental group of a 3-manifold built by attaching thickened annuli to two copies of the handlebody H of genus n , along the attaching curves given by ϕ'_w .

Definition 1.4. Let $\underline{w} = \{[w_1], \dots, [w_r]\}$ be a collection of conjugacy classes in the free group $F = F_n$. For $F' < F$ a finite index subgroup, we define the *lifts of \underline{w}* to be the following set of F' -conjugacy classes:

$$\underline{w}_{F'} = \left\{ [gw^{n(g)}g^{-1}] \mid \begin{array}{l} g \in F, w \in \underline{w}, \text{ and} \\ n(g) \in \mathbb{Z}_{>0} \text{ minimal so that } gw^{n(g)}g^{-1} \in F' \end{array} \right\}$$

The collection \underline{w} is *virtually geometric* if $\underline{w}_{F'}$ is geometric for some finite index $F' < F$. In this case, we say the presentation

$$\langle x_1, \dots, x_n \mid w_1, \dots, w_r \rangle$$

is *virtually geometric*.

Here is an equivalent topological formulation: Realize \underline{w} as an embedded collection of circles $N_{\underline{w}}$ in H . If, for some finite cover $\tilde{H} \xrightarrow{p} H$, the inclusion

$$\tilde{\iota}: p^{-1}(N_{\underline{w}}) \rightarrow \tilde{H}$$

is homotopic to an embedding into $\partial\tilde{H}$, then \underline{w} is virtually geometric.

Remark 1.5. If a presentation $\langle x_1, \dots, x_n \mid w_1, \dots, w_r \rangle$ is virtually geometric, then there are positive integers k_1, \dots, k_r so that

$$\langle x_1, \dots, x_n \mid w_1^{k_1}, \dots, w_r^{k_r} \rangle$$

is virtually a 3-manifold group. Indeed, realizing $\underline{w} = \{w_1, \dots, w_r\}$ as a collection of embedded loops $N_{\underline{w}}$ in a genus n handlebody H , let $\tilde{H} \xrightarrow{p} H$ be a *regular* finite cover in which the preimage $\pi^{-1}(N_{\underline{w}})$ is homotopic to an embedding in the boundary. For $i \in \{1, \dots, r\}$ let $\gamma_i \subset H$ be the loop in $N_{\underline{w}}$ corresponding to w_i . Each component of $p^{-1}(\gamma_i)$ covers γ_i with the same degree; let k_i be this degree. We construct a complex \tilde{K} from \tilde{H} by attaching k_i disks to each component of $p^{-1}(\gamma_i)$, for each i . Since $p^{-1}(N_{\underline{w}})$ is homotopic to an embedding into $\partial\tilde{H}$, the complex \tilde{K} is homotopy equivalent to a 3-manifold. Moreover, the covering map $\tilde{H} \rightarrow H$ extends to a covering map $\tilde{K} \rightarrow K$ where K is a complex with $\pi_1(K) = \langle x_1, \dots, x_n \mid w_1^{k_1}, \dots, w_r^{k_r} \rangle$.¹

¹Actually, one can use the Orbifold Theorem [2, 7] together with a theorem of McCullough and Miller [15] to show that

$$\langle x_1, \dots, x_n \mid w_1^N, \dots, w_r^N \rangle$$

is a virtual 3-manifold group for some positive integer N , but the author does not know an elementary argument.

(An aside: It is possible to show that every finitely presented group admits a presentation which is virtually geometric, but the virtually geometric presentation will nearly always be different from the original one. Briefly, one realizes the group as the fundamental group of a special polyhedron; a finite branched cover of this polyhedron with positive even-degree branching over the center of every face will be thickenable. It follows that a presentation coming from the special polyhedron is virtually geometric. For more on *special polyhedra* and *thickenability*, see Chapter 1 of [14].)

The double $D_n(\underline{w})$ along a virtually geometric collection \underline{w} is also virtually the fundamental group of a 3-manifold, obtained by joining together a pair of handlebodies by thickened annuli. A 3-manifold argument based on Dehn's Lemma [10, Lemma 20] then shows that if $D_n(\underline{w})$ is one-ended, it contains a closed surface subgroup.

Gordon and Wilton gave some examples of virtually geometric but non-geometric words in [10], and asked the following:

Question 1.6. [10, Question 22] *Let F be a free group of rank $n \geq 2$. Is $\{w\}$ virtually geometric for every $w \in F$?*

We give a negative answer to this question in Section 4, by showing that the word $bbaaccabc \in \langle a, b, c \rangle$ is not virtually geometric (See Corollary 4.2).

Remark 1.7. Danny Calegari (personal communication) pointed out a second motivation for studying Question 1.6, connected with *stable commutator length* [4]. If a homologically trivial collection \underline{w} is geometric, then $\text{scl}(\underline{w})$ is in $\mathbb{Z} + \frac{1}{2}$, by [4, Proposition 4.4]. (This Proposition is an application of Gabai's result [9, 6.18] on the proportionality of Thurston and Gromov norms on the second homology of a 3-manifold.) More generally, suppose \underline{w} is virtually geometric, so that $\underline{w}_{F'}$ is geometric for some F' with $[F : F'] < \infty$. A straightforward argument then shows that

$$\text{scl}(\underline{w}) \in \frac{1}{[F : F']}(\mathbb{Z} + \frac{1}{2}).$$

Thus a positive answer to Question 1.6 for words in $[F, F]$ would have given an alternate proof of the rationality of scl in free groups, which is the main theorem of [5]. In fact, the answer to Question 1.6 is negative even in this more restricted setting (see Corollary 4.3).

2. Whitehead graphs

This section describes certain graphs which arise naturally from collections of words together with disk systems in handlebodies. The material in this section is discussed in more detail in Berge [1].

In the graphs we consider, multiple edges may connect a pair of vertices, but no edge joins a vertex to itself. The *valence* of a vertex is the number of incident edges (not the number of neighbors), and a graph will be said to be *regular* if every vertex has the same valence.

A collection of conjugacy classes in a free group $F = \langle x_1, \dots, x_n \rangle$ can also be thought of as a collection of *cyclic words*, i.e., freely and cyclically reduced "words" w_1, \dots, w_r whose letters are indexed mod l_i where l_i is the length of w_i . Given a collection \underline{w} of cyclic words, one can form the *Whitehead graph* $W(\underline{w})$ as follows:

- (1) The $2n$ vertices of $W(\underline{w})$ are the generators x_1, \dots, x_n of F and their inverses.
- (2) For each two-letter sequence xy in some $w_i \in \underline{w}$, attach an edge from x^{-1} to y . (If some $w_i = x$ has length one, attach an edge from x^{-1} to x .)

Example 2.1. If F is the free group $\langle a, b, c \rangle$, and $\underline{w} = \{bbaaccabc\}$, then the Whitehead graph $W(\underline{w})$ is isomorphic to the complete bipartite graph $K_{3,3}$.

Remark 2.2. Again, there is a topological interpretation: Realize the collection \underline{w} as an embedded 1-submanifold N in the handlebody H . There is a collection of disks $\mathcal{D} = \{d_1, \dots, d_n\}$ in H so that for each i there is a loop in H representing the generator x_i which intersects d_i exactly once, and has empty intersection with d_j for all $j \neq i$. After homotoping N to intersect \mathcal{D} minimally, split the handlebody along \mathcal{D} to give a ball with $2n$ disks \mathcal{D}_\pm in the boundary, and N' a 1-submanifold with boundary in \mathcal{D}_\pm . The Whitehead graph can then be obtained as the quotient of $\mathcal{D}_\pm \cup N'$ by crushing each disk to a point.

Notice that if N was homotopic to an embedding into ∂H which still intersects \mathcal{D} minimally, then the Whitehead graph is planar. As shown by an example of Berge [1, Section 12], the assumption that N still intersects \mathcal{D} minimally after being homotoped into ∂H is necessary, even for geometric \underline{w} ; there are examples of geometric collections of words with non-planar Whitehead graph. However, we will see in Theorem 2.3 that if \mathcal{D} is *minimal* in a certain sense, and \underline{w} is geometric, then the Whitehead graph is planar.

A *complete system of disks* for H is a collection \mathcal{C} of properly embedded disks which is maximal with respect to the condition that $H \setminus \cup \mathcal{C}$ is connected. The system \mathcal{D} from Remark 2.2 is an example of a complete system. Different choices of disks determine different choices of free generators for $\pi_1(H)$. (Actually, there are multiple complete disk systems corresponding to any choice of free generators; these disk systems are related to one another by homeomorphisms of H which act trivially on $\pi_1(H)$.)

Any collection of disjoint properly embedded disks in H (complete or not) gives rise to a graph $W(\underline{w}, \mathcal{C})$ in exactly the same manner as described in Remark 2.2. So long as \mathcal{C} is a complete system, the graph $W(\underline{w}, \mathcal{C})$ always has $2n$ vertices, but the number of edges depends on the particular complete system \mathcal{C} . (The number of edges is the number of times \mathcal{C} intersects \underline{w} , after \underline{w} has been homotoped to make this intersection minimal.) The complete system \mathcal{C} is said to be *minimal* (with respect to \underline{w}) if the number of edges is minimal over all complete systems of disks for H .

As explained in [1], the following can be deduced from a result of Zieschang [6, Theorem 1].

Theorem 2.3. [1, Section 11] *If \mathcal{C} is minimal, and \underline{w} is geometric, then $W(\underline{w}, \mathcal{C})$ is planar.*

(In case the hypotheses of Theorem 2.3 are satisfied, Berge [1, Section 13] explains further how to obtain a *Heegaard diagram*, and thus an explicit embedding of $N_{\underline{w}}$ in ∂H , from $W(\underline{w}, \mathcal{C})$.)

The next theorem is explained in [1, Section 8], and follows from the “peak reduction” lemma of Higgins–Lyndon (see [11] or [13, I.4], cf. [8, Section 4] for an account with a different flavor).

Theorem 2.4. [1, Section 8] *If \mathcal{C} is not minimal, then $W(\underline{w}, \mathcal{C})$ contains a pair of vertices v_+ and v_- with the same valence k and a collection of $s < k$ edges $\{e_1, \dots, e_s\}$ which separate v_+ from v_- .*

Remark 2.5. The pair of vertices v_{\pm} in Theorem 2.4 is the pair of vertices coming from some single disk D in \mathcal{C} . One can obtain a new disk system \mathcal{C}' for H by deleting D and adding a disk D' which intersects each of the edges e_1, \dots, e_s exactly once, without intersecting any other edges. The Whitehead graph $W(\underline{w}, \mathcal{C}')$ is obtained from $W(\underline{w}, \mathcal{C})$ by adding a new pair of vertices (corresponding to D'), deleting $\{v_{\pm}\}$, and “splicing together” the edges incident to $\{v_{\pm}\}$. (We will give a precise definition of *splicing* in the next section.) An example of such an operation (called a *Whitehead move*) is shown in Figure 1.

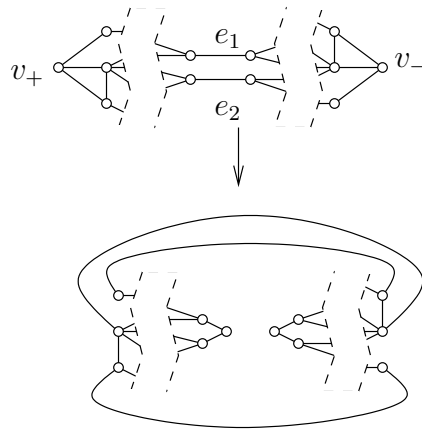


FIGURE 1. Reducing the number of edges in the Whitehead graph by a Whitehead move (an “unsplice” followed by a splice).

3. Splicing

Let G_1 and G_2 be graphs containing vertices $v_1 \in G_1, v_2 \in G_2$ of the same valence. Let σ be a bijection from the edges incident to v_1 to the edges incident to v_2 . We can form a new graph G from this data by the procedure:

- (1) Form the disjoint union of G_1 and G_2 .
- (2) Delete the vertices v_1 and v_2 (and any adjacent edges).
- (3) For each edge e connecting v_1 to some $v' \in G_1$, the corresponding edge $\sigma(e)$ connects v_2 to some $v'' \in G_2$. Add an edge to G connecting v' to v'' .

If G is obtained by such a procedure, we say that G is obtained by *splicing* G_1 and G_2 at (v_1, v_2) . (There may be more than one such G for a given $G_1, G_2, (v_1, v_2)$.)

Lemma 3.1. *Let $v_i \in G_i$ be vertices for $i \in \{1, 2\}$. If $G_2 \setminus v_2$ is connected, and G is obtained by splicing G_1 and G_2 at (v_1, v_2) , then G_1 is a minor of G .*

Proof. Collapse $G_2 \setminus v_2$ to a point to obtain a graph isomorphic to G_1 . □

Definition 3.2. We say a graph is p -edge-connected if no collection of $p - 1$ or fewer edges disconnects the graph.

Lemma 3.3. Let $k > 0$, and suppose that G is obtained by splicing G_1 and G_2 at (v_1, v_2) . If G_1 and G_2 are regular, k -valent, k -edge-connected graphs, then so is G .

Proof. For $i \in \{1, 2\}$, let G'_i be the graph obtained from G_i by removing v_i and all the edges adjacent to v_i . The graph G'_i will be thought of either as a subgraph of G_i or as a subgraph of G , depending on context.

For $i \in \{1, 2\}$ we claim that G'_i is $(\lfloor \frac{k-1}{2} \rfloor + 1)$ -edge-connected. Indeed, suppose that G'_1 is disconnected by $\lfloor \frac{k-1}{2} \rfloor$ or fewer edges into components C_1, \dots, C_c for some $c \geq 2$. For some $1 \leq j \leq c$, the number of edges connecting v_1 to C_j in G_1 is at most $\lfloor \frac{k}{2} \rfloor$. It follows that G_1 can be disconnected by at most $\lfloor \frac{k-1}{2} \rfloor + \lfloor \frac{k}{2} \rfloor = k - 1$ edges, contradicting the k -edge-connectedness of G_1 .

Let E be a set of $(k - 1)$ or fewer edges in G . Either G'_1 or G'_2 contains at most $\lfloor \frac{k-1}{2} \rfloor$ edges in E . Switching labels if necessary, we suppose it is G'_1 . Since G'_1 is $(\lfloor \frac{k-1}{2} \rfloor + 1)$ -edge-connected, the subgraph $G'_1 \setminus E \subset G \setminus E$ is connected.

Let x_1, \dots, x_k be the (not necessarily distinct) vertices of G_1 connected to v_1 , and let E_0 be the set of k edges in G connecting G'_1 to G'_2 . Finally, let $G''_2 \subset G$ be

$$G'_2 \cup \{x_1, \dots, x_k\} \cup E_0.$$

Notice that there is a quotient map $p: G''_2 \rightarrow G_2$ which takes each x_j to v_2 . Since G_2 is k -edge-connected, $p(E \cap G''_2)$ doesn't disconnect G_2 . This implies that every vertex of G''_2 is connected in $G''_2 \setminus E$ to x_j for some j . Since the x_j are all in G'_1 , and $G'_1 \setminus E$ is connected, $G \setminus E$ is connected. \square

We now explain how splicing arises in building Whitehead graphs. Let \underline{w} be a collection of words, as in Section 2. Let \mathcal{C} be a complete system of disks for the handlebody H , and let \tilde{H} be some d -fold cover of H , with $\pi_1(\tilde{H}) = F'$. The preimage $\tilde{\mathcal{C}}$ in \tilde{H} has too many disks to be a complete disk system for \tilde{H} , but we can still form the graph $W(\underline{w}_{F'}, \tilde{\mathcal{C}})$. Each component of the complement of $\tilde{\mathcal{C}}$ looks exactly like the complement of \mathcal{C} in H , so we obtain the following:

Lemma 3.4. The graph $W(\underline{w}_{F'}, \tilde{\mathcal{C}})$ is isomorphic to a disjoint union of d copies of $W(\underline{w}, \mathcal{C})$.

To get a complete system of disks for \tilde{H} , we must delete some disks from $\tilde{\mathcal{C}}$. Each time we delete one of these disks, the number of complementary components goes down by one, and the Whitehead graph changes by splicing two components together:

Lemma 3.5. Let $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subset \mathcal{D}_2$ be disk systems in a handlebody H so that \mathcal{D}_0 is complete and $\mathcal{D}_2 \setminus \mathcal{D}_1$ is a single disk. Let \underline{w} be a collection of conjugacy classes in $\pi_1(H)$. The graph $W_1 = W(\underline{w}, \mathcal{D}_1)$ is obtained from $W_2 = W(\underline{w}, \mathcal{D}_2)$ by splicing two components of W_2 together at a vertex.

(Similarly, the Whitehead move in Figure 1 can be described as an “unsplicing” of the edges e_1, \dots, e_s followed by splicing the resulting components together at v_{\pm} .)

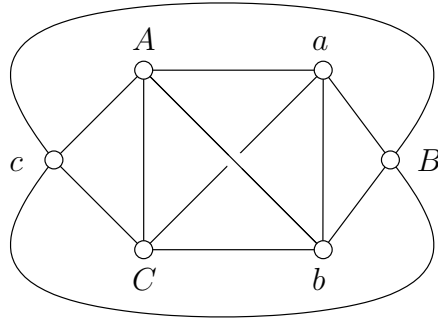


FIGURE 2. The Whitehead graph of the word $w = baabccACBBCA$ is non-planar, 4-valent, and 4-edge-connected.

4. Examples

Theorem 4.1. *Let \mathcal{C} be a complete system of disks for the handlebody H , and let \underline{w} be a collection of conjugacy classes in $\pi_1(H)$. Let $k \geq 3$. If $W(\underline{w}, \mathcal{C})$ is a regular, k -valent, k -edge-connected non-planar graph, then \underline{w} is not virtually geometric.*

Proof. Let \tilde{H} be a d -fold cover of H , corresponding to $F' < F$, where $F = \pi_1(H)$, and let $\tilde{\mathcal{C}}$ be the preimage of \mathcal{C} in \tilde{H} .

By Lemma 3.4, the Whitehead graph $W(\underline{w}_{F'}, \tilde{\mathcal{C}})$ is equal to d disjoint copies of $W(\underline{w}, \mathcal{C})$. If $\mathcal{D} \subset \tilde{\mathcal{C}}$ is a complete system of disks for \tilde{H} , then Lemma 3.5 implies that $W(\underline{w}_{F'}, \mathcal{D})$ is obtained from $W(\underline{w}_{F'}, \tilde{\mathcal{C}})$ by successively splicing together distinct components in some way (determined by the choice of complete system $\mathcal{D} \subset \tilde{\mathcal{C}}$) until only $2 \cdot \text{rank}(F')$ vertices remain. Exactly $d - 1$ splices will be performed, and each splice reduces the number of components by one, by Lemma 3.3, so $W(\underline{w}_{F'}, \mathcal{D})$ will be connected.

Moreover, induction on d together with Lemma 3.3 imply that $W(\underline{w}_{F'}, \mathcal{D})$ is k -valent and k -edge-connected. By Theorem 2.4, the disk system \mathcal{D} is minimal.

A k -valent, k -edge-connected graph cannot be disconnected by removing a single vertex, so induction on d and Lemma 3.1 imply that the graph $W(\underline{w}, \mathcal{C})$ is a minor of $W(\underline{w}_{F'}, \mathcal{D})$. It follows that $W(\underline{w}_{F'}, \mathcal{D})$ is non-planar.

Theorem 2.3 then implies that $\underline{w}_{F'}$ is not geometric in \tilde{H} . Since the finite cover \tilde{H} was arbitrary, \underline{w} is not virtually geometric. \square

A concrete example is the word $bbaaccabc \in F_3 = \langle a, b, c \rangle$; the Whitehead graph $W(\{bbaaccabc\})$ is isomorphic to the complete bipartite graph $K_{3,3}$. This graph is 3-valent, 3-edge-connected, and non-planar. Applying Theorem 4.1 gives the following.

Corollary 4.2. *The word $w = bbaaccabc$ in $F_3 = \langle a, b, c \rangle$ is not virtually geometric.*

For an example lying in the commutator subgroup, we take $w = baabccACBBCA$. (We use the common convention $A = a^{-1}$, $B = b^{-1}$, $C = c^{-1}$.) The Whitehead graph, shown in Figure 2, is non-planar, 4-valent, and 4-edge-connected, so Theorem 4.1 immediately gives the following.

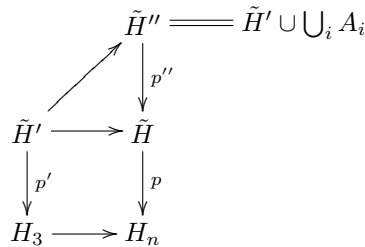
Corollary 4.3. *The word $w = baabccACBBCA$ in $F_3 = \langle a, b, c \rangle$ is not virtually geometric.*

The curious reader should be able to construct numerous similar examples, but it should be noted that either of the above examples gives non-virtually geometric words in F_n for all $n \geq 2$, as the following proposition shows.

Proposition 4.4. *Let $\underline{w} = \{w\}$ be a non-virtually geometric collection of words in F_3 . For all $n \geq 2$ there is an embedding $\phi_n: F_3 \rightarrow F_n$ so that $\phi_n(\underline{w})$ is not virtually geometric in F_n .*

Proof. For $n = 2$ we argue as follows: Let $\phi_2: F_3 \rightarrow F_2$ be any map with image an index 2 subgroup. Indeed, the homomorphism ϕ_2 is realized by a double cover of handlebodies $H_3 \rightarrow H_2$, and any cover \tilde{H} of H_2 either covers H_3 itself or has a double cover which covers H_3 . The lifts of \underline{w} in this cover are contained in the lifts of $\phi_2(\underline{w})$. Thus if $\phi_2(\underline{w})$ were virtually geometric, then \underline{w} would be virtually geometric.

For $n > 3$, we let $\phi_n: F_3 \rightarrow F_n$ be any embedding of F_3 as a free factor of F_n . To see why $\phi_n(\underline{w})$ is non-virtually geometric, it is easiest to argue topologically. Let H_3 be the handlebody of genus 3; we can obtain a handlebody H_n of genus n by attaching $n - 3$ one-handles along $2n - 6$ disjoint disks in the boundary of H_3 . Realize \underline{w} as an embedded 1-manifold N in $H_3 \subset H_n$. Suppose there is some finite cover $\tilde{H} \xrightarrow{p} H_n$ in which the preimage of N is homotopic to an embedded submanifold of the boundary. There is a corresponding finite cover $\tilde{H}' \xrightarrow{p'} H_3$ so \tilde{H}' embeds in \tilde{H} . There is also an infinite-sheeted cover $\tilde{H}'' \xrightarrow{p''} \tilde{H}$ corresponding to $\pi_1(\tilde{H}') \subset \pi_1(\tilde{H})$; the space \tilde{H}'' can be obtained from \tilde{H}' by attaching finitely many non-compact 3-manifolds with boundary A_1, \dots, A_l to \tilde{H}' along disks. Each of the A_i is homeomorphic to a ball with a Cantor set removed from its boundary.



Notice that $p'^{-1}(N) \subseteq (p \circ p'')^{-1}(N)$ coincides with the union of the compact components of $(p \circ p'')^{-1}(N)$. If $p^{-1}(N)$ is homotopic to an embedding in the boundary of \tilde{H} , then $(p \circ p'')^{-1}(N)$ is homotopic to an embedding in the boundary of \tilde{H}'' . Since $\tilde{H}'' = \tilde{H}' \cup \bigcup_i A_i$, this embedding is homotopic to an embedding in the boundary of \tilde{H}' , which covers H_3 . This contradicts the assumption that \underline{w} is not virtually geometric in F_3 . \square

Remark 4.5. Gordon and Wilton’s motivation for asking Question 1.6 was the question of which doubles of free groups contain closed surface subgroups. The double $D_3(\{baabccACBBCA\})$ contains a surface subgroup, by an application of the main theorem of [3]. The double $D_3(\{bbaaccabc\})$ contains a closed surface subgroup by an explicit construction of Sang-hyun Kim [12].

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