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AN ESTIMATE FROM BELOW FOR THE BUFFON NEEDLE PROBABILITY OF THE FOUR-CORNER CANTOR SET

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Abstract. Let $\mathcal{C}_n$ be the n-th generation in the construction of the middle-half Cantor set. The Cartesian square $\mathcal{K}_n = \mathcal{C}_n \times \mathcal{C}_n$ consists of 4^n squares of side-length 4^{-n}. The chance that a long needle thrown at random in the unit square will meet $\mathcal{K}_n$ is essentially the average length of the projections of $\mathcal{K}_n$, also known as the Favard length of $\mathcal{K}_n$. A classical theorem of Besicovitch implies that the Favard length of $\mathcal{K}_n$ tends to zero. It is still an open problem to determine its exact rate of decay. Until recently, the only explicit upper bound was $\exp(-c \log_* n)$, due to Peres and Solomyak. ($\log_* n$ is the number of times one needs to take log to obtain a number less than 1 starting from n). In [11] the power estimate from above was obtained. The exponent in [11] was less than 1/6 but could have been slightly improved. On the other hand, a simple estimate shows that from below we have the estimate $\frac{c}{n}$. Here we apply the idea from [4], [1] to show that the estimate from below can be in fact improved to $c\,\frac{\log n}{n}$. This is in drastic contrast to the case of *random* Cantor sets studied in [13].

1. Introduction

The four-corner Cantor set $\mathcal{K}$ is constructed by replacing the unit square by four sub-squares of side length 1/4 at its corners, and iterating this operation in a self-similar manner in each sub-square. More formally, consider the set $\mathcal{C}_n$ that is the union of 2^n segments:

$$\mathcal{C}_n = \bigcup_{a_j \in \{0,3\}, j=1,..,n} \left[\sum_{j=1}^n a_j 4^{-j}, \sum_{j=1}^n a_j 4^{-j} + 4^{-n}\right],$$

and let the middle half Cantor set be

$$\mathcal{C} := \bigcap_{n=1}^{\infty} \mathcal{C}_n \, .$$

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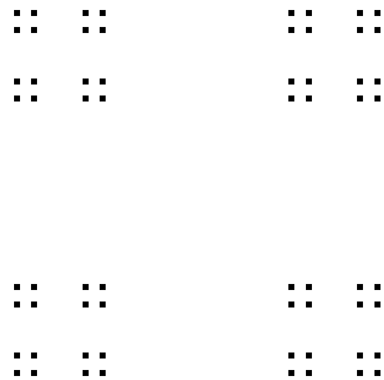

FIGURE 1. $\mathcal{K}_3$, the third stage of the construction of $\mathcal{K}$.

(It can also be written as $\mathcal{C} = \{\sum_{n=1}^{\infty} a_n 4^{-n} : \ a_n \in \{0,3\}\}$.) The four corner Cantor set $\mathcal{K}$ is the Cartesian square $\mathcal{C} \times \mathcal{C}$.

Since the one-dimensional Hausdorff measure of $\mathcal{K}$ satisfies $0 < \mathcal{H}^1(\mathcal{K}) < \infty$ and the projections of $\mathcal{K}$ in two distinct directions have zero length, a theorem of Besicovitch (see [3, Theorem 6.13]) yields that the projection of $\mathcal{K}$ to almost every line through the origin has zero length. This is equivalent to saying that the Favard length of $\mathcal{K}$ equals zero. Recall (see [2, p. 357]) that the **Favard length** of a planar set E is defined by

$$\text{Fav}(E) = \frac{1}{\pi} \int_0^{\pi} |\operatorname{Proj} \mathcal{R}_\theta E| \, d\theta, \tag{1.1}$$

where Proj denotes the orthogonal projection from $\mathbb{R}^2$ to the horizontal axis, $\mathcal{R}_\theta$ is the counterclockwise rotation by angle θ, and $|A|$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The Favard length of a set E in the unit square has a probabilistic interpretation: up to a constant factor, it is the probability that the "Buffon's needle," a long line segment dropped at random, hits E. (More precisely, suppose the needle's length is infinite, pick its direction uniformly at random, and then locate the needle in a uniformly chosen position in that direction, at distance at most $\sqrt{2}$ from the center of the unit square.)

The set $\mathcal{K}_n = \mathcal{C}_n^2$ is a union of 4^n squares with side length 4^{-n} (see Figure 1 for a picture of $\mathcal{K}_3$). By the dominated convergence theorem, $\text{Fav}(\mathcal{K}) = 0$ implies $\lim_{n\to\infty} \text{Fav}(\mathcal{K}_n) = 0$. We are interested in good estimates for $\text{Fav}(\mathcal{K}_n)$ as $n \to \infty$. A lower bound $\text{Fav}(\mathcal{K}_n) \geq \frac{c}{n}$ for some $c > 0$ follows from Mattila [8, 1.4]. Peres and Solomyak [13] proved that

$$\text{Fav}(\mathcal{K}_n) \leq C \exp[-a \log_* n] \quad \text{for all} \quad n \in \mathbb{N},$$

where

$$\log_* n = \min \left\{ k \geq 0 : \underbrace{\log \log \ldots \log}_{k} n \leq 1 \right\} .$$

This result can be viewed as an attempt to make a quantitative statement out of a qualitative Besicovitch projection theorem [2], [15], using this canonical example of a Besicovitch irregular set. It is very interesting to consider quantitative analogs of Besicovitch theorem in general. The reader can find more of that in [15].

In [11] the following estimate from above was obtained

$$\mathrm{Fav}(\mathcal{K}_n) \leq \frac{C_\tau}{n^\tau} ,$$

where τ was strictly less than $1/6$. This can be slightly improved, but it is still a long way till $\tau = 1$. Here we show, using the idea of [1], [4], that $\tau = 1$ is impossible.

Theorem 1. *There exists $c > 0$ such that*

$$\mathrm{Fav}(\mathcal{K}_n) \geq c \, \frac{\log n}{n} \quad \textit{for all} \;\; n \in \mathbb{N}. \tag{1.2}$$

Remark. This result is somewhat surprising in light of the probabilistic result in [13]. There, the authors consider planar Cantor sets constructed randomly as follows. Starting from the unit square U, divide U into four equal squares U_1, U_2, U_3, U_4. Similarly divide each of these into four squares $U_{j1}, U_{j2}, U_{j3}, U_{j4}$. For each j, randomly choose one square U_{jk} (of side length $\frac{1}{16}$). The four chosen squares form the first level $\tilde{\mathcal{K}}_1$. Repeat this process, always choosing the next generation randomly. The authors in [13] show that one expects

$$\frac{1}{Cn} \leq \mathrm{Fav}(\tilde{\mathcal{K}_n}) \leq \frac{C}{n}.$$

Proof of Theorem 1. The proof is an immediate corollary of the idea of [4] if one applies the duality between Cantor sets and Kakeya sets from [9]. As a "warm-up" we are going to prove a much simpler estimate

$$\mathrm{Fav}(\mathcal{K}_n) \geq \frac{c}{n} \quad \text{for all} \;\; n \in \mathbb{N} \,. \tag{1.3}$$

This does not require [1], [4].

In what follows the square means only the Cantor square. Let L_θ be the line passing through the origin at an angle θ with the x-axis. Let $f_{n,\theta}(x)$ denote the number of squares in $\mathcal{K}_n$ whose orthogonal projection onto the line L_θ contains a point x of this line. For each square Q contained in $\mathcal{K}_n$ with sidelength 4^{-n}, let $\chi_{Q,\theta}$ be the characteristic function of the projection of Q onto L_θ. Let $\ell(Q)$ be the sidelength of a square Q. Then $f_{n,\theta}(x) = \sum_{Q, \ell(Q) = 4^{-n}} \chi_{Q,\theta}(x)$. Therefore,

$$\int \int f_{n,\theta}(x) dx \, d\theta \asymp 4^n \cdot 4^{-n} = 1 \,. \tag{1.4}$$

Let us denote the support of $f_{n,\theta}(x)$ by $E_{n,\theta}$, $|E_{n,\theta}|$ being its one-dimensional measure on L_θ.

Knowing the first and second moment of $f_{n,\theta}(x)$ we can estimate $\int |E_{n,\theta}|\,d\theta$ by using Cauchy inequality twice:

$$1 \asymp \int\int f_{n,\theta} dx d\theta \le \int |E_{n,\theta}|^{\frac{1}{2}} (\int f^2_{n,\theta}(x) dx)^{\frac{1}{2}}\, d\theta \le$$
$$(\int |E_{n,\theta}|\, d\theta)^{\frac{1}{2}} (\int\int f^2_{n,\theta}(x) dx\, d\theta)^{\frac{1}{2}}\,.$$

Hence,

$$\int |E_{n,\theta}|\, d\theta \ge c\,\frac{1}{\int\int f^2_{n,\theta}(x) dx\, d\theta}\,. \tag{1.5}$$

Now

$$\int\int f^2_{n,\theta}(x) dx\, d\theta = \sum_{Q,Q',\ell(Q)=\ell(Q')=4^{-n}} \int\int \chi_{Q,\theta}(x)\chi_{Q',\theta}(x) dx\, d\theta\,.$$

So for each pair $P = (Q,Q')$, $\ell(Q) = \ell(Q') = 4^{-n}$ (Q and Q' may coincide) we consider

$$p_P := \int |Proj_\theta Q \cap Proj_\theta Q'|\, d\theta\,. \tag{1.6}$$

We define a distance-type function on pairs of squares. For $k = 0, 1, ..., n$, we call a pair $P = (Q,Q')$ a k–pair if Q, Q' are in a square of sidelength 4^{-k}, but not in any square of sidelength 4^{-k-1}. We have 4^k of 4^{-k}–squares, so we have $\asymp 4^k \cdot (4^{n-k})^2$ different k–pairs. For each k–pair P we obviously have

$$p_P \le C\, 4^{-n} 4^{k-n}\,.$$

Putting this together we get

$$\int\int f^2_{n,\theta}(x) dx\, d\theta = \sum_P p_P = \sum_{k=0}^{n} \sum_{P \text{ is a } k-\text{pair}} p_P$$
$$\le C \sum_{k=0}^{n-1} \sum_{P \text{ is a } k-\text{pair}} 4^{-n} 4^{k-n} \le C \sum_{k=0}^{n-1} 4^k \cdot (4^{n-k})^2 4^{-n} 4^{k-n} \le C\, n\,.$$

This estimate and (1.5) give us

$$\int |E_{n,\theta}|\, d\theta \ge \frac{c}{n}\,.$$

To prove (1.2) one needs to count pairs in a much more interesting way, which one gets from [1].

First we consider axis $0X$, where 0 is the origin and the axis has angle $\arctan \frac{1}{2}$ with the horizontal axis. We also need $0Y$, the orthogonal axis. Project the original unit square onto $0X$. We obtain the segment $I_0 := [0, L], L = \sqrt{2}\cos(\frac{\pi}{4} - \arctan \frac{1}{2})$ on $0X$. Notice that projections of Cantor squares of size 4^{-k}, $k = 0, ..., n$, generate the 4-adic grid on $I_0 = [0, L]$. Intervals of this 4-adic grid will be called I_σ, where σ is the word of length at most n in the alphabet of $\{0, 1, 2, 3\}$.

We have 4^n points that are the projections of the centers of 4^n squares Q of size 4^{-n}. We will call this set S, and use the notation s (maybe with indices) for elements of S. Because of our choice of axis $0X$, this correspondence between elements of S

and squares of sidelength 4^{-n} is one-to-one. Let y_s be the $0Y$ coordinate of the center of Q_s. Note that each s is the center of an interval I_σ, and that the projections of all cubes Q onto this axis are disjoint. This is an important feature of the argument.

Along with the usual Euclidean distance $|s_1 - s_2|$ between the points $s_1, s_2 \in S$, we have another very simple distance which will play a *crucial* role in proving (1.2). Namely,

$$d(s_1, s_2) := \min\{|I_\sigma|,\ s_1 \in I_\sigma\ s_2 \in I_\sigma\}\,.$$

This is just the usual 4-adic distance scaled by L. Of course $|s_1 - s_2| \leq d(s_1, s_2)$.

For $j = 0, 1, .., \log n$, $k \in [-n+j, 0]$, we call pair P a (j,k)-pair , if

$$\frac{|s_1 - s_2|}{|y_{s_1} - y_{s_2}|} \asymp 4^{-j}\,, |s_1 - s_2| \asymp 4^{-k-j}\,.$$

Now the pair $P = (Q, Q')$ of squares of size 4^{-n} is just a pair $(s_1, s_2), s_i \in S$.

For every (j,k)-pair $P = (s_1, s_2)$ one immediately has

$$p_P \leq C\,\frac{1}{4^n} \cdot \frac{4^{-n}}{|y_{s_1} - y_{s_2}|}\,. \tag{1.7}$$

where p_P is as in (1.6). Now we want to estimate the number $A_{j,k}$ of all (j,k)-pairs. If (s_1, s_2) is a (j,k)-pair, then

$$|s_1 - s_2| \asymp 4^{-k-j}$$

But also

$$4^j |s_1 - s_2| \leq C\,|y_{s_1} - y_{s_2}|\,,$$

and

$$|y_{s_1} - y_{s_2}| \leq C' d(s_1, s_2)\,. \tag{1.8}$$

The last inequality is *obvious* but it is the most crucial for the proof!

This is because we just obtained $d(s_1, s_2) \geq c4^{-k}$. How many 4-adic intervals are such that $d(s_1, s_2) \geq c4^{-k} \geq 4^{-k-a}$ (a is absolute), and $|s_1 - s_2| \asymp 4^{-k-j}$? Corresponding two 4-adic intervals of size 4^{-n} should be both in $C\,4^{-k-j}$-neighborhood of the 4-adic points of $1, 2, 3, ..., k, k+1, .., k+a$-generations. We have $4, 4^2, ..., 4^{k+a}$ such points correspondingly.

Therefore,

$$A_{j,k} \leq C \sum_{m=0}^{k+a} 4^m \left(\frac{4^{-k-j}}{4^{-n}}\right)^2 = C\,4^{2n-k-2j}$$

Another way to count the number of (j,k) pairs is as follows.

Recall that we have a 4-adic structure on the axis $0X$, and that each s_j is the center of an interval of length $4^{-n}L$. Hence we may identify each s_j with an n-digit string of numbers in $\{0, 1, 2, 3\}$. Say $s_1 = a_1 a_2 ... a_n$ and $s_2 = b_1 b_2 ... b_n$. If (s_1, s_2) is a (j,k) pair, then there is some $i \in \{0, 1, ..., k+a\}$ such that $d(s_1, s_2) = L4^{-k-a+i}$. Further, we have that $|s_1 - s_2| \leq L4^{-k-j}$. Hence we know that $a_1 = b_1$, $a_2 = b_2$, ... , $a_{k+a-i} = b_{k+a-i}$, because $d(s_1, s_2) = L4^{-k-a+i}$, and we know that the next $j - a + i$ digits of both a and b are almost uniquely determined, because $|s_1 - s_2| \leq L4^{-k-j}$. I.e., we must have either

$$a = a_1 a_2 ... a_{k+a-i} 033...33 a_{k+j} ... a_n \text{ and } b = a_1 a_2 ... a_{k+a} 100...00 b_{k+j} ... b_n$$

or

$$a = a_1 a_2 ... a_{k+a-i} 133...33 a_{k+j} ... a_n \text{ and } b = a_1 a_2 ... a_{k+a} 200...00 b_{k+j} ... b_n$$

or

$$a = a_1 a_2 ... a_{k+a-i} 233...33 a_{k+j} ... a_n \text{ and } b = a_1 a_2 ... a_{k+a} 300...00 b_{k+j} ... b_n,$$

(possibly with the roles of a and b switched). Hence for each $i = 0, 1, ..., k+a$, we have that the number of (j,k) pairs (s_1, s_2) with $d(s_1, s_2) = L4^{-k-a+i}$ is less than $C4^{k+a-i}4^{2(n-k-j)} = C4^{2n-k-2j-i}$. Hence

$$A_{j,k} \le C \sum_{i=0}^{k+a} 4^{2n-k-2j-i} \le C4^{2n-k-2j}.$$

Using this and (1.7) we get

$$\sum_{p \in (j,k)-\text{pairs}} p_P \le C\, 4^{2n-k-2j} \frac{4^{-2n}}{4^{-k}} \asymp 4^{-2j} \,.$$

The union of all (j,k)-pairs over all k is called: $\mathcal{P}'_j$.

So fix j, and get

$$\sum_{p \in \mathcal{P}'_j} p_P = \sum_{k=-n+j}^{0} \sum_{p \in (j,k)-\text{pairs}} p_P \le C \frac{n}{4^{2j}} \,. \tag{1.9}$$

Now let $J_j := [c_1 4^{-j}, c_2 4^{-j}]$, where c_1 is sufficiently small and c_2 is sufficiently large. These are intervals of *angles* θ with respect to the axis $0X$, where zero angle means we are line parallel to the axis $0X$.

Here is a crucial geometric observation:

(1.10) If $P = (Q, Q'), Q \neq Q'$ is so that

$$Proj_\theta Q \cap Proj_\theta Q' \neq \emptyset \,, \theta \in J_j \quad \text{then} \quad P \in \mathcal{P}'_j \,.$$

Let us throw into $\mathcal{P}'_j$ also all (Q,Q) pairs. The resulting collection is called j-pairs: $\mathcal{P}_j$. As

$$\int_{J_j} |E_{n,\theta}|\, d\theta \ge c \frac{(\int_{J_j} \int f_{n,\theta} dx\, d\theta)^2}{\int_{J_j} \int f_{n,\theta}^2 dx\, d\theta} \,,$$

$$\int_{J_j} \int f^2_{n,\theta} dx\, d\theta \le \sum_{p \in \mathcal{P}_j} p_P \le C \frac{n}{4^{2j}} + \tag{1.11}$$

$$\int_{J_j} \int \sum_{P=(Q,Q), \ell(Q)=4^{-n}} \chi_{Q,\theta}(x) dx\, d\theta \le C \frac{n}{4^{2j}} + C\, 4^{-j} 4^n 4^{-n} \le C \frac{n}{4^{2j}} \,,$$

and

$$\int_{J_j} \int f_{n,\theta} dx\, d\theta \ge c\, |J_j| \cdot 4^n \cdot 4^{-n} \asymp 4^{-j} \,,$$

we combine this to obtain

$$\int_{J_j} |E_{n,\theta}|\, d\theta \geq c\, 4^{-2j} \frac{4^{2j}}{n} = \frac{c}{n}\,. \tag{1.12}$$

Remark. Notice that (1.11) stops to be valid if $j > \log_4 n + Const$, because the contribution from the (Q, Q) pairs is no longer negligible. This explains why we did not get a better estimate from below than that in the Theorem.

Summing (1.12) over $j = 0, ..., \log n$ we obtain (1.2). Theorem is completely proved. □

2. Median value of $|E_{n,\theta}|$

Question 1. What is the median value of $|E_{n,\theta}|$?

Let us call this median value M_n. We can prove the following simple theorem, which immediately implies (1.3) of course.

Theorem 2. $M_n \geq \frac{c}{n}$.

Proof. We are going to prove

$$\int \frac{1}{|E_{n,\theta}|}\, d\theta \leq C\, n\,. \tag{2.1}$$

If one uses Tchebyshev's inequality this immediately gives $M_n \geq \frac{c}{n}$.

To prove (2.1) we use [9]. Let us fix a small positive ε, and let μ_n be an equidistributed measure on $\mathcal{C}_n$. Let $Proj_\theta$ stand (as always) for the orthogonal projection onto line L_θ. Notice that given two points $z, \zeta \in \mathbb{C}$ we have

$$\frac{\varepsilon}{|z-\zeta|} \asymp |\{\theta : |Proj_\theta(z) - Proj_\theta(\zeta)| \leq \varepsilon\|| \,.$$

Using this we write

$$\int\int \frac{\varepsilon}{|z-\zeta|} d\mu_n(z) d\mu_n(\zeta) \asymp \int\int |\{\theta : |Proj_\theta(z) - Proj_\theta(\zeta)| \leq \varepsilon\}| d\mu_n(z) d\mu_n(\zeta)$$

Introduce

$$\Phi_\varepsilon(x) = \begin{cases} 1\,, \text{ if } 0 \leq x \leq \varepsilon \\ 0\,, \text{ otherwise} \end{cases}$$

Then we repeat

$$\int\int \frac{\varepsilon}{|z-\zeta|} d\mu_n(z) d\mu_n(\zeta) \asymp \int\int\int \Phi_\varepsilon(|Proj_\theta(z) - Proj_\theta(\zeta)|) d\mu_n(z) d\mu_n(\zeta) d\theta =$$

$$\int\int\int \Phi_\varepsilon(|x-y|) d\mu_{n,\theta}(x) d\mu_{n,\theta}(y) d\theta\,,$$

where $d\mu_{n,\theta}$ is the projection of the measure μ_n on the line L_θ. In our old notation

$$d\mu_{n,\theta} = f_{n,\theta}(x)\, dx\,. \tag{2.2}$$

Of course

$$\int \Phi_\varepsilon(|x-y|) d\mu_{n,\theta}(y) = \mu_{n,\theta}(B(x,\varepsilon))\,,$$

and finally we get

$$\int\int \frac{1}{|z-\zeta|} d\mu_n(z) d\mu_n(\zeta) \geq c \int\int \frac{\mu_{n,\theta}(B(x,\varepsilon))}{\varepsilon}\, d\mu_{n,\theta}(x)\, d\theta\,. \tag{2.3}$$

The left hand side is $\leq C\,n$. One can see this by noting that for each square Q of side length 4^{-n} in $\mathcal{K}_n$, and for each $k = 0, 1, ..., n$, there are 4^{n-k} squares Q' at distance 4^{-k}.

In (2.3) we now use Fatou's lemma:

$$\int \liminf_{\varepsilon\to 0} \frac{\mu_{n,\theta}(B(x,\varepsilon))}{\varepsilon}\, d\mu_{n,\theta}(x)\, d\theta \leq C\,n\,. \tag{2.4}$$

Recalling (2.2) we obtain

$$\int\int_{E_{n,\theta}} f_{n,\theta}(x)^2\, dx\, d\theta \leq C\,n\,. \tag{2.5}$$

Recalling (1.4) we can rewrite it as

$$\int \frac{\int_{E_{n,\theta}} f_{n,\theta}(x)^2\, dx}{(\int_{E_{n,\theta}} f_{n,\theta}(x)\, dx)^2}\, d\theta \leq C\,n\,. \tag{2.6}$$

By Cauchy inequality

$$\frac{1}{|E_{n,\theta}|} \leq \frac{\int_{E_{n,\theta}} f_{n,\theta}(x)^2\, dx}{(\int_{E_{n,\theta}} f_{n,\theta}(x)\, dx)^2}\,.$$

Combine this and (2.6) and obtain the desired estimate

$$\int \frac{1}{|E_{n,\theta}|}\, d\theta \leq C\,n\,.$$

Inequality (2.1) and, therefore, Theorem 2 are completely proved.

□

3. Sierpiński's Cantor set

Consider now another Cantor set, which, by analogy with Sierpiński's gasket, we call Sierpiński's Cantor set $\mathcal{S}$. We take an equilateral triangle with side lenghth 1, leave 3 triangles of size 1/3 at each corner, and then continue this for n generations. On step n we get 3^n equilateral tringles of size 3^{-n}. Call this union of triangles $\mathcal{S}_n$. Its intersection is $\mathcal{S}$,

$$0 < H^1(\mathcal{S}) < \infty\,,$$

and this is a Besicovitch irregular set, so, by Besicovitch projection theorem (see [10])

$$\zeta_n := \int |\mathcal{S}_{n,\theta}|\, d\theta \to 0,\ n\to\infty\,.$$

Question 2. What is the order of magnitude of ζ_n?

This is the same question, which we had for 4-corner Cantor set.

Absolutely the same reasoning as above proves

Theorem 3.

$$\zeta_n = \int |\mathcal{S}_{n,\theta}|\, d\theta \geq c\frac{\log n}{n}\,.$$

In fact, projection of the triangles on the base side generate 3-adic lattice on the base side. Then we notice that (1.8) and (1.10) hold now as well. The proof is the same after these observations.

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