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REALITY AND TRANSVERSALITY FOR SCHUBERT CALCULUS IN OG($n, 2n+1$)

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Abstract. We prove an analogue of the Mukhin-Tarasov-Varchenko theorem (formerly the Shapiro-Shapiro conjecture) for the maximal type B_n orthogonal Grassmannian OG($n, 2n+1$).

1. The Mukhin-Tarasov-Varchenko Theorem

For any non-negative integer k, let $\mathbb{C}_k[z]$ denote the $(k+1)$-dimensional complex vector space of polynomials of degree at most k:

$$\mathbb{C}_k[z] := \{f(z) \in \mathbb{F}[z] \mid \deg f(z) \leq k\}\,.$$

Fix integers $0 \leq d \leq m$, and consider the Grassmannian $X = \mathrm{Gr}(d, \mathbb{C}_{m-1}[z])$, the variety of all d-dimensional linear subspaces of the m-dimensional vector space $\mathbb{C}_{m-1}[z]$. A point $x \in X$ is ***real*** if x is is spanned by polynomials in $\mathbb{R}_{m-1}[z]$; a subset of $S \subset X$ is real if every point in S is real.

The Mukhin-Tarasov-Varchenko theorem (formerly the Shapiro-Shapiro conjecture) asserts that any zero-dimensional intersection of Schubert varieties in X, relative a special family of flags in $\mathbb{C}_{m-1}[z]$, is transverse and real. This theorem is remarkable for two immediate reasons: first, it is a rare example of an algebraic geometry problem in which the solutions are always provably real; second, the usual arguments to prove transversality involve Kleiman's transversality theorem [5], which requires that the Schubert varieties be defined relative to generic flags. We recall the most relevant statements here, and refer the reader to the survey article [14] for a discussion of the history, context, reformulations and applications of this theorem.

To begin, we define a full flag in $\mathbb{C}_{m-1}[z]$, for each $a \in \mathbb{CP}^1$:

$$F_\bullet(a) \;:\; \{0\} \subset F_1(a) \subset \cdots \subset F_{m-1}(a) \subset \mathbb{C}_{m-1}[z]\,.$$

If $a \in \mathbb{C}$,

$$F_i(a) := (z+a)^{m-i}\mathbb{C}[z] \cap \mathbb{C}_{m-1}[z]$$

is the set of polynomials in $\mathbb{C}_{m-1}[z]$ divisible by $(z+a)^{m-i}$. For $a = \infty$, we set $F_i(\infty) := \mathbb{C}_{i-1}[z] = \lim_{a\to\infty} F_i(a)$. The flag $F_\bullet(a)$ is often described as the flag osculating the rational normal curve $\gamma : \mathbb{CP}^1 \to \mathbb{P}(\mathbb{C}_{m-1}[z])$, $\gamma(t) = (z+t)^{m-1}$, which simply means that $F_i(a)$ is the span of $\{\gamma(a), \gamma'(a), \ldots, \gamma^{(i-1)}(a)\}$.

Let $\Lambda = \Lambda_{d,m}$ be the set of all partitions $\lambda : (\lambda^1 \geq \cdots \geq \lambda^d)$, where $\lambda^1 \leq m-d$ and $\lambda^d \geq 0$. We say λ is a partition of k and write $\lambda \vdash k$ or $|\lambda| = k$ if $k = \lambda^1 + \cdots + \lambda^d$.

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For every $\lambda \in \Lambda$, the ***Schubert variety*** in X relative to the flag $F_\bullet(a)$ is

$$X_\lambda(a) := \{x \in X \mid \dim\big(x \cap F_{n-d-\lambda^i+i}(a)\big) \geq i\,,\ \text{for } i = 1, \dots, d\}\,.$$

The codimension of $X_\lambda(a)$ in X is $|\lambda|$.

Theorem 1 (Mukhin-Tarasov-Varchenko [6, 7])**.** *If $a_1, \dots a_s \in \mathbb{RP}^1$ are distinct real points, and $\lambda_1, \dots \lambda_s \in \Lambda$ are partitions with $|\lambda_1| + \cdots + |\lambda_s| = \dim X$, then the intersection*

$$X_{\lambda_1}(a_1) \cap \cdots \cap X_{\lambda_s}(a_s)$$

is finite, transverse, and real.

In [13], Sottile conjectured an analogue of Theorem 1 for $\mathrm{OG}(n, 2n{+}1)$, the maximal orthogonal Grassmannian in type B_n. In Section 2 of this note, we give a proof of this conjecture (our Theorem 3). We discuss some of its consequences in Section 3; in particular, we note that Theorem 3 should yield a geometric proof of the Littlewood-Richardson rule for $\mathrm{OG}(n, 2n{+}1)$.

2. The theorem for $\mathrm{OG}(n, 2n{+}1)$

Fix a positive integer n, and consider the non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the $(2n+1)$-dimensional vector space $\mathbb{C}_{2n}[z]$ given by

$$\Big\langle \sum_{k=0}^{2n} a_k \tfrac{z^k}{k!}\,,\, \sum_{\ell=0}^{2n} b_\ell \tfrac{z^\ell}{\ell!} \Big\rangle = \sum_{m=0}^{2n} (-1)^m a_m b_{2n-m}\,.$$

Let $Y = \mathrm{OG}(n, \mathbb{C}_{2n}[z]))$ be the orthogonal Grassmannian in $\mathbb{C}_{2n}[z]$, which is the variety of all n-dimensional isotropic subspaces of $\mathbb{C}_{2n}[z]$. The dimension of Y is $\frac{n(n+1)}{2}$.

The definition of a Schubert variety in Y requires our reference flags to be *orthogonal flags*. As explained in the next proposition, the bilinear form on $\mathbb{C}_{2n}[z]$ has been chosen so that this is true for the flags $F_\bullet(a)$.

Proposition 2. *For $a \in \mathbb{CP}^1$, then the flag $F_\bullet(a)$ is an orthogonal flag; that is, $F_i(a)^\perp = F_{2n+1-i}(a)$, for $i = 0, \dots, 2n+1$.*

Proof. For $a = 0, \infty$, this is straightforward to verify. We deduce the result for all other a by showing that $\langle f(z), g(z)\rangle = \langle f(z+a), g(z+a)\rangle$.

To see this, note that $\langle \frac{d}{dz}(\frac{z^k}{k!}), \frac{z^\ell}{\ell!}\rangle = -\langle \frac{z^k}{k!}, \frac{d}{dz}(\frac{z^\ell}{\ell!})\rangle$, so $\frac{d}{dz}$ is a skew-symmetric operator on $\mathbb{C}_{2n}[z]$. It follows that $\exp(a\frac{d}{dz})$ is an orthogonal operator on $\mathbb{C}_{2n}[z]$ and so $\langle f(z+a), g(z+a)\rangle = \langle \exp(a\frac{d}{dz})f(z), \exp(a\frac{d}{dz})g(z)\rangle = \langle f(z), g(z)\rangle$. □

The Schubert varieties in Y are indexed by the set Σ of all *strict partitions* σ : $(\sigma^1 > \sigma^2 > \cdots > \sigma^k)$, with $\sigma^1 \leq n$, $\sigma^k > 0$, $k \leq n$. For convenience, we put $\sigma^j = 0$ for $j > k$. We associate to σ a decreasing sequence of integers, $\overline{\sigma}^1 > \cdots > \overline{\sigma}^n$, such that $\overline{\sigma}^i = \sigma^i$ if $\sigma^i > 0$, and $\{|\overline{\sigma}^1|, \dots, |\overline{\sigma}^n|\} = \{1, \dots, n\}$. It is not hard to see that $\overline{\sigma}^i$ is given explicitly by the formula

$$\overline{\sigma}^i = \sigma^i - i + \#\{j \in \mathbb{N} \mid j \leq i < j + \sigma^j\}\,.$$

For $\sigma \in \Sigma$, the ***Schubert variety*** in Y relative to the flag $F_\bullet(a)$ is defined to be

$$Y_\sigma(a) := \{y \in Y \mid \dim\big(y \cap F_{1+n-\overline{\sigma}^i}(a)\big) \geq i\,,\ \text{for } i = 1, \dots, n\}\,.$$

The codimension of $Y_\sigma(a)$ in Y is $|\sigma|$. We refer the reader to [2, 12] for further details.

Theorem 3. *If $a_1, \dots a_s \in \mathbb{RP}^1$ are distinct real points, and $\sigma_1, \dots \sigma_s \in \Sigma$, with $|\sigma_1| + \cdots + |\sigma_s| = \dim Y$, then the intersection*

$$Y_{\sigma_1}(a_1) \cap \cdots \cap Y_{\sigma_s}(a_s)$$

is finite, transverse, and real.

Proof. Let $X = \mathrm{Gr}(n, \mathbb{C}_{2n}[z])$, and let $\Lambda = \Lambda_{n,2n+1}$. We prove this result by viewing Y as a subvariety of X, and the Schubert varieties Y_σ as the intersections of Schubert varieties in X with Y. Note that $\dim X = 2 \dim Y = n(n+1)$.

For a strict partition $\sigma \in \Sigma$, let

$$\widetilde{\sigma}^i := \overline{\sigma}^i + i = \sigma^i + \#\{j \in \mathbb{N} \mid j \le i < j + \sigma^j\}\,.$$

Observe that $\widetilde{\sigma}^i - \widetilde{\sigma}^{i+1} = \overline{\sigma}^i - \overline{\sigma}^{i+1} - 1 \ge 0$, and $\widetilde{\sigma}^1 \le \sigma^1 + 1 \le n+1$; hence we see that

$$\widetilde{\sigma} \;:\; (\widetilde{\sigma}^1 \ge \widetilde{\sigma}^2 \ge \cdots \ge \widetilde{\sigma}^n)$$

is a partition in Λ.

It follows directly from the definitions of Schubert varieties in X and Y that

$$X_{\widetilde{\sigma}}(a) \cap Y = Y_\sigma(a)\,.$$

Moreover, we have,

$$\begin{aligned} |\widetilde{\sigma}| &= |\sigma| + \sum_{i \ge 1} \#\{j \in \mathbb{N} \mid j \le i < j + \sigma^j\} \\ &= |\sigma| + \sum_{j \ge 1} \#\{i \in \mathbb{N} \mid j \le i < j + \sigma^j\} \\ &= |\sigma| + \sum_{j \ge 1} \sigma^j \;=\; 2|\sigma|\,. \end{aligned}$$

Thus, if $|\sigma_1| + \cdots + |\sigma_s| = \dim Y$, then $|\widetilde{\sigma}_1| + \cdots + |\widetilde{\sigma}_s| = 2 \dim Y = \dim X$, and so by Theorem 1 the intersection

$$X_{\widetilde{\sigma}_1}(a_1) \cap \cdots \cap X_{\widetilde{\sigma}_s}(a_s)$$

is finite, transverse, and real; in particular this intersection is a zero-dimensional reduced scheme. It follows immediately that

$$Y_{\sigma_1}(a_1) \cap \cdots \cap Y_{\sigma_s}(a_s) = Y \cap X_{\widetilde{\sigma}_1}(a_1) \cap \cdots \cap X_{\widetilde{\sigma}_s}(a_s)$$

is finite and real. To see that the intersection on the left hand side is also transverse, note that it is proper, so it suffices to show that it is scheme-theoretically reduced. But this is immediate from the fact that the right hand side is the intersection of Y with a zero-dimensional reduced scheme. $\square$

3. Consequences

Let $0 \leq d \leq m$, $X = \mathrm{Gr}(d, \mathbb{C}_{m-1}[z])$, be as in Section 1. We can consider the Wronskian of d polynomials $f_1(z), \dots, f_d(z) \in \mathbb{C}_{m-1}[z]$:

$$\mathrm{Wr}_{f_1,\dots,f_d}(z) := \begin{vmatrix} f_1(z) & \cdots & f_d(z) \\ f_1'(z) & \cdots & f_d'(z) \\ \vdots & \vdots & \vdots \\ f_1^{(d-1)}(z) & \cdots & f_d^{(d-1)}(z) \end{vmatrix}.$$

This is a polynomial of degree at most $\dim X = d(n-d)$. If $f_1, \dots, f_d$ are linearly dependent, the Wronskian is zero; otherwise up to a constant multiple, $\mathrm{Wr}_{f_1,\dots,f_d}(z)$ depends only on the linear span of $f_1(z), \dots, f_d(z)$ in $\mathbb{C}_{m-1}[z]$. Thus the Wronskian gives us a well defined morphism of schemes $\mathrm{Wr} : X \to \mathbb{P}(\mathbb{C}_{d(n-d)}[z])$, called the ***Wronski map***. This morphism is flat and finite [1]. For $x \in X$ we will write $\mathrm{Wr}(x; z)$ for any representative of $\mathrm{Wr}(x)$ in $\mathbb{C}_{d(n-d)}[z]$.

The Wronski map has a deep connection to the Schubert varieties on X relative to the flags $F_\bullet(a)$, $a \in \mathbb{CP}^1$. A proof of the following classical result may be found in [1, 9, 14].

Theorem 4. *The Wronksian $\mathrm{Wr}(x;z)$ is divisible by $(z+a)^k$ if and only if $x \in X_\lambda(a)$ for some partition $\lambda \vdash k$. Also, $x \in X_\mu(\infty)$ for some $\mu \vdash \big(\dim X - \deg \mathrm{Wr}(x;z)\big)$.*

For $X = \mathrm{Gr}(n, \mathbb{C}_{2n}[z])$, and $Y = \mathrm{OG}(n, \mathbb{C}_{2n}[z])$ we deduce the following analogue:

Theorem 5. *If $y \in Y$ then $\mathrm{Wr}(y;z) = P(y;z)^2$ for some polynomial $P(y;z) \in \mathbb{C}_{n(n+1)/2}[z]$. $P(y;z)$ is divisible by $(z+a)^k$ if and only if $y \in Y_\sigma(a)$ for some strict partition $\sigma \vdash k$ in Σ. Also, $y \in Y_\tau(\infty)$ for some strict partition $\tau \vdash \big(\dim Y - \deg P(y;z)\big)$.*

Proof. Let $y \in Y$, and let $(z+a)^\ell$ be the largest power $(z+a)$ that divides $\mathrm{Wr}(x;z)$. By Theorem 4, there exists a partition $\lambda \vdash \ell$ such that $y \in X_\lambda(a)$. Since ℓ is maximal, y is in the Schubert cell

$$X^\circ_\lambda(a) := \big\{x \in X \;\big|\; \dim\big(x \cap F_k(a)\big) \geq i,\; n{+}1{-}\lambda^i{+}i \leq k \leq n{+}1{-}\lambda^{i+1}{+}i,\; 0 \leq i \leq n\big\}$$
$$= X_\lambda(a) \setminus \Bigg(\bigcup_{|\mu|>|\lambda|} X_\mu(a)\Bigg).$$

(Here, by convention, $\lambda^0 = n+1$, $\lambda^{n+1} = 0$.) The Schubert cells in Y are of the form

$$Y^\circ_\sigma(a) := \big\{y \in Y \;\big|\; \dim\big(y \cap F_k(a)\big) \geq i,\; n{+}1{-}\overline{\sigma}^i \leq k \leq n{-}\overline{\sigma}^{i+1},\; 0 \leq i \leq n\big\}$$
$$= X^\circ_{\widetilde{\sigma}}(a) \cap Y$$

(Here, by convention, $\overline{\sigma}^0 = n+1$, $\overline{\sigma}^{n+1} = -n-1$.) Now, the intersection $X^\circ_\lambda(a) \cap Y$ is nonempty, since it contains y, and is therefore a Schubert cell in Y. It follows that $\lambda = \widetilde{\kappa}$ for some strict partition $\kappa \in \Sigma$. Thus $\ell = |\lambda| = 2|\kappa|$ is even, which proves that $\mathrm{Wr}(y;z) = P(y;z)^2$ is a square.

We have shown that $(z+a)^{|\kappa|}$ is the largest power of $(z+a)$ that divides $P(y;z)$, and $y \in Y^\circ_\kappa(a)$. If $y \in Y_\sigma(a)$ then we must have $Y_\sigma(a) \supset Y_\kappa(a)$, which implies that $|\sigma| \leq |\kappa|$, and hence $(z+a)^k$ divides $P(y;z)$. Conversely, for any $k \leq |\kappa|$ there exists $\sigma \vdash k$ such that $Y_\sigma(a) \supset Y_\kappa(a)$, and so $y \in Y_\sigma(a)$. This proves the second assertion. The third is proved by the same argument, taking $\ell = \dim Y - \deg P(y; z)$ and $a = \infty$. □

If we write $P(y)$ for the class of $P(y; z)$ in projective space $\mathbb{P}(\mathbb{C}_{n(n+1)/2}[z])$, then $y \mapsto P(y)$ defines a morphism of schemes $P : Y \to \mathbb{P}(\mathbb{C}_{n(n+1)/2}[z])$.

Theorem 6. *P is a flat, finite morphism.*

Proof. Let $h(z) = (z + a_1)^{k_1} \cdots (z + a_s)^{k_s} \in \mathbb{C}_{n(n+1)/2}[z]$. By Theorem 5,

$$P^{-1}(h(z)) = \bigcap_{i=1}^{s} \left(\bigcup_{\sigma_i \vdash k_i} Y_{\sigma_i}(a_i) \right),$$

which, by Theorem 3, is a finite set. Since P is a projective morphism, this implies that that P is flat and finite [4, Ch. III, Exer. 9.3(a)]. □

In [9] we showed that the properties of the Wronski map and Theorem 1 can be used to give geometric interpretations and proofs of several combinatorial theorems in the jeu de taquin theory, including the Littlewood-Richardson rule for Grassmannians in type A_n. The map P and Theorem 3 are the appropriate analogues for OG($n, 2n{+}1$). With a few modifications, it should be possible to use the arguments in [9] to give geometric proofs of the analogous results in the theory of shifted tableaux, as developed in [3, 8, 10, 11, 15], including the Littlewood-Richardson rule for OG($n, 2n{+}1$). The main ingredients required to adapt these proofs are Theorems 3, 5 and 6, and the Gel′fand-Tsetlin toric degeneration of OG($n, 2n{+}1$), which can be also be computed by considering $Y \subset X$. The complete details should be straightforward but somewhat lengthy, and we will not include them here.

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