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EXPONENTIAL DECAY OF DISPERSION MANAGED SOLITONS FOR VANISHING AVERAGE DISPERSION

M. Burak Erdoğan, Dirk Hundertmark, and Young-Ran Lee

Abstract. We show that any L^2 solution of the Gabitov-Turitsyn equation describing dispersion managed solitons decay exponentially in space and frequency domains. This confirms in the affirmative Lushnikov's conjecture of exponential decay of dispersion managed solitons.

1. Introduction

Consider the one-dimensional non-linear Schrödinger equation (NLS) with periodically varying dispersion coefficient

$$iu_t + d(t)u_{xx} + |u|^2u = 0. \tag{1.1}$$

It describes the amplitude of a signal transmitted via amplitude modulation of a carrier wave through a fiber-optical cable where the dispersion is varied periodically along the fiber, see, e.g., [3, 24, 27, 29, 32]. In (1.1) t corresponds to the distance along the fiber, x denotes the (retarded) time, and $d(t)$ is the dispersion along the wave-guide which, for practical purposes, one can assume to be piecewise constant.

In fiber optic cables, the information can be transmitted using localized soliton pulses in allocated time slots; the presence of a pulse corresponds to "1" and the absence of a pulse corresponds to "0" in binary format. Solitary solutions exist due to a delicate balance between the dispersion and nonlinearity. In order to minimize the interaction between the individual pulses, one needs to keep the pulses sufficiently far apart. A draw-back of solitary solutions for this application is that the soliton solutions with small support have large L^2 norm unless the dispersion constant is small. The technique of dispersion management was invented to overcome this difficulty. The idea is to use alternating sections of constant but (nearly) opposite dispersion. This introduces a rapidly varying dispersion $d(t)$ with small average dispersion, leading to well-localized stable soliton-like pulses changing periodically along the fiber. This idea has been enormously fruitful (see, *e.g.*, [20, 1, 8, 9, 14, 17, 18, 23, 26]). Record breaking transmission rates had been achieved using this technology [25] which is now widely used commercially.

To study strong dispersion management regime, it is convenient to write

$$d(t) = \frac{1}{\varepsilon}d_0(t/\varepsilon) + d_{av},$$

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Here $d_0(t)$ is the mean zero part which we assume to be piecewise constant, and d_{av} the average dispersion over one period , and ε is a small parameter. Rescaling t/ε to t, the envelope equation takes the form

$$iu_t + d_0(t)u_{xx} + \varepsilon d_{\mathrm{av}} u_{xx} + \varepsilon |u|^2 u = 0. \tag{1.2}$$

Since the full equation (1.2) is very hard to study, Gabitov and Turitsyn suggested to separate the free motion given by the solution of $iu_t + d_0(t)u_{xx} = 0$ in (1.2), and to average over one period, see [8, 9]. In the case[1] when $d_0(t) = 1$ on $[-1, 0]$ and $d_0(t) = -1$ on $[0, 1]$ this yields the following equation for the "averaged" solution v

$$iv_t + \varepsilon d_{\mathrm{av}} v_{xx} + \varepsilon Q(v, v, v) = 0, \quad \text{where} \tag{1.3}$$

$$Q(v_1, v_2, v_3) := \int_0^1 T_r^{-1}\Big(T_r v_1 \overline{T_r v_2} T_r v_3\Big) ds, \tag{1.4}$$

and $T_r = e^{ir\partial_x^2}$. In some sense, v is the slowly varying part of the amplitude and the varying dispersion is interpreted as a fast background oscillation, justifying formally the above averaging procedure. This is similar to Kapitza's treatment of the unstable pendulum, see [19]. This averaging procedure yielding (1.3) is well-supported by numerical and theoretical studies, see, for example, [1, 31, 32], and was rigorously justified in [33] in the limit of large local dispersion, i.e., as $\varepsilon \to 0$.

One can find stationary solutions by making the ansatz $v(t, x) = e^{i\varepsilon\omega t} f(x)$ in (1.3). This yields the time independent equation

$$-\omega f = -d_{\mathrm{av}} f_{xx} - Q(f, f, f) \tag{1.5}$$

describing stationary soliton-like solutions, the so-called dispersion managed solitons. Despite the enormous interest in dispersion managed solitons, there are few rigorous results available. One reason for this is that it is a nonlinear and nonlocal equation. Existence and smoothness of weak solutions of (1.5) had first been rigorously established in [33] for positive average dispersion $d_{\mathrm{av}} > 0$. In the case $d_{\mathrm{av}} = 0$, the existence was obtained in [15], also see [11] for a simplified proof. Smoothness in the case $d_{av} = 0$ was established in [28].

Remark 1.1. By a weak solution we mean $f \in H^1$ in the case $d_{av} > 0$, or $f \in L^2$ in the case $d_{av} = 0$, such that

$$-\omega\langle g, f\rangle = d_{av}\langle g', f'\rangle - \langle g, Q(f, f, f)\rangle. \tag{1.6}$$

for all $g \in H^1$. Here $\langle g, f\rangle = \int_{\mathbb{R}} \overline{g(x)} f(x)\, dx$ is the usual scalar product on $L^2(\mathbb{R})$.

By a formal calculation, using the unicity of T_r in L^2, we have

$$\langle g, Q(f, f, f)\rangle = \mathcal{Q}(g, f, f, f),$$

where

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_r f_1(x)} T_r f_2(x) \overline{T_r f_3(x)} T_r f_4(x) dx ds. \tag{1.7}$$

The functional $\mathcal{Q}(f_1, f_2, f_3, f_4)$ is well-defined for $f_j \in L^2(\mathbb{R})$ due to Strichartz inequality, see [33, 10].

[1] In fact, our method can be extended to more general dispersion profiles. We will address this issue together with the case $d_{\mathrm{av}} > 0$ in a later paper.

The decay of the solutions was first addressed by Lusnikov in [21]. He gave convincing but non-rigorous arguments that any solution f of (1.5) for $d_{av} = 0$ satisfies

$$f(x) \sim |x| \cos(a_0 x^2 + a_1 x + a_2) e^{-b|x|} \quad \text{as } x \to \pm\infty \tag{1.8}$$

for some suitable choice of real constants a_j and $b > 0$, see also [22]. In particular, he predicted that f and $\widehat{f}$ decay exponentially at infinity. For $d_{av} = 0$, the first rigorous x-space decay bounds were established in [10], where it was shown that both f and $\widehat{f}$ decay faster than any polynomial in the case $d_{\mathrm{av}} = 0$. In particular any weak solution is a Schwartz function.

Our main result confirms Lusnikov's exponential decay prediction:

Theorem 1.2. *Assume that $d_{av} = 0$. Let $f \in L^2$ be a weak solution of (1.5). Then there exists $\mu > 0$ such that*

$$|f(x)| \lesssim e^{-\mu|x|}, \quad |\widehat{f}(\xi)| \lesssim e^{-\mu|\xi|},$$

where $\widehat{f}$ is the Fourier transform of f.

We have the following immediate corollary.

Corollary 1.3. *Under the conditions of Theorem 1.2, both f and $\widehat{f}$ are analytic in a strip containing the real line.*

Remark 1.4. Weak solutions of 1.5 can be found with the help of a variational principle. For $d_{\mathrm{av}} = 0$ it is given by

$$P_\lambda := \sup\{\mathcal{Q}(f,f,f,f) \,|\, \|f\|_2^2 = \lambda\} \tag{1.9}$$

Note $\mathcal{Q}(f,f,f,f) = \int_0^1 \int |e^{it\partial_x^2} f(x)|^4 \, dxdt = \|e^{it\partial_x^2} f\|^4_{L^4_{[0,1]} L^4_x}$ and $e^{it\partial_x^2} f$ is the space-time Fourier transform of a measure concentrated on the paraboloid $\{\tau = k^2\} \subset \mathbb{R}^2$ with square-integrable density $\hat{f}$,

$$e^{it\partial_x^2} f(x) = \frac{1}{2\pi} \iint e^{ixk+it\tau} \delta(\tau - k^2) \hat{f}(k) \, d\tau dk.$$

Since, by scaling, $P_\lambda = P_1 \lambda^2$, the variational problem (1.9) yields the best constant in the L^4 Fourier extension estimate

$$\|e^{it\partial_x^2} f\|^4_{L^4_{[0,1]} L^4_x} \le P_1 \|f\|^4_{L^2(\mathbb{R})}. \tag{1.10}$$

for measure with an L^2 density on the paraboloid. Existence of maximizers for the variational problem (1.9) was established in [15], see also [11]. Thus our Theorem 1.2 and Corollary 1.3 for any weak solutions of (1.5) for vanishing average dispersion show, in particular, strong regularity properties for any maximizer of the Fourier extension estimate (1.10).

The inequality (1.10) is, of course, closely related to the one-dimensional Strichartz inequality

$$\|e^{it\partial_x^2} f\|_{L^6_t L^6_x} \le S_1 \|f\|_{L^2(\mathbb{R})}, \tag{1.11}$$

for which the sharp constant and the maximizers have been classified in [4, 7, 12], and the Fourier extension problem for the sphere for which the existence of maximizers and their properties has recently been discussed in [6].

In the proof of Theorem 1.2, the central idea is, as in [2, 13], to obtain suitable exponentially weighted a-priori estimates for the weak solution. In [2, 13], the commutator of the exponential weight with the Schrödinger operator is easily calculated since the operator is local. Variations of Agmon's method work also for relativistic Schrödinger operators which are nonlocal. However, in these applications one relies on the pointwise decay of the corresponding kernel. In our case, a major difficulty arises since our operator Q is nonlocal and the kernel of the free Schrödinger evolution has no pointwise decay. We overcome this difficulty by using the multi-linear structure of $\mathcal{Q}$ and the oscillation of the kernel. This is done in Section 2, where we obtain exponentially weighted multi-linear estimates for $\mathcal{Q}$. Multi-linear refinements of the Strichartz estimate where first established in [5] and later systematically studied in [30]. The results of these two papers focus, however, on the Fourier side and do not allow exponential weights. More importantly, we require bounds *independent* of the exponential weights, see Theorems 2.2 and 2.3 below. Our bounds are refinements of the x-space Strichartz estimates which were developed in [10] and used in conjunction with well-known Fourier space Strichartz estimates to prove that any weak solution is a Schwartz function. We would also like to note that our proof uses only the fact that $f \in L^2$ and as such does not require any of the previous smoothness or decay results.

2. A-priori estimates for $\mathcal{Q}$

We start with two alternative representations of $\mathcal{Q}$ inspired by the calculations in [12].

Lemma 2.1.
(2.1)

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \frac{1}{4\pi}\int_0^1 \int_{\mathbb{R}^4} \frac{1}{t} e^{-i(\eta_1^2-\eta_2^2+\eta_3^2-\eta_4^2)/(4t)} \overline{f_1(\eta_1)} f_2(\eta_2) \overline{f_3(\eta_3)} f_4(\eta_4) \delta(\eta_1-\eta_2+\eta_3-\eta_4)\, d\eta dt$$

(2.2)

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \frac{1}{2\pi}\int_0^1 \int_{\mathbb{R}^4} e^{it(\eta_1^2-\eta_2^2+\eta_3^2-\eta_4^2)} \overline{\widehat{f}_1(\eta_1)} \widehat{f}_2(\eta_2) \overline{\widehat{f}_3(\eta_3)} \widehat{f}_4(\eta_4) \delta(\eta_1-\eta_2+\eta_3-\eta_4)\, d\eta dt$$

Proof. Using the formula

$$T_t f(x) = \frac{1}{\sqrt{4\pi i t}} \int_{\mathbb{R}} e^{i(x-y)^2/(4t)} f(y) dy,$$

we get

$$\overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x) = \frac{1}{(4\pi t)^2} \int_{\mathbb{R}^4} e^{ix(\eta_1-\eta_2+\eta_3-\eta_4)/(2t)} e^{-i(\eta_1^2-\eta_2^2+\eta_3^2-\eta_4^2)/(4t)} \overline{f_1(\eta_1)} f_2(\eta_2) \overline{f_3(\eta_3)} f_4(\eta_4) d\eta.$$

From which one obtains (2.1) by performing the x-integration.

Similarly, one obtains (2.2) by using the formula

$$T_t f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} e^{-it\xi^2} \widehat{f}(\xi) d\xi.$$

□

To obtain exponential decay of dispersion managed solitons we need the following 'twisted' dispersion management functionals

$$\mathcal{Q}_{\mu,\varepsilon}(h_1,h_2,h_3,h_4) := \mathcal{Q}(e^{F_{\mu,\varepsilon}(X)}h_1, e^{-F_{\mu,\varepsilon}(X)}h_2, e^{-F_{\mu,\varepsilon}(X)}h_3, e^{-F_{\mu,\varepsilon}(X)}h_4)$$
$$\widetilde{\mathcal{Q}}_{\mu,\varepsilon}(h_1,h_2,h_3,h_4) := \mathcal{Q}(e^{F_{\mu,\varepsilon}(P)}h_1, e^{-F_{\mu,\varepsilon}(P)}h_2, e^{-F_{\mu,\varepsilon}(P)}h_3, e^{-F_{\mu,\varepsilon}(P)}h_4).$$

Here X denotes multiplication by x and $P = -i\partial_x$ is the one-dimensional momentum operator, and

$$F_{\mu,\varepsilon}(x) := \mu \frac{|x|}{1+\varepsilon|x|}, \quad \mu,\varepsilon \geq 0. \tag{2.3}$$

We have the following theorems which are rather surprising at first sight. They are the basis for our proof of exponential decay of dispersion management solitons.

Theorem 2.2. *There exists a constant C such that the bounds*

$$|\mathcal{Q}_{\mu,\varepsilon}(h_1,h_2,h_3,h_4)| \leq C \prod_{j=1}^{4} \|h_j\|_2, \tag{2.4}$$

$$|\widetilde{\mathcal{Q}}_{\mu,\varepsilon}(h_1,h_2,h_3,h_4)| \leq C \prod_{j=1}^{4} \|h_j\|_2 \tag{2.5}$$

hold for all $\mu,\varepsilon \geq 0$.

Theorem 2.3. *There exists a constant C such that if for some $l,k \in \{1,2,3,4\}$ $\tau = \mathrm{dist}(\mathrm{supp}\,(h_l), \mathrm{supp}\,(h_k)) \geq 1$ then*

$$|\mathcal{Q}_{\mu,\varepsilon}(h_1,h_2,h_3,h_4)| \leq \frac{C}{\sqrt{\tau}} \prod_{j=1}^{4} \|h_j\|_2 \tag{2.6}$$

for all $\mu,\varepsilon \geq 0$. Moreover, if $\tau = \mathrm{dist}(\mathrm{supp}\,(\widehat{h_l}), \mathrm{supp}\,(\widehat{h_k})) \geq 1$ then also

$$|\widetilde{\mathcal{Q}}_{\mu,\varepsilon}(h_1,h_2,h_3,h_4)| \leq \frac{C}{\sqrt{\tau}} \prod_{j=1}^{4} \|h_j\|_2 \tag{2.7}$$

Remark 2.4. The point of Theorems 2.2 and 2.3 is that the constant in the bounds is *independent* of $\mu,\varepsilon \geq 0$. We explicitly allow $\varepsilon = 0$, which at first seems to be in conflict with the fact that we need $e^{F_{\mu,0}}h_1 \in L^2$. However, in this case we can restrict ourselves to compactly supported functions h_1 and then use the a-priori bound and the density of these functions in L^2.

Let M be a multiplier in the variables $\eta_1,\eta_2,\eta_3,\eta_4$ and define the oscillatory functionals

$K^1_M(h_1,h_2,h_3,h_4) :=$

$$\int_0^1 \int_{\mathbb{R}^4} \frac{1}{t} e^{-i(\eta_1^2-\eta_2^2+\eta_3^2-\eta_4^2)/(4t)} M(\eta)\overline{h_1(\eta_1)} h_2(\eta_2)\overline{h_3(\eta_3)} h_4(\eta_4)\delta(\eta_1-\eta_2+\eta_3-\eta_4)\, d\eta dt$$

$K_M^2(h_1,h_2,h_3,h_4) :=$

$$\int_0^1 \int_{\mathbb{R}^4} e^{it(\eta_1^2-\eta_2^2+\eta_3^2-\eta_4^2)} M(\eta)\overline{h_1(\eta_1)} h_2(\eta_2)\overline{h_3(\eta_3)} h_4(\eta_4)\delta(\eta_1-\eta_2+\eta_3-\eta_4)\, d\eta dt$$

Note that by Lemma 2.1, we can rewrite the twisted functionals as

$$\mathcal{Q}_{\mu,\varepsilon}(h_1,h_2,h_3,h_4) = \frac{1}{4\pi} K^1_{M_{\mu,\varepsilon}}(h_1,h_2,h_3,h_4),$$
$$\widetilde{\mathcal{Q}}_{\mu,\varepsilon}(h_1,h_2,h_3,h_4) = \frac{1}{2\pi} K^2_{M_{\mu,\varepsilon}}(\widehat{h_1},\widehat{h_2},\widehat{h_3},\widehat{h_4}),$$

where

$$M_{\mu,\varepsilon}(\eta) = e^{F_{\mu,\varepsilon}(\eta_1)-F_{\mu,\varepsilon}(\eta_2)-F_{\mu,\varepsilon}(\eta_3)-F_{\mu,\varepsilon}(\eta_4)}.$$

Note that by the triangle inequality the function $M_{\mu,\varepsilon}$ is bounded by 1 on the set $\eta_1-\eta_2+\eta_3-\eta_4=0$ for any $\mu,\varepsilon \geq 0$. Therefore Theorems 2.2 and 2.3 follow immediately from the Propositions 2.5 and 2.6 below.

Proposition 2.5. *Let* $\widetilde{M} := \sup_{\eta_1-\eta_2+\eta_3-\eta_4=0} M(\eta_1,\eta_2,\eta_3,\eta_4) < \infty$. *Then* K_M^n, $n=1,2$, *is well-defined for all* $h_j \in L^2(\mathbb{R})$. *Moreover,*

$$|K_M^n(h_1,h_2,h_3,h_4)| \lesssim \widetilde{M} \prod_{j=1}^4 \|h_j\|_2 \tag{2.8}$$

where the implicit constant is independent of M *and* h_j, $j=1,2,3,4$.

Proof. By scaling, we can assume $\widetilde{M}=1$. Let $a(\eta) := \eta_1^2-\eta_2^2+\eta_3^2-\eta_4^2$. We write

$$|K_M^1| \leq \int_{\mathbb{R}^4} \Big| \int_0^1 \frac{1}{t} e^{-ia(\eta)/(4t)} dt \Big| |M(\eta)| \prod_{j=1}^4 |h_j(\eta_j)| \delta(\eta_1-\eta_2+\eta_3-\eta_4) d\eta$$
$$\leq \widetilde{M} \int_{\mathbb{R}^4} \Big| \int_0^1 \frac{1}{t} e^{-ia(\eta)/(4t)} dt \Big| \prod_{j=1}^4 |h_j(\eta_j)| \delta(\eta_1-\eta_2+\eta_3-\eta_4) d\eta. \tag{2.9}$$

Now we divide the t-integral into two pieces $t \leq |a(\eta)|$ where oscillations will be important and $t \geq |a(\eta)|$. More precisely,

$$(2.9) \leq \int_{\mathbb{R}^4} \Big| \int_0^{\min(1,|a(\eta)|)} \frac{e^{-ia(\eta)/(4t)}}{t} dt \Big| \prod_{j=1}^4 |h_j(\eta_j)| \delta(\eta_1-\eta_2+\eta_3-\eta_4) d\eta \tag{2.10}$$
$$+ \int_{\mathbb{R}^4} \int_{\min(1,|a(\eta)|)}^1 \frac{dt}{t} \prod_{j=1}^4 |h_j(\eta_j)| \delta(\eta_1-\eta_2+\eta_3-\eta_4) d\eta. \tag{2.11}$$

Let us introduce the following functionals, which, for later flexibility, we define in a little bit more generality than needed at the moment. For any (measurable) subset $A \subset \mathbb{R}^4$ let

$$I_1(A) := \int\limits_A \min(1, |a(\eta)|^{-1}) \prod_{j=1}^4 |h_j(\eta_j)| \delta(\eta_1 - \eta_2 + \eta_3 - \eta_4) d\eta \tag{2.12}$$

$$I_2(A) := \int\limits_0^1 \int\limits_{A \cap \{|a(\eta)| \le t\}} \frac{1}{t} \prod_{j=1}^4 |h_j(\eta_j)| \delta(\eta_1 - \eta_2 + \eta_3 - \eta_4) d\eta dt. \tag{2.13}$$

By Fubini-Tonelli

$$(2.11) = \int\limits_0^1 \frac{1}{t} \int\limits_{|a(\eta)| \le t} \prod_{j=1}^4 |h_j(\eta_j)| \delta(\eta_1 - \eta_2 + \eta_3 - \eta_4) d\eta dt = I_2(\mathbb{R}^4).$$

To estimate (2.10) we employ the following bound, which follows by the change of variable $\tau = 1/t$ and then an integration by parts.

$$\Big| \int\limits_0^b \frac{1}{t} e^{-ia/(4t)} dt \Big| \le \frac{8|b|}{|a|}. \tag{2.14}$$

Using this one sees

$$(2.10) \lesssim I_1(\mathbb{R}^4),$$

hence

$$|K_M^1| \lesssim I_1(\mathbb{R}^4) + I_2(\mathbb{R}^4) \tag{2.15}$$

We start to estimate the second term. Since $|a(\eta)| = 2(\eta_1 - \eta_2)(\eta_2 - \eta_3)$ on the set $\eta_1 - \eta_2 + \eta_3 - \eta_4 = 0$, we can estimate the η-integral in $I_2(\mathbb{R}^4)$ by

$$\begin{aligned} &\int\limits_{|a(\eta)| \le t} \prod_{j=1}^4 |h_j(\eta_j)| \delta(\eta_1 - \eta_2 + \eta_3 - \eta_4) d\eta \le \\ &\qquad \int\limits_{|\eta_1 - \eta_2| \le \sqrt{t}} \prod_{j=1}^4 |h_j(\eta_j)| \delta(\eta_1 - \eta_2 + \eta_3 - \eta_4) d\eta \\ &\qquad + \int\limits_{|\eta_2 - \eta_3| \le \sqrt{t}} \prod_{j=1}^4 |h_j(\eta_j)| \delta(\eta_1 - \eta_2 + \eta_3 - \eta_4) d\eta \end{aligned} \tag{2.16}$$

The first integral on the right hand side of (2.16) can be bounded by

$$\begin{aligned} &\int\limits_{|\eta_1 - \eta_2| \le \sqrt{t}} |h_1(\eta_1)||h_2(\eta_2)||h_3(\eta_3)||h_4(\eta_1 - \eta_2 + \eta_3)| d\eta_1 d\eta_2 d\eta_3 \\ &\qquad \le \|h_3\|_2 \|h_4\|_2 \int\limits_{|\eta_1 - \eta_2| \le \sqrt{t}} |h_1(\eta_1)||h_2(\eta_2)| d\eta_1 d\eta_2 \end{aligned}$$

$$\le \|h_3\|_2\|h_4\|_2\Big(\int\limits_{|\eta_1-\eta_2|\le\sqrt{t}} |h_1(\eta_1)|^2 d\eta_1 d\eta_2\Big)^{1/2}\Big(\int\limits_{|\eta_1-\eta_2|\le\sqrt{t}} |h_2(\eta_2)|^2 d\eta_1 d\eta_2\Big)^{1/2}$$
$$= 2\sqrt{t}\|h_1\|_2\|h_2\|_2\|h_3\|_2\|h_4\|_2,$$

where we used Cauchy-Schwarz inequality first in the $d\eta_3$ integral, then in the $d\eta_1 d\eta_2$ integral. The second integral can be estimated similarly. Thus

$$I_2(\mathbb{R}^4) \le 4\|h_1\|_2\|h_2\|_2\|h_3\|_2\|h_4\|_2 \int_0^1 \frac{1}{\sqrt{t}}dt = 8\|h_1\|_2\|h_2\|_2\|h_3\|_2\|h_4\|_2.$$

To estimate $I_1(\mathbb{R}^4)$ we split the η-integral into two disjoint regions

$$\begin{aligned} A_1 &:= \{\eta \in \mathbb{R}^4 : |\eta_1-\eta_2| \le 1 \text{ or } |\eta_2-\eta_3| \le 1\} \\ &= \{\eta \in \mathbb{R}^4 : |\eta_1-\eta_2| \le 1\} \cup \{\eta \in \mathbb{R}^4 : |\eta_2-\eta_3| \le 1\} \\ A_2 &:= \{\eta \in \mathbb{R}^4 : |\eta_1-\eta_2| > 1 \text{ and } |\eta_2-\eta_3| > 1\}. \end{aligned}$$

Obviously, $I_1(\mathbb{R}^4) = I_1(A_1) + I_1(A_2)$. For $I_1(A_1)$, we bound the minimum in (2.12) by 1 and then estimate the remaining integral as in (2.16) but now for $t = 1$. This shows

$$I_1(A_1) \lesssim \|h_1\|_2\|h_2\|_2\|h_3\|_2\|h_4\|_2.$$

On the other hand

$$\begin{aligned} I_1(A_2) &\le \int\limits_{A_2} \frac{|h_1(\eta_1)||h_2(\eta_2)||h_3(\eta_3)||h_4(\eta_4)|}{|a(\eta)|}\delta(\eta_1-\eta_2+\eta_3-\eta_4)d\eta \\ &\lesssim \int\limits_{\substack{|\eta_1-\eta_2|\ge1\\|\eta_2-\eta_3|\ge1}} \frac{|h_1(\eta_1)||h_2(\eta_2)||h_3(\eta_3)||h_4(\eta_1-\eta_2+\eta_3)|}{|\eta_1-\eta_2||\eta_2-\eta_3|}d\eta_1 d\eta_2 d\eta_3 \\ &\le \|h_1\|_2\|h_2\|_2\|h_4\|_2\Big(\int\limits_{\substack{|\eta_1-\eta_2|\ge1\\|\eta_2-\eta_3|\ge1}} \frac{|h_3(\eta_3)|^2}{|\eta_1-\eta_2|^2|\eta_2-\eta_3|^2}d\eta_1 d\eta_2 d\eta_3\Big)^{1/2} \\ &\lesssim \|h_1\|_2\|h_2\|_2\|h_3\|_2\|h_4\|_2. \end{aligned} \tag{2.17}$$

This finishes the proof for K_M^1. The proof for K_M^2 is simpler. Using the inequality

$$\Big|\int_0^1 e^{iat}dt\Big| \lesssim \min(1, |a|^{-1}), \tag{2.18}$$

one realizes $|K_M^2| \lesssim I_1(\mathbb{R}^4)$ and then proceeds as in the bound of $I_1(\mathbb{R}^4)$. □

A refinement of this proposition, when at least two of the functions, say, h_j and h_k, have separated supports, is

Proposition 2.6. *Assume that* $\widetilde{M} := \sup_{\eta_1-\eta_2+\eta_3-\eta_4=0} M(\eta_1,\eta_2,\eta_3,\eta_4) < \infty$ *and that there exist* $l, k \in \{1,2,3,4\}$ *with* $\tau = \operatorname{dist}(\operatorname{supp}(h_l), \operatorname{supp}(h_k)) \ge 1$. *Then,*

$$|K_M^n(h_1,h_2,h_3,h_4)| \lesssim \frac{\widetilde{M}}{\sqrt{\tau}}\prod_{j=1}^4 \|h_j\|_2, \quad n = 1,2. \tag{2.19}$$

Proof. Again we can assume $\widetilde{M}=1$. For $A\subset\mathbb{R}^4$ let

$$I(A) := \int_A \Big|\int_0^1 \frac{1}{t}e^{-ia(\eta)/(4t)}dt\Big|\prod_{j=1}^4 |h_j(\eta_j)|\delta(\eta_1-\eta_2+\eta_3-\eta_4)d\eta. \tag{2.20}$$

Let $J^\tau_{l,k} := \{\eta\in\mathbb{R}^4 : |\eta_l-\eta_k|\geq\tau\}$. Then $|K^1_M|\leq I(J^\tau_{l,k})$. By symmetry in (2.20), it is enough to consider the cases $(l,k)=(1,2)$ and $(l,k)=(1,3)$. First we consider the case $(l,k)=(1,2)$.

Recalling the definitions (2.12) and (2.13), we can, as in the proof of Proposition 2.5, bound $I(J^\tau_{1,2})$ by

$$I(J^\tau_{1,2}) \lesssim I_1(J^\tau_{1,2}) + I_2(J^\tau_{1,2}) \tag{2.21}$$

In the integral defining $I_2(J^\tau_{1,2})$, we have $t\geq|a(\eta)|=2|\eta_1-\eta_2||\eta_2-\eta_3|\geq 2\tau|\eta_2-\eta_3|$, which implies $|\eta_2-\eta_3|\lesssim t/\tau$. This yields

$$\begin{aligned} I_2(J^\tau_{1,2}) &\lesssim \int_0^1 \frac{1}{t}\int_{|\eta_2-\eta_3|\lesssim t/\tau}\prod_{j=1}^4|h_j(\eta_j)|\delta(\eta_1-\eta_2+\eta_3-\eta_4)d\eta dt \\ &\lesssim \frac{1}{\tau}\|h_1\|_2\|h_2\|_2\|h_3\|_2\|h_4\|_2, \end{aligned}$$

where we obtained the last line as in the estimate of (2.16) with $\sqrt{t}$ replaced by t/τ.

To estimate $I_1(J^\tau_{1,2})$ let

$$A^\tau_1 := \{|\eta_1-\eta_2|\geq\tau\}\cap\{|\eta_2-\eta_3|\leq 1\} \tag{2.22}$$

$$A^\tau_2 := \{|\eta_1-\eta_2|\geq\tau\}\cap\{|\eta_2-\eta_3|> 1\}. \tag{2.23}$$

Then, obviously,

$$I_1(J^\tau_{1,2}) = I_1(A^\tau_1)+I_1(A^\tau_2).$$

Similar to (2.17), we bound $I_1(A^\tau_2)$ by

$$\begin{aligned} I_1(A^\tau_2) &\lesssim \|h_1\|_2\|h_2\|_2\|h_4\|_2\Big(\int_{\substack{|\eta_1-\eta_2|\geq\tau\\ |\eta_2-\eta_3|\geq 1}}\frac{|h_3(\eta_3)|^2}{|\eta_1-\eta_2|^2|\eta_2-\eta_3|^2}d\eta_1d\eta_2d\eta_3\Big)^{1/2} \\ &\lesssim \frac{1}{\sqrt{\tau}}\|h_1\|_2\|h_2\|_2\|h_3\|_2\|h_4\|_2 \end{aligned}$$

For estimating $I_1(A^\tau_1)$ we bound the minimum in (2.12) by $|a(\eta)|^{-1/2}$ to see

$$\begin{aligned} I_1(A^\tau_1) &\lesssim \int_{A^\tau_1}\frac{1}{|\eta_1-\eta_2|^{1/2}|\eta_2-\eta_3|^{1/2}}\prod_{j=1}^4|h_j(\eta_j)|\delta(\eta_1-\eta_2+\eta_3-\eta_4)d\eta \\ &\lesssim \frac{1}{\sqrt{\tau}}\int_{|\eta_2-\eta_3|\leq 1}\frac{|h_1(\eta_1)h_2(\eta_2)h_3(\eta_3)h_4(\eta_1-\eta_2+\eta_3)|}{|\eta_2-\eta_3|^{1/2}}d\eta_1d\eta_2d\eta_3 \\ &\leq \frac{\|h_1\|_2\|h_4\|_2}{\sqrt{\tau}}\int_{|\eta_2-\eta_3|\leq 1}\frac{|h_2(\eta_2)h_3(\eta_3)|}{|\eta_2-\eta_3|^{1/2}}d\eta_2d\eta_3 \end{aligned}$$

$$\le \frac{\|h_1\|_2\|h_4\|_2}{\sqrt{\tau}}\Big(\int\limits_{|\eta_2-\eta_3|\le 1}\frac{|h_3(\eta_3)|^2 d\eta_2 d\eta_3}{|\eta_2-\eta_3|^{1/2}}\Big)^{1/2}\Big(\int\limits_{|\eta_2-\eta_3|\le 1}\frac{|h_2(\eta_2)|^2 d\eta_2 d\eta_3}{|\eta_2-\eta_3|^{1/2}}\Big)^{1/2}$$
$$\lesssim \frac{1}{\sqrt{\tau}}\|h_1\|_2\|h_2\|_2\|h_3\|_2\|h_4\|_2,$$

where in the third inequality we used the Cauchy Schwarz bound with respect to $d\eta_1$ and in the forth inequality with respect to the measure $|\eta_2-\eta_3|^{-1/2}d\eta_2 d\eta_3$. This finishes the proof for K_M^1.

Again the proof for K_M^2 is simpler. Using (2.18) and the separation condition $|\eta_1-\eta_2|\ge\tau$ for all (η_1,η_2) in the support of $h_1(\eta_1)h_2(\eta_2)$ one sees $|K_M^2|\lesssim I_1(J_{1,2}^\tau)$ and then proceeds as in the bound of $I_1(J_{1,2}^\tau)$.

Now we prove the case $(l,k)=(1,3)$, that is, we assume that the supports of h_1 and h_3 are separated by τ. In this case we have $|K_M^1|\le I(J_{1,3}^\tau)$ and $|K_M^2|\le I^1(J_{1,3}^\tau)$. The triangle inequality yields $J_{1,3}^\tau\subset J_{1,2}^{\tau/2}\cup J_{2,3}^{\tau/2}$, as subsets of $\mathbb{R}^4$, hence

$$I(J_{1,3}^\tau)\le I(J_{1,2}^{\tau/2})+I(J_{2,3}^{\tau/2})\lesssim\frac{1}{\sqrt{\tau}}\prod_{j=1}^4\|h_j\|_2,$$

and similarly for $I^1(J_{1,3}^\tau)$. This finishes the proof of the proposition. □

3. Proof of exponential decay.

Let f be a weak solution of the dispersion management equation. Let

$$\|f\|_{\mu,\varepsilon}:=\|e^{F_{\mu,\varepsilon}(X)}f\|_2, \tag{3.1}$$

with $F_{\mu,\varepsilon}$ defined in (2.3). The main step in our argument is to show that for some positive μ, $\|f\|_{\mu,\varepsilon}$ is bounded in $\varepsilon>0$.

Fix $\tau>1$ and define, for an arbitrary function f,

$$f_{\ll}:=f\chi_{[-\tau/3,\tau/3]},\quad f_<:=f\chi_{[-\tau,\tau]},\quad f_>:=f\chi_{[-\tau,\tau]^c},\quad f_\sim:=f_<-f_{\ll}.$$

Lemma 3.1. *Let f be a weak solution of the dispersion management equation for some $\omega>0$ with $\|f\|=1$. Then*

$$\omega\|f_>\|_{\mu,\varepsilon}\lesssim\|f_>\|_{\mu,\varepsilon}^3+e^{\mu\tau}\|f_>\|_{\mu,\varepsilon}^2+\|f_>\|_{\mu,\varepsilon}e^{2\mu\tau}\Big(\frac{1}{\sqrt{\tau}}+\|f_\sim\|\Big)+e^{3\mu\tau}\Big(\frac{1}{\sqrt{\tau}}+\|f_\sim\|\Big).$$

where the implicit constant does not depend on μ, ε, and τ.

Proof. Since f is a weak solution of the dispersion management equation for some $\omega>0$, we have

$$\omega\langle\varphi,f\rangle=\mathcal{Q}(\varphi,f,f,f),\ \text{for any }\varphi\in L^2.$$

Using this with $\varphi=e^{2F_{\mu,\varepsilon}}f_>$, we obtain

$$\begin{aligned}\omega\|f_>\|_{\mu,\varepsilon}^2&=\mathcal{Q}(e^{2F_{\mu,\varepsilon}}f_>,f,f,f)\\&=\mathcal{Q}_{\mu,\varepsilon}(e^{F_{\mu,\varepsilon}}f_>,e^{F_{\mu,\varepsilon}}f,e^{F_{\mu,\varepsilon}}f,e^{F_{\mu,\varepsilon}}f).\end{aligned}$$

Let $h:=e^{F_{\mu,\varepsilon}}f$. Then

$$\omega\|h_>\|^2=\mathcal{Q}_{\mu,\varepsilon}(h_>,h,h,h).$$

Writing $h=h_>+h_<$, and using the multilinearity of $\mathcal{Q}_{\mu,\varepsilon}$, we obtain

$$\omega\|h_>\|^2=\mathcal{Q}_{\mu,\varepsilon}(h_>,h_>,h_>,h_>)+\mathcal{Q}_{\mu,\varepsilon}(h_>,h_<,h_<,h_<) \tag{3.2}$$

$$+ \mathcal{Q}_{\mu,\varepsilon}(h_>, h_>, h_>, h_<) + \mathcal{Q}_{\mu,\varepsilon}(h_>, h_>, h_<, h_<)$$
$$+ \text{(similar terms with permutations of the last three entrees)}$$

Note that by Theorem 2.2, we have

$$|\mathcal{Q}_{\mu,\varepsilon}(h_>, h_>, h_>, h_>)| \lesssim \|h_>\|^4,$$
$$|\mathcal{Q}_{\mu,\varepsilon}(h_>, h_>, h_>, h_<)| \lesssim \|h_>\|^3\|h_<\|.$$

To estimate the remaining terms, we will further split one of the $h_<$ they contain into $h_\ll + h_\sim$:

$$\begin{aligned}|\mathcal{Q}_{\mu,\varepsilon}(h_>, h_<, h_<, h_<)| &\leq |\mathcal{Q}_{\mu,\varepsilon}(h_>, h_<, h_<, h_\ll)| + |\mathcal{Q}_{\mu,\varepsilon}(h_>, h_<, h_<, h_\sim)| \\ &\lesssim \frac{1}{\sqrt{\tau}}\|h_>\|\|h_<\|^2\|h_\ll\| + \|h_>\|\|h_<\|^2\|h_\sim\|,\end{aligned}$$

using Theorem 2.2, Theorem 2.3, and the fact that the supports of $h_>$ and $h_\ll$ are separated by $2\tau/3$. Similarly,

$$\begin{aligned}|\mathcal{Q}_{\mu,\varepsilon}(h_>, h_>, h_<, h_<)| &\leq |\mathcal{Q}_{\mu,\varepsilon}(h_>, h_>, h_<, h_\ll)| + |\mathcal{Q}_{\mu,\varepsilon}(h_>, h_>, h_<, h_\sim)| \\ &\lesssim \frac{1}{\sqrt{\tau}}\|h_>\|^2\|h_<\|\|h_\ll\| + \|h_>\|^2\|h_<\|\|h_\sim\|\end{aligned}$$

Similar estimates hold for the permutations. Using these estimates in (3.2), we obtain

$$\omega\|h_>\|^2 \lesssim \|h_>\|^4 + \|h_>\|^3\|h_<\| + \frac{1}{\sqrt{\tau}}\|h_>\|^2\|h_<\|\|h_\ll\| + \|h_>\|^2\|h_<\|\|h_\sim\| \tag{3.3}$$
$$+ \frac{1}{\sqrt{\tau}}\|h_>\|\|h_<\|^2\|h_\ll\| + \|h_>\|\|h_<\|^2\|h_\sim\|$$

(3.4)

Dividing both sides by $\|h_>\|$ and using $|h_<|, |h_\ll| \leq e^{\mu\tau}|f|$, $|h_\sim| \leq e^{\mu\tau}|f_\sim|$, and $\|f\| = 1$, we obtain

$$\omega\|h_>\| \lesssim \|h_>\|^3 + \|h_>\|^2 e^{\mu\tau} + \|h_>\|e^{2\mu\tau}\Big(\frac{1}{\sqrt{\tau}} + \|f_\sim\|\Big) + e^{3\mu\tau}\Big(\frac{1}{\sqrt{\tau}} + \|f_\sim\|\Big),$$

which finishes the proof. □

Proof of Theorem 1.2. Step 1. We will first determine $\tau > 1$ and we pick μ so that $e^{\mu\tau} = 2$. We can rewrite the bound from Lemma 3.1 as (with the notation $\nu = \|h_>\|$)

$$\Big(\omega - \frac{C}{\sqrt{\tau}} - C\|f_\sim\|\Big)\nu - C\nu^2 - C\nu^3 \leq C\Big(\frac{1}{\sqrt{\tau}} + \|f_\sim\|\Big). \tag{3.5}$$

Step 2. Let $G(\nu) = \frac{w}{2}\nu - C\nu^2 - C\nu^3$. Let $\nu_{\max}$ be the maxima of G on $\mathbb{R}^+$.
Step 3. Let $\nu_0 = \nu_{\max}/2$, and pick $\tau > 1$ so that

i) $C\Big(\frac{1}{\sqrt{\tau}} + \|f_\sim\|\Big) \leq \min(\omega/2, G(\nu_0))$,

ii) $\|f_>\| \leq \nu_0/2$.

With this choice, we rewrite (3.5) as

$$G(\|f_>\|_{\mu,\varepsilon}) \leq G(\nu_0), \tag{3.6}$$

which is valid for any $\varepsilon > 0$. This is depicted in figure 1.

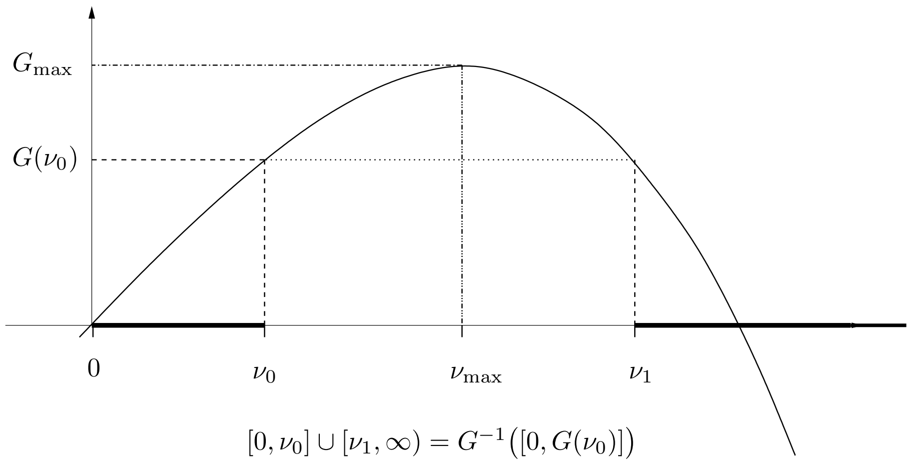

FIGURE 1. Graph of $G(\nu)$ and the trapping region $G^{-1}\big([0,G(\nu_0)]\big)$.

Step 4. Note that by ii) above and our choice of μ in step 1, we have

$$\|f_>\|_{\mu,1} \le \|e^{\mu\frac{|x|}{1+|x|}}\|_\infty \|f_>\| \le e^{\mu}\nu_0/2 < \nu_0. \tag{3.7}$$

Finally, since $\|f_>\|_{\mu,\varepsilon}$ depends continuously on ε for $\varepsilon > 0$, and (3.7), the inequality (3.6) shows that $\|f_>\|_{\mu,\varepsilon}$ is in the same connected component of $G^{-1}([0,G(\nu_0)])$, that is, $\|f_>\|_{\mu,\varepsilon} \in [0,\nu_0]$ for all $\varepsilon > 0$. This implies, by monotone convergence theorem,

$$\|f_>\|_{\mu,0} = \sup_{\varepsilon>0} \|f_>\|_{\mu,\varepsilon} \le \nu_0,$$

which shows $e^{\mu|\cdot|}f \in L^2$. With the obvious change of notation, a similar argument using Theorems 2.2, 2.3 for $\tilde{\mathcal{Q}}_{\mu,\varepsilon}$ shows that $e^{\tilde{\mu}|\cdot|}\widehat{f} \in L^2$, for some $\tilde{\mu} > 0$. Finally, the pointwise exponential bounds follows from the one-dimensional Sobolev embedding theorem, or simply by the following

$$\begin{aligned} e^{\mu|x|}|f(x)|^2 &= e^{\mu|x|}\Big|\int_x^\infty \frac{d}{ds}|f(s)|^2 ds\Big| \\ &\le 2\int_x^\infty e^{\mu|s|}|f(s)||f'(s)|ds \le 2\|e^{\mu|\cdot|}f\|\,\|f'\| < \infty. \end{aligned}$$

Similarly one gets pointwise exponential decay of $\widehat{f}$. □

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References

[1] M. J. Ablowitz and G. Biondini, *Multiscale pulse dynamics in communication systems with strong dispersion management.* Opt. Lett. **23** (1998), 1668–1670.

[2] S. Agmon, *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, Mathematical Notes, **29**, Princeton University Press, Princeton, NJ, 1982.

[3] G. P. Agrawal, *Nonlinear Fiber Optics.* Second Edition (Optics and Photonics), Academic Press, San Diego, 1995.

[4] J. Bennett, N. Bez, A. Carbery and D. Hundertmark, *Heat-flow monotonicity of Strichartz norms.* Anal. PDE **2** (2009), no. 2, 147–158.

[5] J. Bourgain, *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity.* Internat. Math. Res. Notices **1998**, no. 5, 253–283.

[6] M. Christ and S. Shao, *On the extremizers of an adjoint Fourier restriction inequality*, preprint 2010.

[7] D. Foschi, *Maximizers for the Strichartz inequality.* J. Eur. Math. Soc. (JEMS) **9** (2007), no. 4, 739–774.

[8] I. Gabitov and S.K. Turitsyn, *Averaged pulse dynamics in a cascaded transmission system with passive dispersion compensation.* Opt. Lett. **21** (1996), 327–329.

[9] I. Gabitov and S.K. Turitsyn, *Breathing solitons in optical fiber links.* JETP Lett. **63** (1996) 861.

[10] D. Hundertmark and Y.-R. Lee, *Decay estimates and smoothness for solutions of the dispersion managed non-linear Schrödinger equation.* Commun. Math. Phys. **286** (2009), no. 3, 851–873.

[11] D. Hundertmark and Y.-R. Lee, *On non-local variational problems with lack of compactness related to non-linear optics*, Preprint 2010.

[12] D. Hundertmark and V. Zharnitsky, *On sharp Strichartz inequalities for low dimensions.* International Mathematics Research Notices, vol. 2006, Article ID 34080, 18 pages, 2006. doi:10.1155/IMRN/2006/34080

[13] Hunziker and I. S. M. Sigal, *The quantum N-body problem*, J. Math. Phys. **41** (2000), no. 6, 3448–3510.

[14] S. Kumar and A. Hasegawa, *Quasi-soliton propagation in dispersion-managed optical fibers.* Opt. Lett. **22** (1997), 372–374.

[15] M. Kunze, *On a variational problem with lack of compactness related to the Strichartz inequality.* Calc. Var. Partial Differential Equations **19** (2004), no. 3, 307–336.

[16] M. Kunze, J. Moeser and V. Zharnitsky, *Ground states for the higher order dispersion managed NLS equation in the absence of average dispersion,* J. Differential Equations **209** (2005), 77–100.

[17] C. Kurtzke, *Suppression of fiber nonlinearities by appropriate dispersion management.* IEEE Phot. Tech. Lett. **5** (1993), 1250–1253.

[18] T. Lakoba and D. J. Kaup, *Shape of the stationary pulse in the strong dispersion management regime*, Electron. Lett. **34** (1998), 1124–1125.

[19] L.D. Landau and E.M. Lifshitz, *Course of theoretical physics. Vol. 1. Mechanics.* Third edition. Pergamon Press, Oxford-New York-Toronto, Ont., 1976.

[20] C. Lin, H. Kogelnik and L. G. Cohen, *Optical pulse equalization and low dispersion transmission in singlemode fibers in the 1.3–1.7 μm spectral region.* Opt. Lett. **5** (1980), 476–478.

[21] P. M. Lushnikov, *Dispersion-managed soliton in a strong dispersion map limit,* Opt. Lett. **26** (2001), 1535–1537.

[22] P. M. Lushnikov, *Oscillating tails of dispersion managed soliton,* JOSA B **21** (2004), 1913–1918.

[23] P. V. Mamyshev and N. A. Mamysheva, *Pulseoverlapped dispersion-managed data transmission and intrachannel four-wave mixing,* Opt. Lett. **24** (1999), 1454–1456.

[24] F. Mitschke, *Fiber Optics: Physics and Technology,* Springer Verlag, Berlin Heidelberg 2009, 301 pages.

[25] L. F. Mollenauer, A. Grant, X. Liu, X. Wei, C. Xie and I. Kang, *Experimental test of dense wavelengthdivision multiplexing using novel, periodic-group-delaycomplemented dispersion compensation and dispersionmanaged solitons.* Opt. Lett. **28** (2003), 2043–2045.

[26] L. F. Mollenauer, P. V. Mamyshev, J. Gripp, M. J. Neubelt, N. Mamysheva, L. Grüner-Nielsen and T. Veng, *Demonstration of massive wavelength-division multiplexing over transoceanic distances by use of dispersionmanaged solitons.* Opt. Lett. **25** (1999), 704–706.

[27] J. V. Moloney and A. C. Newell, *Nonlinear optics,* Westview Press, 2003, 440 pages.

[28] M. Stanislavova, *Regularity of ground state solutions of DMNLS equations.* J. Diff. Eq. **210** (2005), 87–105.

[29] C. Sulem and P.-L. Sulem, *The non-linear Schrödinger equation. Self-focusing and wave collapse.* Applied Mathematical Sciences, **139**. Springer-Verlag, New York, 1999.

[30] T. Tao, *Multilinear weighted convolution of L^2-functions, and applications to nonlinear dispersive equations.* Amer. J. Math. **123** (2001), no. 5, 839–908.

[31] S. K. Turitsyn, N. J. Doran, J. H. B. Nijhof, V. K. Mezentsev, T. Schäfer and W. Forysiak, in *Optical Solitons: Theoretical challenges and industrial perspectives,* V.E. Zakharov and S. Wabnitz, eds. (Springer Verlag, Berlin, 1999), p. 91.

[32] S. K. Turitsyn, E. G. Shapiro, S. B. Medvedev, M. P. Fedoruk and V. K. Mezentsev, *Physics and mathematics of dispersion-managed optical solitons,* Comptes Rendus Physique, Académie des sciences/Éditions scientifiques et médicales **4** (2003), 145–161.

[33] V. Zharnitsky, E. Grenier, C.K.R.T. Jones and S.K. Turitsyn, *Stabilizing effects of dispersion management,* Physica D **152-153** (2001), 794–817.

Department of Mathematics, Altgeld Hall, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801.

E-mail address: berdogan@math.uiuc.edu

Department of Mathematics, Altgeld Hall, and Institute for Condensed Matter Theory, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801.

E-mail address: dirk@math.uiuc.edu

Department of Mathematics, Sogang University, Shinsu-dong 1, Mapo-gu, Seoul, 121-742, South Korea.

E-mail address: younglee@sogang.ac.kr