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# ASYMPTOTIC GROWTH OF SATURATED POWERS AND EPSILON MULTIPLICITY

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## 1. Introduction

In this paper, we study the growth of saturated powers of modules. In the case of an ideal $I$ in a local ring $(R,\mathfrak{m})$, the saturation of $I^k$ in $R$ is

$$(I^k)^{\rm sat} = I^k :_R \mathfrak{m}^\infty = \cup_{n=1}^\infty I^k :_R \mathfrak{m}^n.$$

There are examples showing that the algebra of saturated powers of $I$, $\bigoplus_{k\ge 0}(I^k)^{\rm sat}$ is not a finitely generated $R$-algebra; for instance, in many cases the saturated powers are the symbolic powers. As such, it cannot be expected that the "Hilbert function", giving the length of the $R$-module $(I^k)^{\rm sat}/I^k$, is very well behaved for large $k$. However, it can be shown that it is bounded above by a polynomial in $k$ of degree $d$, where $d$ is the dimension of $R$. We show that in many cases, there is a reasonable asymptotic behavior of this length.

Suppose that $(R,\mathfrak{m})$ is a Noetherian local domain of dimension $d\ge 1$. Let $L$ be the quotient field of $R$. Let $\lambda(M)$ denote the length of an $R$-module $M$. Let $F$ be a finitely generated free $R$-module, and let $E$ be a submodule of $F$ of rank $e$. Let $S=R[F]={\rm Sym}(F)=\bigoplus_{k\ge 0}F^k$ and let $R[E]=\bigoplus_{k\ge0}E^k$ be the $R$-subalgebra of $S$ generated by $E$. Let

$$E^k :_{F^k} \mathfrak{m}^\infty = \cup_{n=1}^\infty E^k :_{F^k} \mathfrak{m}^n$$

denote the saturation of $E^k$ in $F^k$. We prove the following theorem:

**Theorem 1.1.** *Suppose that $(R,\mathfrak{m})$ is a local domain of depth $\ge 2$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/\mathfrak{m}$ is algebraic over $K$). Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit*

$$\lim_{k\to\infty}\frac{\lambda(E^k :_{F^k} \mathfrak{m}^\infty/E^k)}{k^{d+e-1}}\in\mathbb{R} \qquad (1)$$

*exists.*

The conclusions of this theorem follow from Theorem 3.2 and Remark 3.3.

Theorem 1.1 is proven in the case when $E=I$ is a homogeneous ideal and $R$ is a standard graded normal $K$-algebra in our paper [3] with Hà, Srinivasan and Theodorescu. The theorem is proven with the additional assumptions that $R$ is regular, $E=I$ is an ideal in $F=R$, and the singular locus of ${\rm Spec}(R/I)$ is $\mathfrak{m}$ in our paper [4] with Herzog and Srinivasan. Kleiman [13] has proven Theorem 1.1 in the case that $E$ is

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a direct summand of $F$ locally at every nonmaximal prime of $R$. The theorem is proven for $E$ of low analytic deviation in [4], for the case of ideals, and by Ulrich and Validashti [19] for the case of modules; in the case of low analytic deviation, the limit is always zero. A generalization of this problem to the case of saturations with respect to non $\mathfrak{m}$-primary ideals is investigated by Herzog, Puthenpurakal and Verma in [10]; they show that an appropriate limit exists for monomial ideals.

An example in [3] shows that even in the case when $E$ is an ideal $I$ in a regular local ring $R$, the limit may be irrational.

An important technique in the proof of Theorem 1.1 is to use a theorem of Lazarsfeld [14] showing that the volume of a line bundle on a complex projective variety can be expressed as a limit of numbers of global sections of powers of the line bundle; Lazarsfeld's theorem is deduced from an approximation theorem of Fujita [6] (generalizations of Fujita's result to positive characteristic are given in [17] and [15]).

We can interpret our results in terms of local cohomology. Let $F_L^k = F^k \otimes_R L$, where $L$ is the quotient field of $R$, so that we have natural embeddings $E^k \subset F^k \subset F_L^k$ for all $k$. We have identities

$$H^0_{\mathfrak{m}}(F^k/E^k) \cong E^k :_{F^k} \mathfrak{m}^{\infty}/E^k \text{ and } H^1_{\mathfrak{m}}(E^k) \cong E^k :_{F_L^k} \mathfrak{m}^{\infty}/E^k.$$

Further, these two modules are equal if $R$ has depth $\geq 2$.

We thus obtain the following corollary to Theorem 1.1, which shows that the epsilon multiplicity $\varepsilon(E)$ of a module, defined as a limsup in [19], actually exists as a limit.

**Corollary 1.2.** *Suppose that $(R, \mathfrak{m})$ is a local domain of depth $\geq 2$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/\mathfrak{m}$ is algebraic over $K$). Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit*

$$\lim_{k\to\infty} \frac{(d+e-1)!}{k^{d+e-1}} \lambda(H^0_{\mathfrak{m}}(F^k/E^k)) \in \mathbb{R}$$

*exists. Thus the epsilon multiplicity $\varepsilon(E)$ of $E$ exists as a limit.*

By the above identities of local cohomology, we see that (1) is equivalent to the statement that

$$\lim_{k\to\infty} \frac{H^0_{\mathfrak{m}}(F^k/E^k)}{k^{d+e-1}} = \lim_{k\to\infty} \frac{H^1_{\mathfrak{m}}(E^k)}{k^{d+e-1}} \in \mathbb{R} \tag{2}$$

exists when $\operatorname{depth}(R) \geq 2$.

In Section 4, we extend our results to domains of dimension $d \geq 2$. We prove the following extension of Theorem 1.1, which shows that the second limit of (2),

$$\lim_{k\to\infty} \frac{H^1_{\mathfrak{m}}(E^k)}{k^{d+e-1}} \in \mathbb{R}$$

exists when $R$ is a domain of dimension $d \geq 2$.

**Theorem 1.3.** *Suppose that $(R, \mathfrak{m})$ is a local domain of dimension $d \geq 2$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/\mathfrak{m}$ is algebraic over $K$). Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit*

$$\lim_{k\to\infty} \frac{\lambda\left(E^k :_{F_L^k} \mathfrak{m}^{\infty}/E^k\right)}{k^{d+e-1}} \in \mathbb{R} \tag{3}$$

*exists.*

Theorem 1.3 follows from Theorem 4.1 and equations (24) and (6). We prove that the first limit of (2),

$$\lim_{k\to\infty}\frac{H^0_{\mathfrak{m}}(F^k/E^k)}{k^{d+e-1}}\in\mathbb{R}$$

exists when $R$ is a domain of dimension $d\ge 2$ and $E$ is embedded in $F$ of rank $< d+e$. I thank Craig Huneke, Bernd Ulrich and Javid Validashti for pointing out this interesting consequence of Theorem 1.3.

**Corollary 1.4.** *Suppose that $(R,\mathfrak{m})$ is a local domain of dimension $d\ge 2$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/\mathfrak{m}$ is algebraic over $K$). Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Suppose that $\gamma=\operatorname{rank}(F)<d+e$. Then the limits*

$$\lim_{k\to\infty}\frac{\lambda\left(E^k:_{F^k}\mathfrak{m}^\infty/E^k\right)}{k^{d+e-1}}\in\mathbb{R} \tag{4}$$

*and*

$$\lim_{k\to\infty}\frac{(d+e-1)!}{k^{d+e-1}}\lambda(H^0_{\mathfrak{m}}(F^k/E^k))\in\mathbb{R} \tag{5}$$

*exist. In particular, the epsilon multiplicity $\varepsilon(E)$ of $E$ exists as a limit.*

In the case when $e=1$ and $F=R$, we get the following statement.

**Corollary 1.5.** *Suppose that $(R,\mathfrak{m})$ is a local domain of dimension $d\ge 1$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/\mathfrak{m}$ is algebraic over $K$). Suppose that $I$ is an ideal in $R$. Let $(I^k)^{\rm sat}=I^k:_R\mathfrak{m}^\infty$ be the saturation of $I^k$. Then the limit*

$$\lim_{k\to\infty}\frac{\lambda((I^k)^{\rm sat}/I^k)}{k^d}\in\mathbb{R}$$

*exists.*

Asymptotic polynomial like behavior of the length of extension functions is studied by Katz and Theodorescu [12], Theodorescu [18] and Crabbe, Katz, Striuli and Theodorescu [2]. By the local duality theorem, we obtain the following corollary to Theorem 1.1.

**Corollary 1.6.** *Suppose that $(R,\mathfrak{m})$ is a Gorenstein local domain of dimension $d\ge 2$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/\mathfrak{m}$ is algebraic over $K$). Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit*

$$\lim_{k\to\infty}\frac{\lambda(\operatorname{Ext}^d_R(F^k/E^k,R))}{k^{d+e-1}}\in\mathbb{R}$$

*exists.*

## 2. Preliminaries

Suppose that $(R,\mathfrak{m})$ is a Noetherian local domain of dimension $d\ge 1$ with quotient field $L$. Let $\lambda_R(M)$ denote the length of an $R$-module $M$. When there is no danger of confusion, we will denote $\lambda_R(M)$ by $\lambda(M)$.

Let $F$ be a finitely generated free $R$-module of rank $\gamma$, and let $E$ be a submodule of $F$ of rank $e$. Let $S=R[F]=\mathrm{Sym}(F)=\bigoplus_{k\ge 0}F^k$, and let $R[E]=\bigoplus_{k\ge 0}E^k$ be the $R$-subalgebra of $S$ generated by $E$. Let

$$E^k:_{F^k}\mathfrak{m}^\infty=\cup_{n=1}^\infty E^k:_{F^k}\mathfrak{m}^n$$

denote the saturation of $E^k$ in $F^k$.

Let $F^k_L=F^k\otimes_R L$ (where $L$ is the quotient field of $R$), so that we have natural embeddings $E^k\subset F^k\subset F^k_L$ for all $k$. Let $X=\mathrm{Spec}(R)$, $\widetilde{E^k}$ be the sheafification of $E$ on $X$ and let $u_1,\ldots,u_s$ be generators of the ideal $\mathfrak{m}$.

There are identities

$$H^0(X\setminus\{\mathfrak{m}\},\widetilde{E^k})=\cap_{i=1}^s(E^k)_{u_i}=E^k:_{F^k_L}\mathfrak{m}^\infty. \tag{6}$$

From the exact sequence of cohomology groups

$$0\to H^0_{\mathfrak{m}}(E^k)\to E^k\to H^0_{\mathfrak{m}}(X\setminus\{\mathfrak{m}\},\widetilde{E^k})\to H^1_{\mathfrak{m}}(E^k)\to 0,$$

we deduce that we have isomorphisms of $R$-modules

$$H^1_{\mathfrak{m}}(E^k)\cong E^k:_{F^k_L}\mathfrak{m}^\infty/E^k \tag{7}$$

for $k\ge 0$. The same calculation for $F^k$ shows that

$$H^1_{\mathfrak{m}}(F^k)\cong F^k:_{F^k_L}\mathfrak{m}^\infty/F^k. \tag{8}$$

From the left exact local cohomology sequence

$$0\to H^0_{\mathfrak{m}}(F^k/E^k)\to H^1_{\mathfrak{m}}(E^k)\to H^1_{\mathfrak{m}}(F^k),$$

we have that

$$H^0_{\mathfrak{m}}(F^k/E^k)\cong\left(E^k:_{F^k_L}\mathfrak{m}^\infty)\cap F^k\right)/E^k=E^k:_{F^k}\mathfrak{m}^\infty/E^k. \tag{9}$$

From (6), and the fact that $F^k$ is a free $R$-module, we have that $H^0(X\setminus\{\mathfrak{m}\},\widetilde{F^k})=F^k$ and

$$E^k:_{F^k_L}\mathfrak{m}^\infty=E^k:_{F^k}\mathfrak{m}^\infty\text{ if }R\text{ has depth}\ge 2. \tag{10}$$

Let $ES$ be the ideal of $S$ generated by $E$. We compute the degree $n$ part of $(ES)^n$ from the formula

$$\left[(ES)^n\right]_n=E^n. \tag{11}$$

Let $R[\mathfrak{m}E]=\bigoplus_{n\ge 0}(\mathfrak{m}E)^n$ be the $R$-subalgebra of $S$ generated by $\mathfrak{m}E$.

Let $X=\mathrm{Spec}(R)$, $Y=\mathrm{Proj}(R[\mathfrak{m}E])$ and $Z=\mathrm{Proj}(R[E])$.

Write $R[E]=R[\overline{x}_1,\ldots,\overline{x}_t]$ as a standard graded $R$-algebra, with deg $\overline{x}_i=1$ for all $i$. For $1\le i\le t$, let

$$R_i=R[\frac{\overline{x}_1}{\overline{x}_i},\ldots,\frac{\overline{x}_t}{\overline{x}_i}],$$

and let $V_i = \mathrm{Spec}(R_i)$ for $1 \le i \le t$. $\{V_i\}$ is an affine cover of $Z$. Let $u_1, \ldots, u_s$ be generators of the ideal $\mathfrak{m}$. For $1 \le i \le s$ and $1 \le j \le t$, let

$$R_{i,j} = R[\frac{u_\alpha \overline{x}_\beta}{u_i \overline{x}_j} \mid 1 \le \alpha \le s, 1 \le \beta \le t],$$

and $U_{i,j} = \mathrm{Spec}(R_{i,j})$. Then $\{U_{i,j}\}$ is an affine cover of $Y$. Since

$$R_j[\frac{u_1}{u_i}, \ldots, \frac{u_s}{u_i}] = R_{i,j},$$

we see that $Y$ is the blow up of the ideal sheaf $\mathfrak{m}\mathcal{O}_Z$.

The structure morphism $f : Y \to X$ factors as a sequence of projective morphisms

$$Y \xrightarrow{g} Z \xrightarrow{h} X,$$

where $Y$ is the blow up the ideal sheaf $\mathfrak{m}\mathcal{O}_Z$. Define line bundles on $Y$ by $\mathcal{L} = g^*\mathcal{O}_Z(1)$ and $\mathcal{M} = \mathfrak{m}\mathcal{O}_Y$. Then $\mathcal{O}_Y(1) \cong \mathcal{M} \otimes \mathcal{L}$.

We have $\mathcal{O}_Z(1)|V_j = \overline{x}_j\mathcal{O}_{V_j}$, $\mathcal{L}|U_{i,j} = \overline{x}_j\mathcal{O}_{U_{i,j}}$ and $\mathcal{M}|U_{i,j} = u_i\mathcal{O}_{U_{i,j}}$.

We give three consequences (Proposition 2.1, Proposition 2.2 and Corollary 2.3) of Serre's fundamental theorem for projective morphisms which will be useful.

**Proposition 2.1.** *$\bigoplus_{k\ge0} H^i(Y, \mathcal{L}^k)$ are finitely generated $R[E]$-modules for all $i \in \mathbb{N}$.*

*Proof.* Let $\widetilde{E^k}$ be the sheafication of $E^k$ on $X$. From the natural surjections for $k \ge 0$ of $\mathcal{O}_Z$-modules $g^*(\widetilde{E^k}) \to \mathcal{O}_Z(k)$, we obtain surjections $f^*(\widetilde{E^k}) \to \mathcal{L}^k$ of $\mathcal{O}_Y$-modules, and a surjection $f^*(\bigoplus_{k\ge0} \widetilde{E^k}) \to \bigoplus_{k\ge0} \mathcal{L}^k$. Hence $\bigoplus_{k\ge0} \mathcal{L}^k$ is a finitely generated $f^*(\bigoplus_{k\ge0} \widetilde{E^k})$-module. By Theorem III.2.4.1 [8], $R^i f_*(\bigoplus_{k\ge0} \mathcal{L}^k)$ is a finitely generated $\bigoplus_{k\ge0} \widetilde{E_k}$-module for $i \in \mathbb{N}$. Taking global sections on the affine $X$, we obtain the conclusions of the proposition. □

**Proposition 2.2.** *Suppose that $A$ is a Noetherian ring, and $B = \bigoplus_{k\ge0} B_k$ is a finitely generated graded $A$-algebra, which is generated by $B_1$ as an $A$-algebra. Let $C = Spec(A)$ and $D = Proj(B)$. Let $\alpha : D \to C$ be the structure morphism. Then there exists a positive integer $\overline{k}$ such that $B_k = \Gamma(D, \mathcal{O}_D(k))$ for $k \ge \overline{k}$.*

*Proof.* The ring $\bigoplus_{k\ge0} \Gamma(D, \mathcal{O}_D(k))$ is a finitely generated graded $B$-module by Theorem III.2.4.1 [8]. Hence $(\bigoplus_{k\ge0} \Gamma(D, \mathcal{O}_D(k)))/B$ is a finitely generated graded $B$-module. Since every element of this module is $B_+ = \bigoplus_{k>0} B_k$ torsion, we have that $B_k/E_k = 0$ for $k \gg 0$. □

Taking the maximum over the $\overline{k}$ obtained from the above proposition applied to a finite affine cover of $W$, we obtain the following generalization of Proposition 2.2.

**Corollary 2.3.** *Suppose that $W$ is a Noetherian scheme and $\mathcal{B} = \bigoplus_{k\ge0} \mathcal{B}_k$ is a finitely generated graded $\mathcal{O}_W$-algebra, which is locally generated by $\mathcal{B}_1$ as a $\mathcal{O}_W$-algebra. Let $W' = Proj(\mathcal{B})$ and let $\alpha : W' \to W$ be the structure morphism. Then there exists a positive integer $\overline{k}$ such that $\mathcal{B}_k = \alpha_*\mathcal{O}_{W'}(k)$ for $k \ge \overline{k}$,*

## 3. Asymptotic Growth

**Proposition 3.1.** *Let $(R,\mathfrak{m})$ be a local domain of depth $\ge 2$. Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ $R$-submodule of a finitely generated free $R$-module $F$. Let notation be as above. Then there exist positive integers $k_0$, $k_1$ and $\tau$ such that*

1) *for $k \ge k_0$, $n \in \mathbb{Z}$ and $\mathfrak{p} \in X \setminus \{\mathfrak{m}\}$,*
$$\Gamma(Y, \mathcal{M}^n \otimes \mathcal{L}^k)_{\mathfrak{p}} = (E^k)_{\mathfrak{p}}.$$
2) *For $k \ge k_1$,*
$$E^k :_{F^k} \mathfrak{m}^\infty = \Gamma(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k).$$

*Proof.* We first establish 1). $U_i = \mathrm{Spec}(R_{u_i})$ for $1 \le i \le s$ is an affine cover of $X \setminus \{\mathfrak{m}\}$. $g|f^{-1}(U_i)$ is an isomorphism; in fact
$$f^{-1}(U_i) = \mathrm{Proj}(R[\mathfrak{m}E]_{u_i}) = \mathrm{Proj}(R[E]_{u_i}) = h^{-1}(U_i).$$
By Proposition 2.2, there exist positive integers $a_i$ such that
$$\Gamma(f^{-1}(U_i), \mathcal{M}^{-n} \otimes \mathcal{L}^k) = \Gamma(h^{-1}(U_i), \mathcal{O}_Z(k)) = (E^k)_{u_i}$$
for $k \ge a_i$. Let $k_0 = \max\{a_1, \ldots, a_s\}$. Then for $\mathfrak{p} \in U_i$
$$\Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k)_{\mathfrak{p}} = \Gamma(f^{-1}(U_i), \mathcal{M}^n \otimes \mathcal{L}^k)_{\mathfrak{p}} = (E^k)_{\mathfrak{p}}$$
for $k \ge k_0$, establishing 1).

We now establish 2). Suppose that $n \ge 0$, and $k \ge 0$. Suppose that $\sigma \in E^k :_{F^k} \mathfrak{m}^n$. Let $i, j$ be such that $1 \le i \le s$ and $1 \le j \le t$. $\sigma\mathfrak{m}^n \subset E^k$ implies $u_i^n\sigma \in E^k$ which implies there is an expansion
$$u_i^n \sigma = \sum_{n_1 + \cdots + n_t = k} r_{n_1,\ldots,n_t} \overline{x}_1^{n_1} \cdots \overline{x}_t^{n_t}$$
with $r_{n_1,\ldots,n_t} \in R$. Thus
$$u_i^n \sigma = \overline{x}_j^k \left( \sum_{n_1+\cdots+n_t=k} r_{n_1,\ldots,n_t} (\frac{\overline{x}_1}{\overline{x}_j})^{n_1} \cdots (\frac{\overline{x}_t}{\overline{x}_j})^{n_t} \right),$$
so that $\sigma \in u_i^{-n}\overline{x}_j^k R_{i,j}$. Thus
$$\sigma \in \cap_{i,j} u_i^{-n} \overline{x}_j^k R_{i,j} = \Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k).$$
We have established that for $k \ge 0$ and $n \ge 0$,
$$E^k :_{F^k} \mathfrak{m}^n \subset \Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k). \tag{12}$$

Recall that $S$ is a polynomial ring $S = R[y_1, \ldots, y_\gamma]$ over $R$, where $\gamma$ is the rank of $F$. Let $W = \mathrm{Proj}(S)$, with natural morphism $\alpha : W \to X$. Let $\mathcal{I}$ be the sheafication of the graded ideal $ES$ on $W$. We have expansions
$$\overline{x}_i = \sum_{l=1}^{\gamma} f_{il} y_l$$
with $f_{il} \in R$.

The inclusion $R[E] \subset S$ induces a rational map from $W$ to $Z$.

Let $\beta : W' \to W$ be the blow up of the ideal sheaf $\mathcal{I}$. Let $\mathcal{N} = \mathcal{I}\mathcal{O}_{W'}$ be the induced line bundle. $W'$ has an affine cover $A_{i,j} = \mathrm{Spec}(T_{ij})$ for $1 \le i \le t$ and $1 \le j \le \gamma$ with

$$T_{ij} = R[\frac{y_1}{y_j}, \ldots, \frac{y_\gamma}{y_j}][\frac{\overline{x}_1}{\overline{x}_i}, \ldots, \frac{\overline{x}_t}{\overline{x}_i}].$$

From the inclusions

$$R_i = R[\frac{\overline{x}_1}{\overline{x}_i}, \ldots, \frac{\overline{x}_t}{\overline{x}_i}] \subset T_{ij}$$

we have induced morphisms $A_{i,j} \to V_i = \mathrm{Spec}(R_i)$ which patch to give a morphism $\varphi : W' \to Z$ which is a resolution of indeterminacy of the rational map from $W$ to $Z$.

We calculate for all $i, j$,

$$\varphi^*(\mathcal{O}_Z(1)) \mid A_{i,j} = \overline{x}_i \mathcal{O}_{A_{ij}} = y_j(\sum_l f_{i,l}\frac{y_l}{y_j})\mathcal{O}_{A_{ij}} = (\beta^*\mathcal{O}_W(1))\,\mathcal{I}|A_{i,j},$$

to see that

$$(\beta^*\mathcal{O}_W(1)) \otimes \mathcal{N} \cong \varphi^*\mathcal{O}_Z(1).$$

By Corollary 2.3, there exists a positive integer $k_1 \ge k_0$ such that $\beta_*\mathcal{N}^k = \mathcal{I}^k$ for $k \ge k_1$. From the natural inclusion $\mathcal{O}_Z(k) \subset \varphi_*\varphi^*\mathcal{O}_Z(k)$, we have by the projection formula that for $k \ge k_1$,

$$\begin{aligned} h_*\mathcal{O}_Z(k) &\subset h_*\varphi_*(\varphi^*\mathcal{O}_Z(k)) = \alpha_*\beta_*(\beta^*\mathcal{O}_W(k) \otimes \mathcal{N}^k) \\ &= \alpha_*[\mathcal{O}_W(k) \otimes \beta_*\mathcal{N}^k] = \alpha_*[\mathcal{O}_W(k) \otimes \mathcal{I}^k] \\ &\subset \alpha_*\mathcal{O}_W(k) = \widetilde{F^k}, \end{aligned} \tag{13}$$

where $\widetilde{F^k}$ is the sheafication of the $R$-module $F$ on $X$. Now we have

$$\begin{aligned} \Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k) &= \Gamma(X, f_*(\mathcal{M}^{-n} \otimes \mathcal{L}^k)) \\ &\subset \Gamma(X \setminus \{\mathfrak{m}\}, f_*(\mathcal{M}^{-n} \otimes \mathcal{L}^k)) = \Gamma(X \setminus \{\mathfrak{m}\}, h_*\mathcal{O}_Z(k)) \\ &\subset \Gamma(X \setminus \{\mathfrak{m}\}, \widetilde{F^k}) = F^k \end{aligned} \tag{14}$$

since $R$, and hence the free $R$-module $F^k$, have depth $\ge 2$.

From (11), we deduce that for $k, n \ge 0$,

$$\big((ES)^k :_S \mathfrak{m}^n S\big) \cap F^k = E^k :_{F^k} \mathfrak{m}^n. \tag{15}$$

By 1.5 [11] or Theorem 1.3 [16], there exists a positive integer $\tau$ such that

$$(ES)^k :_S \mathfrak{m}^{k\tau} S = (ES)^k :_S (\mathfrak{m}S)^\infty$$

for all $k \ge 0$. Thus from (15) we have that

$$E^k :_{F^k} \mathfrak{m}^{k\tau} = E^k :_{F^k} \mathfrak{m}^\infty \tag{16}$$

for $k \ge 0$. From (16), (12) and (14), we have inclusions

$$E^k :_{F^k} \mathfrak{m}^\infty \subset \Gamma(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k) \subset F^k$$

for $k \ge k_1$. The conclusions of 2) of the proposition now follow from 1) of the proposition since $E^k :_{F^k} \mathfrak{m}^\infty$ is the largest $R$-submodule $N$ of $F^k$ which has the property that $N_\mathfrak{p} = (E^k)_\mathfrak{p}$ for $\mathfrak{p} \in X \setminus \{\mathfrak{m}\}$.

□

**Theorem 3.2.** *Suppose that $(R,\mathfrak{m})$ is a local domain of depth $\geq 2$ which is essentially of finite type over a field $K$ of characteristic zero. Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit*

$$\lim_{k\to\infty}\frac{\lambda\left(E^k :_{F^k} \mathfrak{m}^\infty/E^k\right)}{k^{d+e-1}}\in\mathbb{R}$$

*exists.*

*Proof.* Let notation be as above.

First consider the short exact sequences

$$0\to\Gamma(Y,\mathcal{L}^k)/E^k\to E^k:_{F^k}\mathfrak{m}^\infty/E^k\to E^k:_{F^k}\mathfrak{m}^\infty/\Gamma(Y,\mathcal{L}^k)\to 0. \tag{17}$$

$\bigoplus_{k\geq 0}\Gamma(Y,\mathcal{L}^k)$ is a finitely generated $R[E]$-module by Lemma 2.1. By 1) of Proposition 3.1, the support of the $R$-module $\Gamma(Y,\mathcal{L}^k)/E^k$ is contained in $\{\mathfrak{m}\}$ for all $k$. Since $(\bigoplus_{k\geq 0}\Gamma(Y,\mathcal{L}^k))/R[E]$ is a finitely generated $R[E]$-module, there is a positive integer $r$ such that $\mathfrak{m}^r(\Gamma(Y,\mathcal{L}^k)/E^k)=0$ for all $k$. Since $\dim R[E]/\mathfrak{m}R[E]\leq\dim R+\text{rank } E-1=d+e-1$, and $R/\mathfrak{m}^r$ is an Artin local ring, we have that $\lambda(\Gamma(Y,\mathcal{L}^k)/E^k)$ is a polynomial of degree less than or equal to $d+e-2$ for $k\gg 0$ by the Hilbert-Serre theorem. Thus there exists a constant $\alpha$ such that $\lambda(\Gamma(Y,\mathcal{L}^k)/E^k)\leq\alpha k^{d+e-2}$ for all $k$. From (17), we are now reduced to showing that the limit

$$\lim_{k\to\infty}\frac{\lambda(E^k:_{F^k}\mathfrak{m}^\infty/\Gamma(Y,\mathcal{L}^k))}{k^{d+e-1}}$$

exists, from which we will have

$$\lim_{k\to\infty}\frac{\lambda(E^k:_{F^k}\mathfrak{m}^\infty/\Gamma(Y,\mathcal{L}^k))}{k^{d+e-1}}=\lim_{k\to\infty}\frac{\lambda(E^k:_{F^k}\mathfrak{m}^\infty/E^k)}{k^{d+e-1}}. \tag{18}$$

Taking global sections of the short exact sequences

$$0\to\mathcal{L}^k\to\mathcal{M}^{-k\tau}\otimes\mathcal{L}^k\to\mathcal{M}^{-k\tau}\otimes\mathcal{L}^k\otimes(\mathcal{O}_Y/\mathfrak{m}^{k\tau}\mathcal{O}_Y)\to 0,$$

we obtain by Proposition 3.1 left exact sequences

$$0\to E^k:_{F^k}\mathfrak{m}^\infty/\Gamma(Y,\mathcal{L}^k)\to\Gamma(Y,\mathcal{M}^{-k\tau}\otimes\mathcal{L}^k\otimes(\mathcal{O}_Y/\mathfrak{m}^{k\tau}\mathcal{O}_Y))\to H^1(Y,\mathcal{L}^k) \tag{19}$$

for $k\geq k_1$.

Let $u_1,\ldots,u_s$ be generators of the ideal $\mathfrak{m}$, and set $U_i=\text{Spec}(R_{u_i})$, so that $\{U_1,\ldots,U_s\}$ is an affine cover of $X\setminus\{\mathfrak{m}\}$. Then $\mathcal{L}|f^{-1}(U_i)$ is ample, so there exist positive integers $b_i$ such that $R^1f_*(\mathcal{L}^k)\mid U_i=0$ for $k\geq b_i$. Let $k_2=\max\{b_1,\ldots,b_s\}$. We have that the support of $H^1(Y,\mathcal{L}^k)$ is contained in $\{\mathfrak{m}\}$ for $k\geq k_2$.

$\bigoplus_{k\geq 0}H^1(Y,\mathcal{L}^k)$ is a finitely generated $R[E]$-module by Lemma 2.1. Hence the submodule $M=\bigoplus_{k\geq k_2}H^1(Y,\mathcal{L}^k)$ is a finitely generated graded $R[E]$-module. We have that $\mathfrak{m}^rM=0$ for some positive integer $r$. Since

$$\dim R[E]/\mathfrak{m}R[E]\leq\dim R+\text{rank } E-1=d+e-1,$$

and $R/\mathfrak{m}^r$ is an Artin local ring, we have that $\lambda(H^1(Y,\mathcal{L}^k))$ is a polynomial of degree less than or equal to $d+e-2$ for $k\gg 0$ by the Hilbert-Serre theorem. Thus there exists a constant $c$ such that

$$\lambda(H^1(Y,\mathcal{L}^k))\leq ck^{d+e-2}$$

for all $k \geq 0$. By consideration of (18) and (19), we are reduced to proving that the limit

$$\lim_{k\to\infty} \frac{\lambda(H^0(Y,\mathcal{M}^{-k\tau}\otimes\mathcal{L}^k\otimes\mathcal{O}_Y/\mathfrak{m}^{k\tau}\mathcal{O}_Y))}{k^{d+e-1}} \tag{20}$$

exists.

If $R/\mathfrak{m}$ is algebraic over $K$, let $K' = K$. If $R/\mathfrak{m}$ is transcendental over $K$, let $t_1,\ldots,t_r$ be a lift of a transcendence basis of $R/\mathfrak{m}$ over $K$ to $R$. The rational function field $K(t_1,\ldots,t_r)$ is contained in $R$. Let $K' = K(t_1,\ldots,t_r)$. We have that $R/\mathfrak{m}$ is finite algebraic over $K'$.

There exists an affine $K'$-variety $X' = \mathrm{Spec}(A)$ such that $R$ is the local ring of a closed point $\alpha$ of $X'$, and $E$ extends to a submodule $E'$ of $A^\gamma$, where $\gamma$ is the rank of the free $R$-module $F$. We then have an inclusion of graded $A$-algebras $A[E'] \subset \mathrm{Sym}(A^\gamma)$ which extends $R[E]$. Identify $\mathfrak{m}$ with its extension to a maximal ideal of $A$. The structure morphism $Y' = \mathrm{Proj}(A[\mathfrak{m}E']) \to X'$ is projective and its localization at $\mathfrak{m}$ is $f : Y \to X$. Let $\overline{X}$ be a projective closure of $X'$ and let $\tilde{Y}$ be a projective closure of $Y'$. $X'$ is an open subset of $\overline{X}$ and $Y'$ is an open subset of $\tilde{Y}$. Let $\overline{Y} \to \tilde{Y}$ be the blow up of an ideal sheaf which gives a resolution of indeterminancy of the rational map from $\tilde{Y}$ to $\overline{X}$. We may assume that the morphism $\overline{Y} \to \tilde{Y}$ is an isomorphism over the locus where the rational map is a morphism, and thus an isomorphism over the subset $Y'$ of $\tilde{Y}$. Let $\overline{f} : \overline{Y} \to \overline{X}$ be the resulting morphism. We now establish that $\overline{f}^{-1}(X') = Y'$. Suppose that $p \in X'$ and $q \in \overline{f}^{-1}(p)$. Let $V$ be a valuation ring of the function field $L$ of $\overline{Y}$ (which is also the function field of $Y'$) which dominates the local ring $\mathcal{O}_{\overline{Y},q}$. By assumption, $V$ dominates the local ring $\mathcal{O}_{X',p}$. $V$ dominates the local ring of a point on $Y'$, by the valuative criterion for properness (Theorem II.4.7 [9]) applied to the proper morphism $Y' \to X'$. Since $V$ dominates the local ring of a unique point on $\overline{Y}$, we have that $q \in Y'$.

After possibly replacing $\overline{Y}$ with the blow up of an ideal sheaf on $\overline{Y}$ whose support is disjoint from $Y'$, we may assume that $\mathcal{L}$ extends to a line bundle on $\overline{Y}$ which we will also denote by $\mathcal{L}$. We will identify $\mathfrak{m}$ with its extension to the ideal sheaf of the point $\alpha$ on $\overline{X}$, and identify $\mathcal{M}$ with its extension $\mathfrak{m}\mathcal{O}_{\overline{Y}}$ to a line bundle on $\overline{Y}$. Let $\mathcal{A}$ be an ample divisor on $\overline{X}$. Then there exists $l > 0$ such that $\mathcal{C} = \overline{f}^*(\mathcal{A}^l) \otimes \mathcal{L}$ is generated by global sections and is big.

Set $\mathcal{B} = \mathcal{C} \otimes \mathcal{M}^{-\tau}$. Tensor the short exact sequences

$$0 \to \mathcal{M}^{k\tau} \to \mathcal{O}_{\overline{Y}} \to \mathcal{O}_{\overline{Y}}/\mathfrak{m}^{k\tau}\mathcal{O}_{\overline{Y}} \cong \mathcal{O}_Y/\mathfrak{m}^{k\tau}\mathcal{O}_Y \to 0$$

with $\mathcal{B}^k$ to obtain the short exact sequences

$$0 \to \mathcal{C}^k \to \mathcal{B}^k \to \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/\,\mathfrak{m}^{k\tau}\mathcal{O}_Y \to 0$$

for $k \geq 0$. Taking global sections, we have exact sequences

$$0 \to H^0(\overline{Y},\mathcal{C}^k) \to H^0(\overline{Y},\mathcal{B}^k) \to H^0(Y,\mathcal{M}^{-k\tau}\otimes\mathcal{L}^k\otimes\mathcal{O}_Y/\,\mathfrak{m}^{k\tau}\mathcal{O}_Y) \to H^1(\overline{Y},\mathcal{C}^k). \tag{21}$$

For a coherent sheaf $\mathcal{F}$ on $\overline{Y}$, let

$$h^i(\overline{Y},\mathcal{F}) = \dim_{K'} H^i(\overline{Y},\mathcal{F}).$$

Since $\mathcal{C}$ is semiample (generated by global sections and big) and $\overline{Y}$ has dimension $d+e-1$, we have that

$$\lim_{k\to\infty}\frac{h^1(\overline{Y},\mathcal{C}^k)}{k^{d+e-1}}=0.$$

This follows for instance from [5]. Since $\bigoplus_{k\geq 0}H^0(\overline{Y},\mathcal{C}^k)$ is a finitely generated $K'$ algebra of dimension $d+e$, as $\mathcal{C}$ is generated by global sections and is big (or by the Riemann Roch theorem and the vanishing theorem of [5]) we have that the limit

$$\lim_{k\to\infty}\frac{h^0(\overline{Y},\mathcal{C}^k)}{k^{d+e-1}}\in\mathbb{Q}$$

exists. Since $\mathcal{B}$ is big, by the corollary to [6] given in Example 11.4.7 [14] or [3], we have that the limit

$$\lim_{k\to\infty}\frac{h^0(\overline{Y},\mathcal{B}^k)}{k^{d+e-1}}\in\mathbb{R}$$

exists. From the sequence (21), we see that

$$\lim_{k\to\infty}\frac{h^0(Y,\mathcal{M}^{-k\tau}\otimes\mathcal{L}^k\otimes\mathcal{O}_Y/\,\mathfrak{m}^{k\tau}\mathcal{O}_Y)}{k^{d+e-1}}\in\mathbb{R}$$

exists. The conclusions of the theorem now follow from (20) and the formula

$$\begin{aligned}&h^0(Y,\mathcal{M}^{-k\tau}\otimes\mathcal{L}^k\otimes\mathcal{O}_Y/\,\mathfrak{m}^{k\tau}\mathcal{O}_Y)=\dim_{K'}H^0(Y,\mathcal{M}^{-k\tau}\otimes\mathcal{L}^k\otimes\mathcal{O}_Y/\,\mathfrak{m}^{k\tau}\mathcal{O}_Y)\\&\quad=[R/\mathfrak{m}:K']\lambda(H^0(Y,\mathcal{M}^{-k\tau}\otimes\mathcal{L}^k\otimes\mathcal{O}_Y/\,\mathfrak{m}^{k\tau}\mathcal{O}_Y)).\end{aligned}$$

□

**Remark 3.3.** *The conclusions of Theorem 3.2 are also true if $K$ is a perfect field of positive characteristic and $R/\mathfrak{m}$ is algebraic over $K$. In this case we have that $K'=K$ in the proof of Theorem 3.2. Let $\overline{K}$ be an algebraic closure of $K$. Since $K$ is perfect, $\overline{Y}\times_K\overline{K}$ is reduced, and to compute the limit, we reduce to computing the sections of the pullback of $\mathcal{B}^k$ on the disjoint union of the irreducible (integral) components of $\overline{Y}\times_K\overline{K}$. Fujita's approximation theorem is valid on varieties over an algebraically closed field of positive characteristic, as was shown by Takagi* [17], *from which the existence of the limit now follows.*

**Remark 3.4.** *Theorem 3.2 is proven for graded ideals in* [3]. *An example where the limit is an irrational number is given in* [3]. *The theorem is proven with the additional assumptions that $R$ is regular, $E=I$ is an ideal in $F=R$, and the singular locus of Spec$(R/I)$ is $\mathfrak{m}$ in* [4]. *Kleiman* [13] *has proven Theorem 3.2 in the case that $E$ is a direct summand of $F$ locally at every nonmaximal prime of $R$.*

**Corollary 3.5.** *Suppose that $(R,\mathfrak{m})$ is a local domain of depth $\geq 2$ which is essentially of finite type over a field $K$ of characteristic zero. Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit*

$$\lim_{k\to\infty}\frac{(d+e-1)!}{k^{d+e-1}}\lambda(H^0_{\mathfrak{m}}(F^k/E^k))\in\mathbb{R}$$

*exists. Thus the epsilon multiplicity $\varepsilon(E)$ of the module $E$, defined in* [19] *as a limsup, actually exists as a limit.*

The example of [3] shows that $\varepsilon(E)$ may be an irrational number.

*Proof.* The corollary is immediate from Theorem 3.2 and (9). □

**Remark 3.6.** *The conclusions of Corollary 3.5 are valid if $K$ is a perfect field of positive characteristic and $R/\mathfrak{m}$ is algebraic over $K$, by Remark 3.3.*

## 4. Extension to domains of dimension $\geq 2$.

In this section, we prove extensions of Theorem 1.1 and Corollary 1.2 to domains of dimension $\geq 2$. Let notation be as in Section 2.

Suppose that $R$ is a domain of dimension $d \geq 2$ with a dualizing module. By the Theorem of Finiteness, Theorem VIII.2.1 (and footnote) [7],

$$\overline{R} = \Gamma(X \setminus \{\mathfrak{m}\}, \mathcal{O}_X) = \cap_{\mathfrak{p} \in X \setminus \{\mathfrak{m}\}} R_{\mathfrak{p}} \tag{22}$$

is a finitely generated $R$-module, which lies between $R$ and its quotient field. Since $\overline{R}/R$ is $\mathfrak{m}$-torsion,

$$\lambda_R(\overline{R}/R) < \infty. \tag{23}$$

Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_\alpha$ be the maximal ideals of $\overline{R}$ which lie over $\mathfrak{m}$. By our construction,

$$0 = H^1_{\mathfrak{m}}(\overline{R}) = H^1_{\mathfrak{m}\overline{R}}(\overline{R}) = \bigoplus_{i=1}^{\alpha} H^1_{\mathfrak{m}_i\overline{R}}(\overline{R}),$$

so

$$H^1_{\mathfrak{m}_i\overline{R}_{\mathfrak{m}_i}}(\overline{R}_{\mathfrak{m}_i}) = H^1_{\mathfrak{m}_i\overline{R}}(\overline{R}) \otimes_{\overline{R}} \overline{R}_{\mathfrak{m}_i} = 0$$

for $1 \leq i \leq \alpha$, and thus $\text{depth}(\overline{R}_{\mathfrak{m}_i}) \geq 2$ for $1 \leq i \leq \alpha$.

Let $\overline{F} = F \otimes_R \overline{R}$ and $\overline{R}[\overline{F}] = \bigoplus_{k\geq 0} \overline{F}^k$, so that $\overline{F}^k \cong F^k \otimes_R \overline{R}$ for all $k$. Let $\overline{E} = \overline{R}E$ be the $\overline{R}$-submodule of $\overline{F}$ generated by $E$. Let $\overline{R}[\overline{E}] = \bigoplus_{k\geq 0} \overline{E}^k$ be the $\overline{R}$-subalgebra of $\overline{R}[\overline{F}]$ generated by $\overline{E}$.

Let $u_1, \ldots, u_s$ be generators of the ideal $\mathfrak{m}$. For $k \in \mathbb{N}$, let $\widetilde{E^k}$ be the sheafification of $E^k$ on $X = \text{Spec}(R)$.

There are identities

$$H^0(X \setminus \{\mathfrak{m}\}, \widetilde{E^k}) = \cap_{i=1}^{s} (E^k)_{u_i} = E^k :_{\overline{F}^k} \mathfrak{m}^\infty. \tag{24}$$

From the exact sequence of cohomology groups

$$0 \to H^0_{\mathfrak{m}}(E^k) \to E^k \to H^0_{\mathfrak{m}}(X \setminus \{\mathfrak{m}\}, \widetilde{E^k}) \to H^1_{\mathfrak{m}}(E^k) \to 0,$$

we deduce that we have isomorphisms of $R$-modules

$$H^1_{\mathfrak{m}}(E^k) \cong E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k \tag{25}$$

for $k \geq 0$. The same calculation for $F^k$ shows that

$$H^1_{\mathfrak{m}}(F^k) \cong F^k :_{\overline{F}^k} \mathfrak{m}^\infty / F^k. \tag{26}$$

From the left exact local cohomology sequence

$$0 \to H^0_{\mathfrak{m}}(F^k/E^k) \to H^1_{\mathfrak{m}}(E^k) \to H^1_{\mathfrak{m}}(F^k),$$

we have that

$$H^0_{\mathfrak{m}}(F^k/E^k) \cong \left(E^k :_{\overline{F}^k} \mathfrak{m}^\infty) \cap F^k\right) / E^k = E^k :_{F^k} \mathfrak{m}^\infty / E^k. \tag{27}$$

**Theorem 4.1.** *Suppose that $(R,\mathfrak{m})$ is a local domain of dimension $d\ge 2$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/\mathfrak{m}$ is algebraic over $K$). Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit*

$$\lim_{k\to\infty}\frac{\lambda\left(E^k :_{\overline{F}^k} \mathfrak{m}^\infty/E^k\right)}{k^{d+e-1}}\in\mathbb{R} \tag{28}$$

*exists.*

*Proof.* Since $\overline{E}^k :_{\overline{F}^k} \mathfrak{m}^\infty/\overline{E}^k$ are finitely generated $\mathfrak{m}\overline{R}$-torsion $\overline{R}$-modules, we have that

$$\overline{E}^k :_{\overline{F}^k} \mathfrak{m}^\infty/\overline{E}^k \cong \bigoplus_{i=1}^{\alpha}\left(\overline{E}^k_{\mathfrak{m}_i} :_{\overline{F}^k_{\mathfrak{m}_i}} \mathfrak{m}_i^\infty/\overline{E}^k_{\mathfrak{m}_i}\right).$$

By Theorem 1.1, we have that

$$\lim_{k\to\infty}\frac{\lambda_{\overline{R}_{\mathfrak{m}_i}}\left(\overline{E}^k_{\mathfrak{m}_i} :_{\overline{F}^k_{\mathfrak{m}_i}} \mathfrak{m}_i^\infty/\overline{E}^k_{\mathfrak{m}_i}\right)}{k^{d+e-1}}$$

exists for $1\le i\le\alpha$. Since for any $\overline{R}_{\mathfrak{m}_i}$ module $M$ we have that

$$\lambda_R(M)=[\overline{R}/\mathfrak{m}_i : R/\mathfrak{m}]\lambda_{\overline{R}_{\mathfrak{m}_i}}(M),$$

we conclude that

$$\lim_{k\to\infty}\frac{\lambda_R(\overline{E}^k :_{\overline{F}^k} \mathfrak{m}^\infty/\overline{E}^k)}{k^{d+e-1}} \tag{29}$$

exists. We have

$$\overline{E}^k :_{\overline{F}^k} \mathfrak{m}^\infty = \cap_{i=1}^s(\overline{E}^k)_{u_i} = \cap_{i=1}^s(E^k)_{u_i} = E^k :_{\overline{F}^k} \mathfrak{m}^\infty.$$

Consider the short exact sequences

$$0\to \overline{E}^k/E^k \to E^k :_{\overline{F}^k} \mathfrak{m}^\infty/E^k \to \overline{E}^k :_{\overline{F}^k} \mathfrak{m}^\infty/\overline{E}^k \to 0. \tag{30}$$

Now $\overline{R}[\overline{E}]/R[E]$ is a finitely generated $R[E]$-module, and the support of the $R$-module $\overline{E}^k/E^k$ is contained in $\{\mathfrak{m}\}$ for all $k$, so there exists a positive integer $r$ such that $\mathfrak{m}^r$ annihilates $\overline{R}[\overline{E}]/R[E]$. Thus (as in the argument following equation (17) in the proof of Theorem 3.2), we have that there exists a constant $\beta$ such that

$$\lambda_R(\overline{E}^k/E^k)\le \beta k^{d+e-2} \tag{31}$$

for all $k$. The conclusions of the proposition now follow from (29), (31) and (30). □

I thank Craig Huneke, Bernd Ulrich and Javid Validashti for pointing out the following consequence of Theorem 4.1.

**Corollary 4.2.** *Suppose that $(R,\mathfrak{m})$ is a local domain of dimension $d\ge 2$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/\mathfrak{m}$ is algebraic over $K$). Suppose that $E$ is a rank $e$ submodule of a* *finitely generated free $R$-module $F$. Suppose that $\gamma = \operatorname{rank}(F) < d+e$. Then the limits*

$$\lim_{k\to\infty} \frac{\lambda\left(E^k :_{F^k} \mathfrak{m}^\infty / E^k\right)}{k^{d+e-1}} \in \mathbb{R} \tag{32}$$

*and*

$$\lim_{k\to\infty} \frac{(d+e-1)!}{k^{d+e-1}} \lambda(H^0_{\mathfrak{m}}(F^k/E^k)) \in \mathbb{R} \tag{33}$$

*exist. In particular, the epsilon multiplicity $\varepsilon(E)$ of $E$ exists as a limit.*

*Proof.* We will establish that the limit (32) exists. We have exact sequences

$$0 \to E^k :_{F^k} \mathfrak{m}^\infty / E^k \to E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k \to E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k :_{F^k} \mathfrak{m}^\infty \to 0 \tag{34}$$

and inclusions

$$E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k :_{F^k} \mathfrak{m}^\infty = E^k :_{\overline{F}^k} \mathfrak{m}^\infty / \left( (E^k :_{\overline{F}^k} \mathfrak{m}^\infty) \cap F^k \right) \to F^k :_{\overline{F}^k} \mathfrak{m}^\infty / F^k \tag{35}$$

for $k \geq 0$.

We have

$$F^k :_{\overline{F}^k} \mathfrak{m}^\infty / F^k = \overline{F}^k / F^k \cong (\overline{R}/R)^{\binom{k+\gamma-1}{\gamma-1}}. \tag{36}$$

Since $\gamma = \operatorname{rank}(F) < d+e$, we have

$$\lim_{k\to\infty} \frac{\lambda_R\left(E^k :_{\overline{F}^k} \mathfrak{m}^\infty / E^k :_{F^k} \mathfrak{m}^\infty\right)}{k^{d+e-1}} = 0.$$

The existence of the limit (32) now follows from (34) and Theorem 4.1. The existence of the limit (33) is immediate from (32) and (27). □

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