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THE p-ADIC LEFSCHETZ $(1,1)$ THEOREM IN THE SEMISTABLE CASE, AND THE PICARD NUMBER JUMPING LOCUS

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Abstract. We prove a semistable version of the so-called p-adic Lefschetz $(1,1)$ theorem. As an application, we show a generalization of the Maulik-Poonen result on Picard number jumping locus.

0. Introduction.

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ whose residue field k is perfect. Let X be a proper semistable model over $S := \operatorname{Spec} O_K$, X_K be its generic fiber $X \otimes_{O_K} K$, and Y be its special fiber $X \otimes_{O_K} k$. For an element $[L] \in \operatorname{Pic}^{\log}(Y) := H^1(Y, (\mathcal{O}_Y^\times)^{\log}) := H^1(Y, M_Y^{\mathrm{gp}})$ or $[L] \in \operatorname{Pic}(Y)$, we have a log-crystalline first Chern class

$$c_{\mathrm{crys}}([L]) \in H^2_{\mathrm{crys}}((Y, M_Y)/(W, N^0)),$$

where W is the ring of Witt vectors with coefficient k, and M_Y and N^0 are log-structures on Y and $\operatorname{Spec} W$ respectively (the precise meaning of notations will be explained later). We also have Hyodo-Kato isomorphism ([HK])

$$\rho_\pi : H^m_{\mathrm{crys}}((Y, M_Y)/(W, N^0)) \otimes_W K \cong H^m_{\mathrm{dR}}(X_K/K).$$

This isomorphism depends on the choice of a uniformizer $\pi \in K$. However, we can show that $\rho_\pi(c_{\mathrm{crys}}([L]))$ is independent of the choice of π (Corollary 2.3). In this paper, we first show the following generalization of the Berthelot-Ogus theorem ([BO, Theorem (3.8)]):

Theorem 0.1. (=*Theorem 3.1*) *The element* $[L] \in \operatorname{Pic}^{\log}(Y)_{\mathbb{Q}}$ (*resp.* $[L] \in \operatorname{Pic}(Y)_{\mathbb{Q}}$) *lifts to* $\operatorname{Pic}^{\log}(X)_{\mathbb{Q}}$ (*resp.* $\operatorname{Pic}(X)_{\mathbb{Q}}$), *if and only if*

$$\rho_\pi(c_{\mathrm{crys}}([L])) \in H^2_{\mathrm{dR}}(X_K/K)$$

is in $F^1 H^2_{\mathrm{dR}}(X_K/K)$.

Next, by using this theorem, we deduce a generalization of the Maulik-Poonen result ([MP]) (see Section 4 for precise meaning of the notations):

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Theorem 0.2. (=*Theorem 4.3*) *Let C be the completion of an algebraic closure of K, and O_C be its valuation ring. Let B be an irreducible separated O_K-scheme of finite type, and let $f : \mathcal{X} \to B$ be a proper semistable morphism such that $f_K : \mathcal{X}_K \to B_K$ is smooth (i.e., for any point $b \in B(O_K)$, the pull-back $\mathcal{X}_b \to \operatorname{Spec} O_K$ is a semistable model of the proper smooth variety $\mathcal{X}_b \otimes_{O_K} K$), where $()_K$'s mean $K \otimes_{O_K}$.*

Then, the set

$$B(O_C)_{\mathrm{jumping}} := \{b \in B(O_C) \mid \rho(\mathcal{X}_b \otimes_{O_C} \overline{k}) > \rho(\mathcal{X}_{\bar{\eta}})\}$$

is nowhere dense in $B(O_C)$ for the analytic topology, where ρ's mean (log) Picard numbers.

In Section 1, we review Hyodo-Kato isomorphism, introducing some notations. In Section 2, we study log-crystalline first Chern class and de Rham first Chern class. In Section 3, we prove Theorem 0.1. In Section 4, we prove Theorem 0.2 by using Theorem 0.1.

Notations.

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ whose residue field k is perfect. Let O_K denote the valuation ring of K. Let W be the ring of Witt vectors with coefficient k, and K_0 be its fraction field. Let σ denote the Frobenius on W or K_0. We use the convention that the subscript $()_n$ of rings, schemes, log-structures, etc. means $\otimes \mathbb{Z}/p^n\mathbb{Z}$. Here, "log-structure" means the Fontaine-Illusie-Kato log-structure ([K1]). *When we use the word "log-structure" in this paper, then this means fine saturated log-structure.*

Let X be a proper semistable model over $S := \operatorname{Spec} O_K$, X_K be its generic fiber $X \otimes_{O_K} K$, and Y be its special fiber $X \otimes_{O_K} k$. Let M be the log-structure on X defined by the special fiber Y, and M_Y be the pull-back of M to Y. Let N be the log-structure on S defined by its special fiber $\operatorname{Spec} k$. Let N_1 be the pull-back of N to $\operatorname{Spec} k$, and N_n^0 be the log-structure on $\operatorname{Spec} W_n$ associated to $\Gamma(\operatorname{Spec} k, N_1) \to k \stackrel{[\cdot]}{\to} W_n$, where $[\cdot]$ is the Teichmüller representative. We have a natural exact closed immersion $(\operatorname{Spec} W_n, N_n^0) \hookrightarrow (\operatorname{Spec} W_{n+1}, N_{n+1}^0)$.

We will use the PD-structures γ on (W_n, pW_n) and $(S_n, p\mathcal{O}_{S,n})$ etc. induced by the unique PD-structure on (W, pW) and $(S, p\mathcal{O}_S)$ etc. We abbreviate the log-crystalline site $((Y, M_Y)/(W_n, N_n^0, pW_n, \gamma))^{\log}_{\mathrm{crys}}$ as $((Y, M_Y)/(W_n, N_n^0))^{\log}_{\mathrm{crys}}$ etc. If we abbreviate the coefficient of crystalline cohomology, then it means that the coefficient is the structure sheaf.

1. Review of Hyodo-Kato isomorphism.

First, we briefly review the Hyodo-Kato isomorphism ([HK]). We also prepare some notations. Choose a uniformizer π of K. Let (V, M_V) be the scheme $\operatorname{Spec} W[t]$ with log-structure defined by the divisor "$t = 0$". Let F_V be the Frobenius lift defined by $t \mapsto t^p$ and σ. We have an W-exact closed immersion $i_{V,\pi} : (S, N) \hookrightarrow (V, M_V)$ defined by $t \mapsto \pi$. Let

$$i_{E_n,\pi} : (S_n, N_n) \hookrightarrow (E_n, M_{E_n})$$

be the PD-envelope of $i_{V_n,\pi} := (i_{V,\pi})_n$. The lifting $F_{V_n} := (F_V)_n$ induces a lifting of Frobenius $F_{E_n} : (E_n, M_{E_n}) \to (E_n, M_{E_n})$ The ring $R_{E_n} := \Gamma(E_n, \mathcal{O}_{E_n})$ is isomorphic to $W[t, t^{me}/m!(m \geq 1)] \otimes_W W_n$, where $e := [K : K_0]$. Put $R_E := \varprojlim_n R_{E_n}$. We also have an W_n-exact closed immersion $i_{V_n,0} : (\operatorname{Spec} W_n, N_n^0) \hookrightarrow (V_n, M_{V_n})$ defined by $t \mapsto 0$. It induces a W_n-exact closed immersion

$$i_{E_n,0} : (\operatorname{Spec} W_n, N_n^0) \hookrightarrow (E_n, M_{E_n}).$$

We have a commutative diagram of exact closed immersions:

$$\begin{array}{ccc} (\operatorname{Spec} k, N_1^0) & \hookrightarrow & (S_n, N_n) \\ \big\downarrow & & \big\downarrow {\scriptstyle i_{E_n,\pi}} \\ (\operatorname{Spec} W_n, N_n^0) & \overset{i_{E_n,0}}{\hookrightarrow} & (E_n, M_{E_n}) \end{array}$$

The morphism $i_{E_n,\pi}$ induces a homomorphism

$$\mathrm{pr}_\pi : H^m_{\mathrm{crys}}((X_n, M_n)/(E_n, M_{E_n})) \to H^m_{\mathrm{crys}}((X_n, M_n)/(S_n, N_n)).$$

The natural morphism $(Y, M_Y) \hookrightarrow (X_n, M_n)$ and $i_{E_n,0}$ induce a homomorphism

$$\mathrm{pr}_0 : H^m_{\mathrm{crys}}((X_n, M_n)/(E_n, M_{E_n})) \to H^m_{\mathrm{crys}}((Y, M_Y)/(W_n, N_n^0)).$$

Here we have

$$H^m_{\mathrm{crys}}((X_n, M_n)/(E_n, M_{E_n})) \cong H^m_{\mathrm{crys}}((X_1, M_1)/(E_n, M_{E_n}))$$

and $H^m_{\mathrm{crys}}((Y, M_Y)/(W_n, N_n^0))$ are equipped with σ-semilinear endomorphism φ, which are induced by absolute Frobenius of X_1 and Y and lifting of Frobenius F_{E_n} and σ respectively. Hyodo-Kato ([HK]) proved φ is a bijection after $\mathbb{Q} \otimes \varprojlim_n$ (for more general log-schemes) by using the theory of de Rham-Witt complex (In the case of crystalline cohomology, Berthelot-Ogus showed the analogous statement by the theory of "gauges". These bijections were essential for proving Berthelot-Ogus isomorphism and Hyodo-Kato isomorphism respectively). Put

$$H^m_{\mathrm{crys}}((Y, M_Y)/(W, N^0)) := \varprojlim_n H^m_{\mathrm{crys}}((Y, M_Y)/(W_n, N_n^0)),$$

$$H^m_{\mathrm{crys}}((X, M)/(S, N)) := \varprojlim_n H^m_{\mathrm{crys}}((X_n, M_n)/(S_n, N_n)),$$

$$H^m_{\mathrm{crys}}((X, M)/(E, M_E)) := \varprojlim_n H^m_{\mathrm{crys}}((X_n, M_n)/(E_n, M_{E_n})).$$

We have ([K1, Theorem (6.4)])

$$H^m_{\mathrm{crys}}((X_n, M_n)/(S_n, N_n)) \cong \mathbb{H}^m(X_n, \Omega^\bullet_{X_n/S_n}(\log(M_n/N_n))),$$

where $\mathbb{H}$ means a hypercohomology of a complex. So, we have

$$H^m_{\mathrm{crys}}((X, M)/(S, N)) \cong \mathbb{H}^m(X, \Omega^\bullet_{X/S}(\log(M/N)))$$

since X is proper over S. Thus, we have

$$\mathbb{Q} \otimes H^m_{\mathrm{crys}}((X, M)/(S, N)) \cong \mathbb{H}^m(X_K, \Omega^\bullet_{X_K/K}) =: H^m_{\mathrm{dR}}(X_K/K).$$

Lemma (5.2) of [HK] or Proposition 4.4.6 of [Tsu] tell us the existence of a K_0-linear section of $\mathbb{Q} \otimes \mathrm{pr}_0$ compatible with φ:

$$s : H^m_{\mathrm{crys}}((Y, M_Y)/(W, N^0)) \to \mathbb{Q} \otimes H^m_{\mathrm{crys}}((X, M)/(E, M_E)),$$

which induces an R_E-isomorphism:

$$R_E \otimes_W H^m_{\text{crys}}((Y, M_Y)/(W, N^0)) \xrightarrow{\sim} \mathbb{Q} \otimes H^m_{\text{crys}}((X, M)/(E, M_E)).$$

(Here, we need $\mathbb{Q}\otimes$ to get the section s, because we need the inverse of φ^r, where r is an integer satisfying $\pi^{p^r} \in pO_K$. They used that the r-powered absolute Frobenius $F^r_{S_1} : S_1 \to S_1$ factors through $S_1 \xrightarrow{f} \operatorname{Spec} k \to S_1$, and $X_1 \times_{S_1, F^r_{X_1}} S_1$ is isomorphic to $Y \times_{\operatorname{Spec} k, f} S_1$. See the proof of Lemma 2.5.) By using s and $\mathbb{Q} \otimes \mathrm{pr}_\pi$, we have a homomorphism

$$\mathbb{Q} \otimes \mathrm{pr}_\pi \circ s : H^m_{\text{crys}}((Y, M_Y)/(W, N^0)) \to H^m_{\text{dR}}(X_K/K).$$

the Hyodo-Kato theorem ([HK, Theorem (5.1)]) tells us that this induces a K-isomorphism

$$\rho_\pi : K \otimes_W H^m_{\text{crys}}((Y, M_Y)/(W, N^0)) \xrightarrow{\sim} H^m_{\text{dR}}(X_K/K).$$

2. log-crystalline and de Rham first Chern Classes.

Secondly, I recall the definitions of crystalline first Chern class and de Rham Chern class.

Let $(\mathcal{O}_X^\times)^{\log}$ and $(\mathcal{O}_Y^\times)^{\log}$ denote M^{gp} and M_Y^{gp} respectively (they can be regarded as log-version of multiplicative group). Put $\mathrm{Pic}^{\log}(X)$ and $\mathrm{Pic}^{\log}(Y)$ to be $H^1(X, (\mathcal{O}_X^\times)^{\log})$ and $H^1(Y, (\mathcal{O}_Y^\times)^{\log})$ respectively. The inclusions $\mathcal{O}_X^\times \hookrightarrow (\mathcal{O}_X^\times)^{\log}$ and $\mathcal{O}_Y^\times \hookrightarrow (\mathcal{O}_Y^\times)^{\log}$ induce $\mathrm{Pic}(X) \to \mathrm{Pic}^{\log}(X)$ and $\mathrm{Pic}(Y) \to \mathrm{Pic}^{\log}(Y)$ respectively. The inclusions $\mathcal{O}_Y^\times \hookrightarrow (\mathcal{O}_Y^\times)^{\log}$ and $\mathcal{O}_X^\times \hookrightarrow (\mathcal{O}_X^\times)^{\log}$ induce $\mathrm{Pic}(Y) \to \mathrm{Pic}^{\log}(Y)$ and $\mathrm{Pic}(X) \to \mathrm{Pic}^{\log}(X)$ respectively. By definition, we have $M^{\text{gp}} = \mathcal{O}_{X_K}^\times$. So, $\mathrm{Pic}^{\log}(X)$ is nothing but $\mathrm{Pic}(X_K)$. When we choose a uniformizer, we have an isomorphism $(\mathcal{O}_Y^\times)^{\log} \cong j_*\mathcal{O}_{Y^{\text{sm}}}^\times \cdot \pi^{\mathbb{Z}}$, where $Y^{\text{sm}}(\overset{j}{\hookrightarrow} Y)$ is the smooth locus of Y, and we regard $\pi(\neq 0)$ as living in M_Y, which goes to 0 in $\mathcal{O}_Y$. So, we have an isomorphism $\mathrm{Pic}^{\log}(Y) \cong \mathrm{Pic}(Y^{\text{sm}})$. This isomorphism is independent of the choice of π, since changing π by $u\pi$ ($u \in O_K^\times$) gives only a difference of principal divisor $(u) = 0$. Let $\mathrm{NS}(X_K)$ be $\mathrm{Pic}(X_K)/\mathrm{Pic}^0(X_K)$, where $\mathrm{Pic}^0(X_K)$ is the isomorphism class of line bundles of algebraically equivalent to 0. Let $\mathrm{Pic}^{\log,0}(Y)$ be the pull back of $\mathrm{Pic}^0(Y^{\text{sm}})$ under the isomorphism $\mathrm{Pic}^{\log}(Y) \cong \mathrm{Pic}(Y^{\text{sm}})$, and $\mathrm{NS}^{\log}(Y)$ be $\mathrm{Pic}^{\log}(Y)/\mathrm{Pic}^{\log,0}(Y)$. We call $\rho(X_K) := \mathrm{rank}\mathrm{NS}(X_K)$, and $\rho(Y) := \mathrm{rank}\mathrm{NS}^{\log}(Y)$, their Picard number. The homomorphism $\mathrm{Pic}(X_K) = \mathrm{Pic}^{\log}(X) \to \mathrm{Pic}^{\log}(Y)$ induces a homomorphism

$$\mathrm{NS}(X_K) \to \mathrm{NS}^{\log}(Y). \tag{1}$$

Let $(\mathcal{O}^\times_{(Y,M_Y)/(W_n,N_n^0)})^{\log}$ be a sheaf on the log-crystalline site

$$((Y, M_Y)/(W_n, N_n^0, pW_n, \gamma))^{\log}_{\text{crys}}$$

defined by $(\delta, (U, M_Y|_U) \hookrightarrow (T, M_T)) \mapsto M_T^{\text{gp}}$. Let

$$i_{(Y,M_Y)/(W_n,N_n^0)} : Y_{\text{ét}} \to ((Y, M_Y)/(W_n, N_n^0))^{\log}_{\text{crys}}$$

be the canonical morphism (we use similar notations $i_{(-,-)/(-,-)}$ for log-schemes).

Lemma 2.1. *We have exact sequences*

$$0 \to 1 + J_{(Y,M_Y)/(W_n,N_n^0)} \to (\mathcal{O}^{\times}_{(Y,M_Y)/(W_n,N_n^0)})^{\log} \to i_{(Y,M_Y)/(W_n,N_n^0),*}(\mathcal{O}_Y^{\times})^{\log} \to 0,$$

and

$$0 \to 1 + J_{(Y,M_Y)/(W_n,N_n^0)} \to \mathcal{O}^{\times}_{(Y,M_Y)/(W_n,N_n^0)} \to i_{(Y,M_Y)/(W_n,N_n^0),*}\mathcal{O}_Y^{\times} \to 0$$

on $(((Y,M_Y)/(W_n,N_n^0))^{\log}_{\mathrm{crys}})^{\sim}$.

Proof. The second one is classical. The first one comes from the exactness of the closed immersion $(U, M_Y|_U) \hookrightarrow (T, M_T)$ for any

$$(\delta, (U, M_Y|_U) \hookrightarrow (T, M_T)) \in ((Y,M_Y)/(W_n,N_n^0))^{\log}_{\mathrm{crys}}.$$

□

By using PD-structure on $J_{(Y,M_Y)/(W_n,N_n^0)}$, we have an isomorphism

$$\log : 1 + J_{(Y,M_Y)/(W_n,N_n^0)} \xrightarrow{\sim} J_{(Y,M_Y)/(W_n,N_n^0)}.$$

So, by combining the boundary map of Lemma 2.1, log and the inclusion

$$i : J_{(Y,M_Y)/(W_n,N_n^0)} \hookrightarrow \mathcal{O}_{(Y,M_Y)/(W_n,N_n^0)},$$

we get homomorphisms

$$\begin{aligned}\mathrm{Pic}^{\log}(Y) &\xrightarrow{\partial} H^2_{\mathrm{crys}}((Y,M_Y)/(W_n,N_n^0), 1 + J_{(Y,M_Y)/(W_n,N_n^0)})\\ &\overset{\log}{\cong} H^2_{\mathrm{crys}}((Y,M_Y)/(W_n,N_n^0), J_{(Y,M_Y)/(W_n,N_n^0)})\\ &\xrightarrow{i} H^2_{\mathrm{crys}}((Y,M_Y)/(W_n,N_n^0)),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Pic}(Y) &\xrightarrow{\partial} H^2_{\mathrm{crys}}((Y,M_Y)/(W_n,N_n^0), 1 + J_{(Y,M_Y)/(W_n,N_n^0)})\\ &\overset{\log}{\cong} H^2_{\mathrm{crys}}((Y,M_Y)/(W_n,N_n^0), J_{(Y,M_Y)/(W_n,N_n^0)})\\ &\xrightarrow{i} H^2_{\mathrm{crys}}((Y,M_Y)/(W_n,N_n^0)).\end{aligned}$$

We call them log-crystalline first Chern class homomorphism, and let c_{crys} denote them. Here,

$$\begin{array}{ccc}\mathrm{Pic}(Y) & \longrightarrow & \mathrm{Pic}^{\log}(Y)\\ & \underset{c_{\mathrm{crys}}}{\searrow} & \downarrow c_{\mathrm{crys}}\\ & & H^2_{\mathrm{crys}}((Y,M_Y)/(W_n,N_n^0))\end{array}$$

is commutative, because

$$\begin{array}{ccc}(\mathcal{O}^{\times}_{(Y,M_Y)/(W_n,N_n^0)})^{\log} & \twoheadrightarrow & i_{(Y,M_Y)/(W_n,N_n^0),*}(\mathcal{O}_Y^{\times})^{\log}\\ \uparrow & & \uparrow\\ \mathcal{O}^{\times}_{(Y,M_Y)/(W_n,N_n^0)} & \twoheadrightarrow & i_{(Y,M_Y)/(W_n,N_n^0),*}\mathcal{O}_Y^{\times}\end{array}$$

is commutative.

Proposition 2.2. *The image of the map*

$$c_{\rm crys}:{\rm Pic}^{\log}(Y)\to H^2_{\rm crys}((Y,M_Y)/(W_n,N^0_n))$$

is in

$$H^2_{\rm crys}((Y,M_Y)/(W_n,N^0_n))^{\varphi=p,\mathcal{N}=0},$$

where $\mathcal{N}$ is the monodromy operator on it ([HK, (3.5)], [Tsu, §4.4]).

Proof. First, we have

$$\varphi(c_{\rm crys}([L]))=c_{\rm crys}(\varphi([L]))=c_{\rm crys}([L^p])=c_{\rm crys}(p[L])=pc_{\rm crys}([L]).$$

Next, we will show that $\mathcal{N}(c_{\rm crys}([L]))=0$. Let $(\operatorname{Spec}W[t],N_{W[t]})$ be a log-scheme whose log-structure is defined by "$t=0$", and $(\operatorname{Spec}W_n\langle t\rangle,N_{W_n\langle t\rangle})$ be the PD-envelope of $(\operatorname{Spec}W_n,N^0_n)\hookrightarrow(\operatorname{Spec}W_n[t],N_{W_n[t]})$. We take an embedding system $(Y^\bullet,M_Y|_{Y^\bullet})\hookrightarrow(Z^\bullet,M_{Z^\bullet})$ for (Y,M_Y) over $(\operatorname{Spec}W[t],N_{W[t]})$. Let $(D^\bullet_n,M_{D^\bullet_n})$ be the PD-envelope of $(Y^\bullet,M_Y|_{Y^\bullet})\hookrightarrow(Z^\bullet_n,M_{Z^\bullet_n})$. Then, we have an exact sequence

$$0\to\mathcal{O}_{D^\bullet_n}\otimes_{\mathcal{O}_{Z^\bullet_n}}\omega^\bullet_{Z^\bullet_n/W_n\langle t\rangle}[-1]\overset{\wedge{\rm dlog}\,(t)}{\to}\mathcal{O}_{D^\bullet_n}\otimes_{\mathcal{O}_{Z^\bullet_n}}\omega^\bullet_{Z^\bullet_n/W_n}\to\mathcal{O}_{D^\bullet_n}\otimes_{\mathcal{O}_{Z^\bullet_n}}\omega^\bullet_{Z^\bullet_n/W_n\langle t\rangle}\to 0$$

on $(Y^\bullet)^\sim_{\rm \acute{e}t}$, where $\omega^\bullet_{Z^\bullet_n/W_n}$ and $\omega^\bullet_{Z^\bullet_n/W_n\langle t\rangle}$ denote

$$\Omega^\bullet_{Z^\bullet_n/W_n}(\log(M_{Z^\bullet_n}))\quad\text{and}\quad\Omega^\bullet_{Z^\bullet_n/W_n\langle t\rangle}(\log(M_{Z^\bullet_n}/N_{W_n\langle t\rangle}))$$

respectively. The boundary homomorphism associated to the above exact sequence after tensoring W_n over $W_n\langle t\rangle$ with respect to $t^{[i]}\mapsto 0$ $(i\geq 0)$ gives

$$\begin{aligned}H^m_{\rm crys}((Y,M_Y)/(W_n,\mathcal{O}^\times_{W_n}))&\to H^m_{\rm crys}((Y,M_Y)/(W_n,N^0_n))\\&\overset{\partial}{\to}H^m_{\rm crys}((Y,M_Y)/(W_n,N^0_n)),\end{aligned}$$

where $\mathcal{O}^\times_{W_n}$ is the trivial log-structure on $\operatorname{Spec}W_n$. The above boundary homomorphism ∂ is the monodromy operator ([HK, (3.6)], [K2, Lemma (4.2)])

$$\mathcal{N}:H^m_{\rm crys}((Y,M_Y)/(W_n,N^0_n))\to H^m_{\rm crys}((Y,M_Y)/(W_n,N^0_n)).$$

So, it suffices to show that there is a lift of

$$c_{\rm crys}:{\rm Pic}^{\log}(Y)\to H^2_{\rm crys}((Y,M_Y)/(W_n,N^0_n))$$

to ${\rm Pic}^{\log}(Y)\to H^2_{\rm crys}((Y,M_Y)/(W_n,\mathcal{O}^\times_{W_n}))$. This can be done by a similar construction of $c_{\rm crys}$, that is, the combination the boundary homomorphism of

$$\begin{aligned}0\to 1+J_{(Y,M_Y)/(W_n,\mathcal{O}^\times_{W_n})}&\to(\mathcal{O}^\times_{(Y,M_Y)/(W_n,\mathcal{O}^\times_{W_n})})^{\log}\\&\to i_{(Y,M_Y)/(W_n,\mathcal{O}^\times_{W_n}),*}(\mathcal{O}^\times_Y)^{\log}\to 0,\end{aligned}$$

the isomorphism $1+J_{(Y,M_Y)/(W_n,\mathcal{O}^\times_{W_n})}\overset{\log}{\cong}J_{(Y,M_Y)/(W_n,\mathcal{O}^\times_{W_n})}$, and the inclusion

$$J_{(Y,M_Y)/(W_n,\mathcal{O}^\times_{W_n})}\hookrightarrow\mathcal{O}_{(Y,M_Y)/(W_n,\mathcal{O}^\times_{W_n})}.$$

The commutativity of the following diagrams

$$\begin{array}{ccc}
(\mathcal{O}^{\times}_{(Y,M_Y)/(W_n,\mathcal{O}^{\times}_{W_n})})^{\log} & \twoheadrightarrow & i_{(Y,M_Y)/(W_n,\mathcal{O}^{\times}_{W_n}),*}(\mathcal{O}^{\times}_Y)^{\log} \\
\downarrow & & \downarrow = \\
\alpha_*(\mathcal{O}^{\times}_{(Y,M_Y)/(W_n,N^0_n)})^{\log} & \twoheadrightarrow & \alpha_* i_{(Y,M_Y)/(W_n,N^0_n),*}(\mathcal{O}^{\times}_Y)^{\log},
\end{array}$$

and

$$\begin{array}{ccc}
1+J_{(Y,M_Y)/(W_n,\mathcal{O}^{\times}_{W_n})} & \xrightarrow[\log]{\sim} & J_{(Y,M_Y)/(W_n,\mathcal{O}^{\times}_{W_n})} \\
\downarrow & & \downarrow \\
\alpha_*(1+J_{(Y,M_Y)/(W_n,N^0_n)}) & \xrightarrow[\log]{\sim} & \alpha_* J_{(Y,M_Y)/(W_n,N^0_n)}
\end{array}$$

ensure that it's a lift of c_{crys}, where

$$\alpha : ((Y,M_Y)/(W_n,N^0_n))^{\log}_{\mathrm{crys}} \to ((Y,M_Y)/(W_n,\mathcal{O}^{\times}_{W_n}))^{\log}_{\mathrm{crys}}$$

is the morphism induced by the log-forgetting morphism

$$(\operatorname{Spec} W_n, N^0_n) \to (\operatorname{Spec} W_n, \mathcal{O}^{\times}_{W_n}).$$

□

Corollary 2.3. *For any* $[L] \in \mathrm{Pic}^{\log}(Y)$*,*

$$\rho_\pi(c_{\mathrm{crys}}([L])) \in H^2_{\mathrm{dR}}(X_K/K)$$

does not depend on the choice of π*.*

Proof. For any $u \in O^{\times}_K$, $\rho_{u\pi}$ is given by

$$\rho_{u\pi} = \rho_\pi \circ \exp(\log(u)\mathcal{N})$$

by [HK, Theorem (5.1)] or [Tsu, Proposition 4.4.17]. Now, $\mathcal{N}c_{\mathrm{crys}}([L]) = 0$ by the Proposition. We are done. □

Next, let's recall the de Rham first Chern class. We have a homomorphism of complexes

$$\mathcal{O}^{\times}_{X_K} \to F^1\Omega^{\bullet}_{X_K/K}[1] \to \Omega^{\bullet}_{X_K/K}[1],$$

where the first map is given by $\mathcal{O}^{\times}_{X_K} \ni f \mapsto df/f \in \Omega^1_{X_K/K}$, and $F^1\Omega^{\bullet}_{X_K/K}$ is the first Hodge filtration $\Omega^{\bullet \geq 1}_{X_K/K}$. So, this homomorphism induces a homomorphism

$$\mathrm{Pic}(X_K) \to H^2(X_K, F^1\Omega^{\bullet}_{X_K/K}) \to H^2_{\mathrm{dR}}(X_K/K).$$

The composite is the first de Rham chern class, and let c_{dR} denote it.

Proposition 2.4. *The following diagram is commutative:*

$$
\begin{array}{ccccc}
\mathrm{Pic}(Y) & \longleftarrow & \mathrm{Pic}(X) & \searrow & \\
\downarrow & & \downarrow & & \\
\mathrm{Pic}^{\log}(Y) & \longleftarrow & \mathrm{Pic}^{\log}(X) & \xrightarrow{=} & \mathrm{Pic}(X_K) \\
\downarrow{\scriptstyle c_{\mathrm{crys}}} & & & & \downarrow{\scriptstyle c_{\mathrm{dR}}} \\
K \otimes_W H^2_{\mathrm{crys}}((Y, M)/(W, N^0)) & & \xrightarrow[\rho_\pi]{\sim} & & H^2_{\mathrm{dR}}(X_K/K).
\end{array}
$$

The non-trivial part is the bottom square.

We needs some preparations. Let $c^{W_n}_{\mathrm{crys}}$ be the composition

$$\mathrm{Pic}^{\log}(X) \to \mathrm{Pic}^{\log}(Y) \overset{c_{\mathrm{crys}}}{\to} H^2_{\mathrm{crys}}((Y, M_Y)/(W_n, N^0_n)).$$

We can define

$$c^{E_n}_{\mathrm{crys}} : \mathrm{Pic}^{\log}(X) \to H^2_{\mathrm{crys}}((X_n, M_n)/(E_n, M_{E_n}))$$

and

$$c^{S_n}_{\mathrm{crys}} : \mathrm{Pic}^{\log}(X) \to H^2_{\mathrm{crys}}((X_n, M_n)/(S_n, N_n))$$

by the same way as c_{crys}. By the commutativity of the following diagrams

$$
\begin{array}{ccc}
(\mathcal{O}^\times_{(X_n,M_n)/(E_n,M_{E_n})})^{\log} & \twoheadrightarrow & i_{(X_n,M_n)/(E_n,M_{E_n}),*}(\mathcal{O}^\times_{X_n})^{\log} \\
\downarrow & & \downarrow \\
(i_{E_n,0})_*(\mathcal{O}^\times_{(Y,M_Y)/(W_n,N^0_n)})^{\log} & \twoheadrightarrow & (i_{E_n,0})_* i_{(Y,M_Y)/(W_n,N^0_n),*}(\mathcal{O}^\times_Y)^{\log},
\end{array}
$$

$$
\begin{array}{ccc}
1 + J_{(X_n,M_n)/(E_n,M_{E_n})} & \xrightarrow[\log]{\sim} & J_{(X_n,M_n)/(E_n,M_{E_n})} \\
\downarrow & & \downarrow \\
(i_{E_n,0})_*(1 + J_{(Y,M_Y)/(W_n,N^0_n)}) & \xrightarrow[\log]{\sim} & (i_{E_n,0})_* J_{(Y,M_Y)/(W_n,N^0_n)},
\end{array}
$$

$$
\begin{array}{ccc}
(\mathcal{O}^\times_{(X_n,M_n)/(E_n,M_{E_n})})^{\log} & \twoheadrightarrow & i_{(X_n,M_n)/(E_n,M_{E_n}),*}(\mathcal{O}^\times_{X_n})^{\log} \\
\downarrow & & \downarrow{\scriptstyle =} \\
(i_{E_n,\pi})_*(\mathcal{O}^\times_{(X_n,M_n)/(S_n,N_n)})^{\log} & \twoheadrightarrow & (i_{E_n,\pi})_* i_{(X_n,M_n)/(S_n,N_n),*}(\mathcal{O}^\times_{X_n})^{\log},
\end{array}
$$

and

$$
\begin{array}{ccc}
1 + J_{(X_n,M_n)/(E_n,M_{E_n})} & \xrightarrow[\log]{\sim} & J_{(X_n,M_n)/(E_n,M_{E_n})} \\
\downarrow & & \downarrow \\
(i_{E_n,\pi})_*(1 + J_{(X_n,M_n)/(S_n,N_n)}) & \xrightarrow[\log]{\sim} & (i_{E_n,\pi})_* J_{(X_n,M_n)/(S_n,N_n)},
\end{array}
$$

the following diagram is commutative:

$$\begin{array}{ccccc} & \swarrow^{c^{W_n}_{\mathrm{crys}}} & \mathrm{Pic}^{\log}(X) & \searrow^{c^{S_n}_{\mathrm{crys}}} & \\ & & \downarrow{\scriptstyle c^{E_n}_{\mathrm{crys}}} & & \\ H^2_{\mathrm{crys}}((Y,M_Y)/(W_n,N_n^0)) & \xleftarrow[\mathrm{pr}_0]{} & H^2_{\mathrm{crys}}((X_n,M_n)/(E_n,M_{E_n})) & \xrightarrow[\mathrm{pr}_\pi]{} & H^2_{\mathrm{crys}}((X_n,M_n)/(S_n,N_n)). \end{array}$$

Here,

$$i_{E_n,0} : ((Y,M_Y)/(W_n,N_n^0))^{\log}_{\mathrm{crys}} \to ((X_n,M_n)/(E_n,M_{E_n}))^{\log}_{\mathrm{crys}},$$

and

$$i_{E_n,\pi} : ((X_n,M_n)/(S_n,N_n))^{\log}_{\mathrm{crys}} \to ((X_n,M_n)/(E_n,M_{E_n}))^{\log}_{\mathrm{crys}}$$

are morphisms induced by $(Y,M_Y) \to (X_n,M_n)$ over $i_{E_n,0} : (\mathrm{Spec}\, W_n, N_n^0) \hookrightarrow (E_n,M_{E_n})$ and $(X_n,M_n) \xrightarrow{\mathrm{id}} (X_n,M_n)$ over $i_{E_n,\pi} : (S_n,N_n) \hookrightarrow (E_n,M_{E_n})$ respectively (we use the same symbols by the abuse of notations. the author hope that there would be no confusions).

Lemma 2.5. *the following diagram is commutative:*

$$\begin{array}{ccc} & \swarrow^{c^{W}_{\mathrm{crys}}} & \mathrm{Pic}^{\log}(X) \\ & & \downarrow{\scriptstyle c^{E}_{\mathrm{crys}}} \\ H^2_{\mathrm{crys}}((Y,M_Y)/(W,N^0)) & \xrightarrow{s} & \mathbb{Q} \otimes H^2_{\mathrm{crys}}((X,M)/(E,M_E)). \end{array}$$

Proof. We recall the definition of the section s ([HK, Lemma (5.2)], [Tsu, Proposition 4.4.6]). Let r be an integer satisfying $\pi^{p^r} \in pO_K$. Then, the r-powered absolute Frobenius $F^r_{S_1} : S_1 \to S_1$ factors through $S_1 \xrightarrow{f} \mathrm{Spec}\, k \to S_1$. Thus, $X_1 \times_{S_1, F^r_{X_1}} S_1$ is isomorphic to $Y \times_{\mathrm{Spec}\, k, f} S_1$. Consider the following commutative diagram:

$$\begin{array}{ccccc} X_1 & \xrightarrow{g} & Y & \hookrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ S_1 & \xrightarrow{f} & \mathrm{Spec}\, k & \hookrightarrow & S_1 \\ \downarrow & & \downarrow & & \downarrow \\ E_n & \xrightarrow{g} & \mathrm{Spec}\, W_n\langle t\rangle & \longrightarrow & E_n, \end{array}$$

where g is defined by $t \mapsto t^{p^r}$ and σ^r, and three composite horizontal arrows are r-powered absolute Frobenii. The left big squre with g's and the morphism

$$(\mathrm{Spec}\, W_n\langle t\rangle, N_{W_n\langle t\rangle}) \to (\mathrm{Spec}\, W_n, N_n^0)$$

defined by $t^{[n]} \mapsto 0$ induce

$$g : ((X_1,M_1)/(E_n,M_{E_n}))^{\log}_{\mathrm{crys}} \to ((Y,M_Y)/(W_n\langle t\rangle, N_{W_n\langle t\rangle}))^{\log}_{\mathrm{crys}}$$

and

$$\beta : ((Y,M_Y)/(W_n\langle t\rangle, N_{W_n\langle t\rangle}))^{\log}_{\mathrm{crys}} \to ((Y,M_Y)/(W_n,N_n^0))^{\log}_{\mathrm{crys}}$$

respectively (we use the same symbol g by the abuse of notations).

Then, we have isomorphisms ([HK, Lemma (5.2), Lemma (5.3)]):

$$
\begin{aligned}
&R_E \otimes_W H^m_{\mathrm{crys}}((Y, M_Y)/(W, N^0))_{\mathbb{Q}} \overset{\sim, \varphi^r}{\longleftarrow} R_E \otimes_{\varphi^r, W} H^m_{\mathrm{crys}}((Y, M_Y)/(W, N^0))_{\mathbb{Q}} \\
&\cong R_E \otimes_{g, W\langle t \rangle} H^m_{\mathrm{crys}}((Y, M_Y)/(W\langle t \rangle, N_{W\langle t \rangle}))_{\mathbb{Q}} \\
&\cong R_E \otimes_{\varphi^r, R_E} H^m_{\mathrm{crys}}((X_1, M_1)/(E, M_E))_{\mathbb{Q}} \\
&\overset{\sim, \varphi^r}{\longrightarrow} H^m_{\mathrm{crys}}((X_1, M_1)/(E, M_E))_{\mathbb{Q}} \cong H^m_{\mathrm{crys}}((X, M)/(E, M_E))_{\mathbb{Q}},
\end{aligned}
$$

where the subscript $\mathbb{Q}$ means $\mathbb{Q}\otimes$. The composition of the above isomorphisms does not depend on r. The section s is the composite of the above isomorphisms and $H^m_{\mathrm{crys}}((Y, M_Y)/(W, N^0)) \to R_E \otimes_W H^m_{\mathrm{crys}}((Y, M_Y)/(W, N^0))_{\mathbb{Q}}$. By the same way as c_{crys}, we can define

$$c^{W_n\langle t \rangle}_{\mathrm{crys}} : \mathrm{Pic}^{\log}(X) \to H^2_{\mathrm{crys}}((Y, M_Y)/(W_n\langle t \rangle, N_{W\langle t \rangle})).$$

So, it suffices to show that the following diagrams are commutative:

$$
\begin{array}{ccc}
 & & \mathrm{Pic}^{\log}(X) \\
 & \swarrow^{c^{W_n}_{\mathrm{crys}}} & \downarrow c^{W\langle t \rangle}_{\mathrm{crys}} \\
H^2_{\mathrm{crys}}((Y, M_Y)/(W_n, N^0_n)) & \longleftarrow & H^2_{\mathrm{crys}}((Y, M_Y)/(W_n\langle t \rangle, N_{W_n\langle t \rangle})),
\end{array}
$$

and

$$
\begin{array}{ccc}
\mathrm{Pic}^{\log}(X) & \searrow^{c^{W_n}_{\mathrm{crys}}} & \\
\downarrow c^{W\langle t \rangle}_{\mathrm{crys}} & & \\
H^2_{\mathrm{crys}}((Y, M_Y)/(W_n\langle t \rangle, N_{W_n\langle t \rangle})) & \xrightarrow{g^*} & H^2_{\mathrm{crys}}((X_1, M_1)/(E_n, M_{E_n})).
\end{array}
$$

The commutativity of the diagrams follows from the commutativity of the following diagrams:

$$
\begin{array}{ccc}
(\mathcal{O}^{\times}_{(Y, M_Y)/(W_n, N^0_n)})^{\log} & \twoheadrightarrow & i_{(Y, M_Y)/(W_n, N^0_n), *}(\mathcal{O}^{\times}_Y)^{\log} \\
\downarrow & & \downarrow = \\
\beta_*(\mathcal{O}^{\times}_{(Y, M_Y)/(W_n\langle t \rangle, N_{W_n\langle t \rangle})})^{\log} & \twoheadrightarrow & \beta_* i_{(Y, M_Y)/(W_n\langle t \rangle, N_{W_n\langle t \rangle}), *}(\mathcal{O}^{\times}_Y)^{\log},
\end{array}
$$

$$
\begin{array}{ccc}
1 + J_{(Y, M_Y)/(W_n, N^0_n)} & \underset{\log}{\overset{\sim}{\longrightarrow}} & J_{(Y, M_Y)/(W_n, N^0_n)} \\
\downarrow & & \downarrow \\
\beta_*(1 + J_{(Y, M_Y)/(W_n\langle t \rangle, N_{W_n\langle r \rangle})}) & \underset{\log}{\overset{\sim}{\longrightarrow}} & \beta_* J_{(Y, M_Y)/(W_n\langle t \rangle, N_{W_n\langle r \rangle})},
\end{array}
$$

$$
\begin{array}{ccc}
(\mathcal{O}^{\times}_{(Y, M_Y)/(W_n\langle t \rangle, N_{W_n\langle t \rangle})})^{\log} & \twoheadrightarrow & (\mathcal{O}^{\times}_Y)^{\log} \\
\downarrow g^* & & \downarrow g^* \\
g_*(\mathcal{O}^{\times}_{(X_1, M_1)/(E_n, M_{E_n})})^{\log} & \twoheadrightarrow & g_*(\mathcal{O}^{\times}_{X_1})^{\log},
\end{array}
$$

and

$$\begin{array}{ccc} 1+J_{(Y,M_Y)/(W_n\langle t\rangle, N_{W_n\langle t\rangle})} & \xrightarrow[\log]{\sim} & J_{(Y,M_Y)/(W_n\langle t\rangle, N_{W_n\langle t\rangle})} \\ \downarrow g^* & & \downarrow g^* \\ g_*(1+J_{(X_1,M_1)/(E_n,M_{E_n})}) & \xrightarrow[\log]{\sim} & g_*J_{(X_1,M_1)/(E_n,M_{E_n})}. \end{array}$$

□

Proof. (Proof of Proposition 2.4) By Lemma 2.5, the following diagram is commutative:

(2)

$$\begin{array}{ccccc} & & \mathrm{Pic}^{\log}(X) & & \\ & \swarrow c^W_{\mathrm{crys}} & \downarrow c^E_{\mathrm{crys}} & \searrow c^S_{\mathrm{crys}} & \\ H^2_{\mathrm{crys}}((Y,M_Y)/(W,N^0))_{\mathbb{Q}} & \xrightarrow[s]{} & H^2_{\mathrm{crys}}((X,M)/(E,M_E))_{\mathbb{Q}} & \xrightarrow[\mathrm{pr}_\pi]{} & H^2_{\mathrm{crys}}((X,M)/(S,N))_{\mathbb{Q}}. \end{array}$$

So, it suffices to show the following diagram is commutative (see [BO, Lemma (3.3), Proposition (3.4)] for the good reduction case):

$$\begin{array}{ccc} \mathrm{Pic}^{\log}(X) & & \\ \downarrow c^{S_n}_{\mathrm{crys}} & \searrow c_{\mathrm{dR}} & \\ H^2_{\mathrm{crys}}((X_1,M_1)/(S_n,N_n)) & \xrightarrow[\cong]{} & H^2_{\mathrm{dR}}(X_n/S_n), \end{array}$$

where c_{dR} is defined by

$$(\mathcal{O}^\times_{X_n})^{\log} \overset{f\mapsto df/f}{\longrightarrow} F^1\Omega^\bullet_{X_n/S_n}(\log(M_n/N_n))[1] \to \Omega^\bullet_{X_n/S_n}(\log(M_n/N_n))[1].$$

Put $\omega^\bullet_{X_n/S_n} := \Omega^\bullet_{X_n/S_n}(\log(M_n/N_n))$. Let $L(\omega^\bullet_{X_n/S_n})$ be the complex on

$$((X_1,M_1)/(S_n,N_n))^{\log}_{\mathrm{crys}}$$

deduced from $\omega^\bullet_{X_n/S_n}$ by linearization. We have a canonical homomorphism

$$\mathcal{O}_{(X_1,M_1)/(S_n,N_n)} \to L(\mathcal{O}_{X_n}),$$

and $L(\omega^\bullet_{X_n/S_n})$ is a resolution of $\mathcal{O}_{(X_1,M_1)/(S_n,N_n)}$ by PD-Poincaré lemma. There is a surjective homomorphism $L(\mathcal{O}_{X_n}) \twoheadrightarrow \mathcal{O}_{X_1}$. Let $\mathcal{K}$ be the kernel. It is a PD-ideal in $L(\mathcal{O}_{X_n})$ such that $(\mathcal{O}_{(X_1,M_1)/(S_n,N_n)}, J_{(X_1,M_1)/(S_n,N_n)}) \to (L(\mathcal{O}_{X_n}),\mathcal{K})$ is a PD-homomorphism. Then $\mathcal{K}^\times$ is the kernel of $L(\mathcal{O}_{X_n})^\times \twoheadrightarrow \mathcal{O}^\times_{X_1}$, and it is isomorphic to the kernel of $L((\mathcal{O}^\times_{X_n})^{\log}) \twoheadrightarrow (\mathcal{O}^\times_{X_1})^{\log}$, since $(U, M_1|_U) \hookrightarrow (T, M_T)$ is exact for any $(\delta, (U, M_1|_U) \hookrightarrow (T, M_T)) \in ((X_1,M_1)/(S_n,N_n))^{\log}_{\mathrm{crys}}$. Let $L(\omega^\bullet_{X_n/S_n})^\times$ (resp. $\mathcal{K}^\bullet$, $\mathcal{K}^{\bullet,\times}$) denote the complex

$$L((\mathcal{O}^\times_{X_n})^{\log}) \overset{\mathrm{dlog}}{\to} L(\omega^1_{X_n/S_n}) \overset{L(d)}{\to} L(\omega^2_{X_n/S_n}) \to \cdots$$

$$(\text{resp. } \mathcal{K} \overset{L(d)}{\to} L(\omega^1_{X_n/S_n}) \overset{L(d)}{\to} L(\omega^2_{X_n/S_n}) \to \cdots,$$

$$1+\mathcal{K} \overset{\mathrm{dlog}}{\to} L(\omega^1_{X_n/S_n}) \overset{L(d)}{\to} L(\omega^2_{X_n/S_n}) \to \cdots).$$

Then, we have the commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 1+J_{(X_1,M_1)/(S_n,N_n)} & \longrightarrow & (\mathcal{O}^{\times}_{(X_1,M_1)/(S_n,N_n)})^{\log} & \longrightarrow & i_*(\mathcal{O}^{\times}_{X_1})^{\log} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow {\scriptstyle =} & & \\ 0 & \longrightarrow & \mathcal{K}^{\bullet,\times} & \longrightarrow & L(\omega^{\bullet}_{(X_n,M_n)/(S_n,N_n)})^{\times} & \longrightarrow & i_*(\mathcal{O}^{\times}_{X_1})^{\log} & \longrightarrow & 0, \end{array}$$

(where two i_*'s mean $i_{(X_1,M_1)/(S_n,N_n),*}$) and

$$\begin{array}{ccc} 1+J_{(X_1,M_1)/(S_n,N_n)} & \xrightarrow[\log]{\sim} & J_{(X_1,M_1)/(S_n,N_n)} \\ \downarrow & & \downarrow \\ 1+\mathcal{K} & \xrightarrow[\log]{\sim} & \mathcal{K}. \end{array}$$

Thus, the following diagram is commutative:

$$\begin{array}{ccccccc} \mathrm{Pic}^{\log}(X_1) & \xrightarrow{\partial} & H^2_{\mathrm{crys}}(1+J) & \xrightarrow[\log]{\sim} & H^2_{\mathrm{crys}}(J) & \xrightarrow{i} & H^2_{\mathrm{crys}}((X_1,M_1)/(S_n,N_n)) \\ \downarrow {\scriptstyle =} & & \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \cong} \\ \mathrm{Pic}^{\log}(X_1) & \xrightarrow{\partial} & H^2_{\mathrm{crys}}(\mathcal{K}^{\bullet,\times}) & \xrightarrow[\log]{\sim} & H^2_{\mathrm{crys}}(\mathcal{K}^{\bullet}) & \xrightarrow{i} & H^2_{\mathrm{crys}}(L(\omega^{\bullet}_{X_n/S_n})), \end{array}$$

where $H^2_{\mathrm{crys}}(-)$'s mean $H^2_{\mathrm{crys}}((X_1,M_1)/(S_n,N_n),-)$, and J means $J_{(X_1,M_1)/(S_n,N_n)}$. Here, the composition of the upper horizontal arrows is the definition of $c^{S_n}_{\mathrm{crys}}$. Thus, $c^{S_n}_{\mathrm{crys}}$ is equal to the composition of the lower horizontal arrows under the identification with $H^2_{\mathrm{crys}}((X_1,M_1)/(S_n,N_n)) \cong H^2_{\mathrm{crys}}(L(\omega^{\bullet}_{X_n/S_n}))$.

The kernel of $\mathcal{O}^{\times}_{X_n} \to \mathcal{O}^{\times}_{X_1}$ is $1+p\mathcal{O}_{X_n}$, and this is also isomorphic to the kernel of $(\mathcal{O}^{\times}_{X_n})^{\log} \to (\mathcal{O}^{\times}_{X_1})^{\log}$. Let $J^{\bullet}_{X_n,S_n}$ (resp. $J^{\bullet,\times}_{X_n,S_n}$) denote the complex

$$p\mathcal{O}_{X_n} \xrightarrow{d} \omega^1_{X_n/S_n} \xrightarrow{d} \omega^2_{X_n/S_n} \to \cdots$$

$$(\text{resp. } 1+p\mathcal{O}_{X_n} \xrightarrow{\mathrm{dlog}} \omega^1_{X_n/S_n} \xrightarrow{d} \omega^2_{X_n/S_n} \to \cdots).$$

Then, we have the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{Pic}^{\log}(X_1) & \xrightarrow{\partial} & H^2_{\mathrm{crys}}(\mathcal{K}^{\bullet,\times}) & \xrightarrow[\log]{\sim} & H^2_{\mathrm{crys}}(\mathcal{K}^{\bullet}) & \xrightarrow{i} & H^2_{\mathrm{crys}}(L(\omega^{\bullet}_{X_n/S_n})) \\ \downarrow {\scriptstyle =} & & \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \cong} \\ \mathrm{Pic}^{\log}(X_1) & \xrightarrow{\partial} & \mathbb{H}^2(J^{\bullet,\times}_{X_n/S_n}) & \xrightarrow[\log]{\sim} & \mathbb{H}^2(J^{\bullet}_{X_n/S_n}) & \xrightarrow{i} & \mathbb{H}^2(\omega^{\bullet}_{X_n/S_n}), \end{array}$$

where $\mathbb{H}^2(-)$'s mean $\mathbb{H}^2(X_n,-)$ and ∂ in the bottom line is the boundary homomorphism induced by the exact sequence:

$$0 \to J^{\bullet,\times}_{X_n/S_n} \to [(\mathcal{O}^{\times}_{X_n})^{\log} \to \omega^1_{X_n/S_n} \to \omega^1_{X_n/S_n} \to \cdots] \to (\mathcal{O}^{\times}_{X_1})^{\log} \to 0.$$

Thus, $c^{S_n}_{\mathrm{crys}}$ is equal to the composition of the lower horizontal arrows under the identification with $H^2_{\mathrm{crys}}((X_1,M_1)/(S_n,N_n)) \cong H^2_{\mathrm{crys}}(L(\omega^{\bullet}_{X_n/S_n})) \cong \mathbb{H}^2(\omega^{\bullet}_{X_n/S_n})$.

We have the following commutative diagram whose horizontal lines are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^1\omega^\bullet_{X_n/S_n} & \longrightarrow & [(\mathcal{O}^\times_{X_n})^{\log} \overset{\mathrm{dlog}}{\to} \omega^1_{X_n/S_n} \to \omega^1_{X_n/S_n} \to \cdots] & \longrightarrow & (\mathcal{O}^\times_{X_n})^{\log} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^{\bullet,\times}_{X_n/S_n} & \longrightarrow & [(\mathcal{O}^\times_{X_n})^{\log} \overset{\mathrm{dlog}}{\to} \omega^1_{X_n/S_n} \to \omega^1_{X_n/S_n} \to \cdots] & \longrightarrow & (\mathcal{O}^\times_{X_1})^{\log} & \longrightarrow & 0
\end{array}$$

This gives the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Pic}^{\log}(X_n) & \xrightarrow{\partial} & \mathbb{H}^2(F^1\omega^\bullet_{X_n/S_n}) & \xrightarrow{=} & \mathbb{H}^2(F^1\omega^\bullet_{X_n/S_n}) & \xrightarrow{i} & \mathbb{H}^2(\omega^\bullet_{X_n/S_n}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow = \\
\mathrm{Pic}^{\log}(X_1) & \xrightarrow{\partial} & \mathbb{H}^2(J^{\bullet,\times}_{X_n/S_n}) & \xrightarrow[\log]{\sim} & \mathbb{H}^2(J^\bullet_{X_n/S_n}) & \xrightarrow{i} & \mathbb{H}^2(\omega^\bullet_{X_n/S_n}).
\end{array}$$

The composition of the upper horizontal arrows is the definition of c_{dR}. We are done. □

The following corollary is the semistable version of [BO, Corollary (3.7)].

Corollary 2.6. *If* $[L] \in \mathrm{Pic}^{\log}(Y)$ (*resp.* $[L] \in \mathrm{Pic}(Y)$) *lifts to* $\mathrm{Pic}^{\log}(X)$ (*resp.* $\mathrm{Pic}(X)$)*, then*

$$\rho_\pi(c_{\mathrm{crys}}([L])) \in H^2_{\mathrm{dR}}(X_K/K)$$

is in $F^1H^2_{\mathrm{dR}}(X_K/K)$.

3. p-adic Lefschetz (1,1) theorem in semistable case.

The following theorem is the semistable version of [BO, Theorem (3.8)].

Theorem 3.1. *The element* $[L] \in \mathrm{Pic}^{\log}(Y)_{\mathbb{Q}}$ (*resp.* $[L] \in \mathrm{Pic}(Y)_{\mathbb{Q}}$) *lifts to*

$$\mathrm{Pic}^{\log}(X)_{\mathbb{Q}},\ (\textit{resp. } \mathrm{Pic}(X)_{\mathbb{Q}}),$$

if and only if

$$\rho_\pi(c_{\mathrm{crys}}([L])) \in H^2_{\mathrm{dR}}(X_K/K)$$

is in $F^1H^2_{\mathrm{dR}}(X_K/K)$.

Lemma 3.2. *The element* $[L] \in \mathrm{Pic}^{\log}(Y)_{\mathbb{Q}}$ (*resp.* $[L] \in \mathrm{Pic}(Y)_{\mathbb{Q}}$) *lifts to* $\mathrm{Pic}^{\log}(X_1)_{\mathbb{Q}}$ (*resp.* $\mathrm{Pic}(X_1)_{\mathbb{Q}}$).

Proof. Let r be an integer satisfying $\pi^{p^r} \in pO_K$. Then, the r-powered absolute Frobenius $F^r_{S_1} : S_1 \to S_1$ (resp. $F_{X_1}X_1 \to X_1$) factors through $S_1 \overset{f}{\to} \mathrm{Spec}\, k \to S_1$ (resp. $X_1 \overset{g}{\to} Y \to X_1$). Thus, $X_1 \times_{S_1, F^r_{X_1}} S_1$ is isomorphic to $Y \times_{\mathrm{Spec}\, k, f} S_1$. So, $g^*[L] \in \mathrm{Pic}^{\log}(X_1)$ (resp. $g^*[L] \in \mathrm{Pic}^{\log}(X_1)$) is a lift of $[L^{p^r}]$. Inverting p^r, we get a lift. □

Proof. (Proof of Theorem 3.1) The direction of "only if " is Corollary 2.6. We show the other direction. Assume $\rho_\pi(c_{\mathrm{crys}}([L]))$ is in $F^1H^2_{\mathrm{dR}}(X_K/K)$. By Lemma 3.2, we can take a lift $[L']$ of $[L]$ in $\mathrm{Pic}^{\log}(X_1)_{\mathbb{Q}}$ (resp. $\mathrm{Pic}(X_1)_{\mathbb{Q}}$). Then by the commutative diagram (2) in the proof of Proposition 2.4, the image of $c^S_{\mathrm{crys}}([L'])$ in $H^2_{\mathrm{dR}}(X_K/K)$ is in $F^1H^2_{\mathrm{dR}}(X_K/K)$. By multiplying suitable integer, we can assume that the image of

$c^S_{\mathrm{crys}}([L'])$ in $H^2_{\mathrm{dR}}(X_K/K) \cong H^2_{\mathrm{dR}}(X/S)_{\mathbb{Q}}$ comes from $F^1 H^2_{\mathrm{dR}}(X/S)$. By the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Pic}^{\log}(X_1)(\text{resp. } \mathrm{Pic}(X_1)) & \xrightarrow{\partial} & \mathbb{H}^2(J^{\bullet,\times}_{X/S}) & \xrightarrow[\log]{\sim} & \mathbb{H}^2(J^{\bullet}_{X/S}) & \xrightarrow{i} & \mathbb{H}^2(\omega^{\bullet}_{X/S}) \\
\downarrow{=} & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}^{\log}(X_1)(\text{resp. } \mathrm{Pic}(X_1)) & \xrightarrow{\partial} & H^2(1+p\mathcal{O}_X) & \xrightarrow[\log]{\sim} & H^2(p\mathcal{O}_X) & \xrightarrow{i} & H^2(\mathcal{O}_X),
\end{array}$$

the image of $c^S_{\mathrm{crys}}([L']) = i \circ \log \circ \partial([L'])$ in $H^2(X, \mathcal{O}_X)$ is zero, where $H^2(-)$'s in the lower horizontal line mean $H^2(X, -)$. The second log in the above diagram is the comoposite

$$H^2(X, 1+p\mathcal{O}_X) \cong H^2(\widehat{X}, 1+p\mathcal{O}_{\widehat{X}}) \overset{\log,\sim}{\longrightarrow} H^2(\widehat{X}, p\mathcal{O}_{\widehat{X}}) \cong H^2(X, p\mathcal{O}_X),$$

where $\widehat{X}$ is the p-adic formal completion of X, and we used formal GAGA twice here. The composition

$$p\mathcal{O}_X \hookrightarrow \mathcal{O}_X \overset{p,\sim}{\longrightarrow} p\mathcal{O}_X$$

is the multiplication by p. Thus, the image of $p \cdot \log \circ \partial([L'])$ in $H^2(X, p\mathcal{O}_X)$ is zero. The homomorphism log is an isomorphism, so the image of $p\partial([L'])$ in $H^2(X, 1+p\mathcal{O}_X)$ is zero. On the other hand, the exact sequence

$$0 \to 1+p\mathcal{O}_X \to (\mathcal{O}_X^{\times})^{\log} \to (\mathcal{O}_{X_1}^{\times})^{\log} \to 0$$
$$(\text{resp. } 0 \to 1+p\mathcal{O}_X \to \mathcal{O}_X^{\times} \to \mathcal{O}_{X_1}^{\times} \to 0\ )$$

induces an exact sequence

$$\mathrm{Pic}^{\log}(X) \to \mathrm{Pic}^{\log}(X_1) \to H^2(X, 1+p\mathcal{O}_X)$$
$$(\text{resp. } \mathrm{Pic}(X) \to \mathrm{Pic}(X_1) \to H^2(X, 1+p\mathcal{O}_X)\ ).$$

Here, $p[L']$ goes to zero in $H^2(X, 1+p\mathcal{O}_X)$. Therefore, $p[L']$ comes from $\mathrm{Pic}^{\log}(X)$ (resp. $\mathrm{Pic}(X)$). So, $[L']$ comes from $\mathrm{Pic}^{\log}(X)_{\mathbb{Q}}$ (resp. $\mathrm{Pic}(X)_{\mathbb{Q}}$). □

4. An application to Picard number jumping locus.

We consider a generalization of the Maulik-Poonen result ([MP]).

First, we set up a situation. Let C be the completion of an algebraic closure of K, and O_C be its valuation ring. Let B be an irreducible separated O_K-scheme of finite type, and let $f : \mathcal{X} \to B$ be a proper semistable morphism such that $f_K : \mathcal{X}_K \to B_K$ is smooth, where $()_K$'s mean $K \otimes_{O_K}$. Let $M_{\mathcal{X}}$ and M_B be log-structures on $\mathcal{X}$ and B defined by $\mathcal{X} \otimes_{O_K} k$ and $B \otimes_{O_K} k$ respectively. Let $s, t \in B$ be such that s is a specialization of t (i.e., s is in the closure of $\{t\}$), $\mathrm{char}\,\kappa(t) = 0$ and $\mathrm{char}\,\kappa(s) = p$. Let $(\mathcal{X}_{\bar{t}}, M_{\mathcal{X}_{\bar{t}}})$ and $(\mathcal{X}_{\bar{s}}, M_{\mathcal{X}_{\bar{s}}})$ be the fiber of $(\mathcal{X}, M_{\mathcal{X}})$ at $\bar{t}$ and $\bar{s}$ respectively. By the same way as homomorphism (1), we have a homomorphism

$$\mathrm{sp}_{\bar{t},\bar{s}} : \mathrm{NS}(\mathcal{X}_{\bar{t}}) \to \mathrm{NS}^{\log}(\mathcal{X}_{\bar{s}}).$$

Lemma 4.1. (1) *The homomorphism*

$$\mathbb{Z}[1/p] \otimes \mathrm{sp}_{\bar{t},\bar{s}} : \mathrm{NS}(\mathcal{X}_{\bar{t}})[1/p] \to \mathrm{NS}^{\log}(\mathcal{X}_{\bar{s}})[1/p].$$

is injective, and its cokernel is torsion-free.

(2) *We have* $\rho(\mathcal{X}_{\bar{t}}) \geq \rho(\mathcal{X}_{\bar{s}})$.

(3) *If* $\mathrm{sp}_{\bar{t},\bar{s}}$ *maps a class* $[L]$ *to an ample class, then* L *is ample.*

Proof. These can be shown by a similar way as [MP, Proposition 3.6 (b), (c), (d)]. We will give a rough sketch here. For the details, see [MP, Proposition 3.6]. By using the following diagram (here, we replaced $H^2_{\text{ét}}(\mathcal{X}_{\bar{t}}, \mathbb{Z}_\ell(1)) \cong H^2_{\text{ét}}(\mathcal{X}_{\bar{s}}, \mathbb{Z}_\ell(1))$ by $H^2_{\text{ét}}(\mathcal{X}_{\bar{t}}, \mathbb{Z}_\ell(1)) \cong H^2_{\text{log-ét}}((\mathcal{X}_{\bar{s}}, M_{\mathcal{X}_{\bar{s}}}), \mathbb{Z}_\ell(1))$ (see, [N])) for any $\ell \neq p$:

$$\begin{array}{ccc} \mathrm{NS}(\mathcal{X}_{\bar{t}}) \otimes \mathbb{Z}_\ell & \hookrightarrow & H^2_{\text{ét}}(\mathcal{X}_{\bar{t}}, \mathbb{Z}_\ell(1)) \\ \downarrow{\scriptstyle \mathrm{sp}_{\bar{t},\bar{s}}} & & \downarrow{\scriptstyle \cong} \\ \mathrm{NS}^{\log}(\mathcal{X}_{\bar{s}}) \otimes \mathbb{Z}_\ell & \hookrightarrow & H^2_{\text{log-ét}}((\mathcal{X}_{\bar{s}}, M_{\mathcal{X}_{\bar{s}}}), \mathbb{Z}_\ell(1)), \end{array}$$

we have

$$\mathrm{coker}(\mathrm{sp}_{\bar{t},\bar{s}} \otimes \mathbb{Z}_\ell) \subset \mathrm{coker}\{\mathrm{NS}(\mathcal{X}_{\bar{t}}) \otimes \mathbb{Z}_\ell \to H^2_{\text{ét}}(\mathcal{X}_{\bar{t}}, \mathbb{Z}_\ell(1))\} \cong T_\ell \mathrm{Br}\mathcal{X},$$

where the last term is automatically torsion-free. This induces part (1) and (2). The part (3) comes from [MP, Proposition 3.3], which is essentially [EGA III.I, 4.7.1]. □

Let M_{B_k} be the pull-back of M_B on B_k. For a p-adic formal O_K-log-scheme (T, M_T), let T_1 be the closed subscheme defined by $p\mathcal{O}_T$, and let T_0 be the associated reduced subscheme $(T_1)_{\mathrm{red}}$. Let M_{T_1} and M_{T_0} be the pull-back of M_T to T_1 and T_0 respectively. The notion of enlargement of [O, Definition 2.1] is generalized to the semistable case by Shiho [S, Definition 2.1.1]. We use his definition (note that we use fine log (formal) schemes, not fine saturated log (formal) schemes in his definition). By the same way as [O, Theorem 3.1 and 3.7], we can show that for any $q \in \mathbb{Z}_{\geq 0}$, there exists a log-convergent isocrystal $E := R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{X}, M_{\mathcal{X}})/(W, N^0)} \otimes_W K$ on B_k with isomorphism of K-vector spaces

$$E_{[s]} \cong H^q_{\mathrm{crys}}((\mathcal{X}_s, M_{\mathcal{X}_s})/(W, N^0)) \otimes_W K$$

for each $s \in B(k)$, where $[s]$ is a enlargement given by $((\mathrm{Spf}\, O_K, N), (\mathrm{Spec}\, k, N^0_1) \xrightarrow{s} (B_k, M_{B_k}))$.

Proposition 4.2. *Let* $((T, M), z)$ *be an enlargement of* (B_k, M_{B_k})*. Let*

$$f_0 : (\mathcal{X}_0, M_{\mathcal{X}_0}) \to (T_0, M_{T_0})$$

be obtained from $f : (\mathcal{X}, M_{\mathcal{X}}) \to (B, M_B)$ *by base change along* $z : T_0 \to B_k \hookrightarrow B$*. Let* $g : \mathcal{Y} \to T$ *be a proper semistable lifting of* f_0*. Let* $M_{\mathcal{Y}}$ *be the log-structure defined by* $\mathcal{Y} \otimes_{O_K} k$*. Then for each* $q \in \mathbb{Z}_{\geq 0}$*, there is a canonical isomorphism*

$$(R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{X}_0, M_{\mathcal{X}_0})/(W, N^0)} \otimes_W K)_{(T, M_T)} \cong K \otimes R^q g_* \omega^\bullet_{\mathcal{Y}/T},$$

where $\omega^\bullet_{\mathcal{Y}/T}$ *denotes* $\Omega^\bullet_{\mathcal{Y}/T}(\log(M_{\mathcal{Y}}/M_T))$*.*

Proof. We may assume that we have a Frobenius lift F_T and $F_{\mathcal{Y}}$ on T and $\mathcal{Y}$ respectively, since the problem is local. Let r be an integer satisfying $\pi^{p^r} \in pO_K$. Then, the r-powered absolute Frobenius $F^r_{S_1} : S_1 \to S_1$ factors through $S_1 \xrightarrow{h} \mathrm{Spec}\, k \to$

S_1. Thus, $\mathcal{Y}_0 \times_{S_1, F^r_{Y_1}} S_1$ and $T_1 \times_{S_1, F^r_{T_1}} S_1$ are isomorphic to $\mathcal{X}_0 \times_{\operatorname{Spec} k, h} S_1$ and $T_0 \times_{\operatorname{Spec} k, h} S_1$ respectively. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{Y}_1 & \xrightarrow{h} & \mathcal{X}_0 & \hookrightarrow & \mathcal{Y}_1 \\
\downarrow & & \downarrow & & \downarrow \\
T_1 & \xrightarrow{h} & T_0 & \hookrightarrow & T_1 \\
\downarrow & & \downarrow & & \downarrow \\
S_1 & \xrightarrow{h} & \operatorname{Spec} k & \hookrightarrow & S_1 \\
\downarrow & & \downarrow & & \downarrow \\
E_n & \xrightarrow{h'} & \operatorname{Spec} W_n\langle t\rangle & \longrightarrow & E_n,
\end{array}$$

where h' is defined by $t \mapsto t^{p^r}$ and σ^r, and three composite horizontal arrows are r-powered absolute Frobenii. The composition of the above isomorphisms does not depend on r. Then, we have isomorphisms

$$\begin{aligned}
& R_E \otimes_W R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{X}_0, M_{\mathcal{X}_0})/(W,N^0)} \otimes \mathbb{Q} \xleftarrow{\sim,\varphi^r} R_E \otimes_{\varphi^r, W} R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{X}_0, M_{\mathcal{X}_0})/(W,N^0)} \otimes \mathbb{Q} \\
& \cong R_E \otimes_{h, W\langle t\rangle} R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{X}_0, M_{\mathcal{X}_0})/(W\langle t\rangle, N_{W\langle t\rangle})} \otimes \mathbb{Q} \\
& \cong R_E \otimes_{\varphi^r, R_E} R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{Y}_1, M_{\mathcal{Y}_1})/(E,M_E)} \otimes \mathbb{Q} \\
& \xrightarrow{\sim,\varphi^r} R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{Y}_1, M_{\mathcal{Y}_1})/(E,M_E))} \otimes \mathbb{Q} \cong R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{Y}_n, M_{\mathcal{Y}_n})/(E,M_E)} \otimes \mathbb{Q}.
\end{aligned}$$

This induces the following isomorphism:

$$\begin{aligned}
& (R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{X}_0, M_{\mathcal{X}_0})/(W,N^0)} \otimes_W K)_{(T,M_T)} \cong (R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{Y}_n, M_{\mathcal{Y}_n})/(E,M_E)} \otimes_W K)_{(T,M_T)} \\
& \cong (R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{Y}_n, M_{\mathcal{Y}_n})/(S,N)} \otimes_W K)_{(T,M_T)} \\
& \cong \varprojlim_n (R^q f_{\mathrm{crys}*}\mathcal{O}_{(\mathcal{Y}_n, M_{\mathcal{Y}_n})/(S,N)} \otimes_W K)_{(T,M_T)} \cong K \otimes R^q g_* \omega^\bullet_{\mathcal{Y}/T}.
\end{aligned}$$

□

The following theorem is a semistable version of the Maulik-Poonen result.

Theorem 4.3. *The set*

$$B(O_C)_{\mathrm{jumping}} := \{ b \in B(O_C) \mid \rho(\mathcal{X}_b \otimes_{O_C} \overline{k}) > \rho(\mathcal{X}_{\overline{\eta}}) \}$$

is nowhere dense in $B(O_C)$ for the analytic topology.

Proof. It can be shown by the same way as [MP, Theorem 1.7]. Here, we have to replace [MP, Theorem 4.24] and [MP, Theorem 4.21] by Theorem 3.1 and Proposition 4.2 respectively. We will give a rough idea here.

Let E be the above log-convergent isocrystal E for $q = 2$. We have the canonical isomorphism

$$E_{[s]} \cong H^2_{\mathrm{crys}}((\mathcal{X}_s, M_{\mathcal{X}_s})/(W, N^0)) \otimes_W K$$

for each $s \in B(k)$, Take $[L_k] \in \operatorname{Pic}^{\log}(\mathcal{X}_s)$. Then

$$c_{\mathrm{crys}}([L_k]) \in H^2_{\mathrm{crys}}((\mathcal{X}_s, M_{\mathcal{X}_s})/(W, N^0)) \otimes_W K$$

gives rise to a constant section $\gamma_{\rm crys}([L_k])_T$ of $H^2_{\rm crys}((\mathcal{X}_s, M_{\mathcal{X}_s})/(W, N^0))\otimes_W \mathcal{O}_T \cong E_T$ for a morphism of enlargement $T \to [s]$. By using Proposition 4.2, this gives a section $\gamma_{\rm dR}([L_k])_T$ of $K \otimes R^2 f_* \omega^\bullet_{(\mathcal{X}_T, M_{\mathcal{X}_T})/(T, M_T)}$, which can be mapped to a section $\gamma_{02}([L_k])_T$ of the quotient sheaf $K \otimes R^2 f_* \omega^\bullet_{(\mathcal{X}_T, M_{\mathcal{X}_T})/(T, M_T)}/{\rm Fil}^1$. We can "evaluate" $\gamma_{\rm crys}([L_k])_T$, $\gamma_{\rm dR}([L_k])_T$, and $\gamma_{02}([L_k])_T$ at $b' : {\rm Spf}\, O_K \to T$. By using Theorem 3.1 the locus where $[L_k]$ is in the image of ${\rm sp}_{\bar{t},\bar{s}}$ is the vanishing locus of $\gamma_{02}([L_k])$. By using this fact and the finitely generatedness of Néron-Severi groups, the Picard number jumping locus on a polydisk neighborhood U (see [MP, Definition 4.1] for the definition) is written in the form of

$$\bigcup_{\lambda \in \Lambda,\ \lambda \neq 0} (\text{zeros of } \lambda \text{ in } U),$$

where Λ is a finitely generated $\mathbb{Z}$-submodule of (convergent power series ring on $U)^n$ (see [MP, Lemma 4.2]). Finally, by using linear algebraic arguments, they showed the above union is nowhere dense (see [MP, Proposition 5.1]). □

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