

## THE STABLE MONOMORPHISM CATEGORY OF A FROBENIUS CATEGORY

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ABSTRACT. For a Frobenius abelian category  $\mathcal{A}$ , we show that the category  $\text{Mon}(\mathcal{A})$  of monomorphisms in  $\mathcal{A}$  is a Frobenius exact category; the associated stable category  $\underline{\text{Mon}}(\mathcal{A})$  modulo projective objects is called the stable monomorphism category of  $\mathcal{A}$ . We show that a tilting object in the stable category  $\underline{\mathcal{A}}$  of  $\mathcal{A}$  modulo projective objects induces naturally a tilting object in  $\underline{\text{Mon}}(\mathcal{A})$ . We show that if  $\mathcal{A}$  is the category of (graded) modules over a (graded) self-injective algebra  $A$ , then the stable monomorphism category is triangle equivalent to the (graded) singularity category of the (graded)  $2 \times 2$  upper triangular matrix algebra  $T_2(A)$ . As an application, we give two characterizations to the stable category of Ringel-Schmidmeier.

### 1. Introduction

Let  $\mathcal{A}$  be an abelian category. Denote by  $\text{Mor}(\mathcal{A})$  the category of morphisms in  $\mathcal{A}$  ([3, p.101]): the objects are morphisms in  $\mathcal{A}$  and the morphisms are given by commutative squares in  $\mathcal{A}$ . It is an abelian category ([17, Proposition 1.1]). We are mainly concerned with the full subcategory  $\text{Mon}(\mathcal{A})$  of  $\text{Mor}(\mathcal{A})$  consisting of monomorphisms in  $\mathcal{A}$ , which is called the *monomorphism category* of  $\mathcal{A}$ . It is an additive subcategory of  $\text{Mor}(\mathcal{A})$  which is closed under extensions, thus it becomes an exact category in the sense of Quillen ([22, Appendix A]).

In the case that the abelian category  $\mathcal{A}$  is the module category over a ring, the monomorphism category  $\text{Mon}(\mathcal{A})$  is known as the *submodule category*. Recently it is studied intensively by Ringel and Schmidmeier ([34, 35, 36]). If the ring is  $\mathbb{Z}/(q^p)$  with  $p \geq 2$  and  $q$  a prime number, the study of the submodule category goes back to Birkhoff ([8]; see also [1]). The case that the ring is  $k[t]/(t^p)$  with  $k$  a field is studied by Simson ([37]) and also by Beligiannis ([7]). In this case, the study of indecomposable objects in  $\text{Mon}(\mathcal{A})$  shows an example of the typical trichotomy phenomenon “finite/tame/wild” in the representation theory of finite dimensional algebras, where the trichotomy depends on the parameter  $p$ ; see [36, Section 6]. Moreover, the case where the abelian category  $\mathcal{A}$  is given by the graded module category over the graded algebra  $k[t]/(t^p)$  with  $\deg t = 1$  plays an important role in [36]; in this case, the

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monomorphism category  $\text{Mon}(\mathcal{A})$  is denoted by  $\mathcal{S}(\tilde{p})$ . It is a Frobenius exact category ([27]; also see Lemma 2.1 and compare [22, Section 5]). Then by [18, Chapter I, Theorem 2.6] its stable category  $\underline{\mathcal{S}}(\tilde{p})$  modulo projective objects is triangulated. A very recent and remarkable result due to Kussin, Lenzing and Meltzer claims that the stable category  $\underline{\mathcal{S}}(\tilde{p})$  is triangle equivalent to the stable category of vector bundles on the weighted projective lines of type  $(2, 3, p)$ ; see [27]. Recall that a similar trichotomy phenomenon “domestic/tubular/wild” occurs in the classification of indecomposable vector bundles on the weighted projective lines of type  $(2, 3, p)$ , while the trichotomy again depends on the parameter  $p$ ; see [29, 26]. In this paper, we will call the triangulated category  $\underline{\mathcal{S}}(\tilde{p})$  the *stable category of Ringel-Schmidmeier*.

The present paper studies the monomorphism category  $\text{Mon}(\mathcal{A})$  of a Frobenius abelian category  $\mathcal{A}$ , in particular, the stable category  $\underline{\mathcal{A}}$  modulo projective objects is triangulated. We show that  $\text{Mon}(\mathcal{A})$  is a Frobenius exact category and then the stable category  $\underline{\text{Mon}}(\mathcal{A})$  modulo projective objects is triangulated; it is called the *stable monomorphism category* of  $\mathcal{A}$ . Recently this category is also studied by Iyama, Kato and Miyachi ([21]). Observe that the triangulated categories above are algebraical in the sense of Keller. We have a well-behaved notion of *tilting object* for an algebraical triangulated category ([24]). We prove that a tilting object in  $\underline{\mathcal{A}}$  induces naturally a tilting object in  $\underline{\text{Mon}}(\mathcal{A})$ ; see Theorem 3.2. Moreover, if the category  $\mathcal{A}$  is the (graded) module category over a (graded) self-injective algebra  $A$ , we relate the category  $\underline{\text{Mon}}(\mathcal{A})$  to the category of (graded) Gorenstein-projective modules and then to the (graded) singularity category of the  $2 \times 2$  upper triangular matrix algebra  $T_2(A)$  of  $A$  (for  $T_2(A)$ , see [17, p.115] and [3, Chapter III, Section 2]); see Theorem 4.1. We are inspired by a computational result by Li and Zhang on Gorenstein-projective modules ([30]; compare [7, 21]). Here, the Gorenstein-projective module is in the sense of Enochs and Jenda ([16, Chapter 10]), and the singularity category is in the sense of Orlov ([32, 33]; compare [10, 19]).

Combining all these together, we give two characterizations to the stable category  $\underline{\mathcal{S}}(\tilde{p})$  of Ringel-Schmidmeier. We characterize the stable category  $\underline{\mathcal{S}}(\tilde{p})$  as the bounded derived category of  $T_2(k\mathbb{A}_{p-1}) \simeq k\mathbb{A}_2 \otimes_k k\mathbb{A}_{p-1}$ ; see Corollary 3.4. Here, for each  $n \geq 1$ ,  $\mathbb{A}_n$  is the linear quiver with  $n$  vertices and linear orientation, and  $k\mathbb{A}_n$  is the path algebra. We characterize the stable category  $\underline{\mathcal{S}}(\tilde{p})$  as the graded singularity category of  $T_2(k)[t]/(t^p)$ , where the algebra  $T_2(k)[t]/(t^p)$  is graded such that  $\deg T_2(k) = 0$  and  $\deg t = 1$ ; see Corollary 4.7.

For the convention, throughout we fix a commutative artinian ring  $R$ . All artin algebras are artin  $R$ -algebras, and all categories and functors are  $R$ -linear. For an artin algebra  $A$ , denote by  $\text{mod } A$  the category of finitely generated right  $A$ -modules and by  $\text{proj } A$  the full subcategory consisting of projective modules. We denote by  $A_A$  and  ${}_A A$  the right and left regular modules of the artin algebra  $A$ , respectively. For triangulated categories and derived categories, we refer to [20, 18, 23, 24].

## 2. Monomorphism Category

Let  $\mathcal{A}$  be a Frobenius abelian category. Thus  $\mathcal{A}$  has enough projective objects and enough injective objects, and the class of projective objects coincides with the class of injective objects. Denote by  $\mathcal{P}$  the full subcategory of  $\mathcal{A}$  consisting of projective

objects. Denote by  $\underline{\mathcal{A}}$  the *stable category* of  $\mathcal{A}$  modulo  $\mathcal{P}$ : the objects are the same as  $\mathcal{A}$ , and the morphism spaces are factors of the morphism spaces in  $\mathcal{A}$  modulo those factoring through projective objects ([3, p.101]). The stable category  $\underline{\mathcal{A}}$  is a triangulated category such that its shift functor is given by the quasi-inverse of the syzygy functor on  $\underline{\mathcal{A}}$  and triangles are induced by short exact sequences in  $\mathcal{A}$ ; for details, see [18, Chapter I, Section 2].

Recall that  $\text{Mor}(\mathcal{A})$  is the category of morphisms in  $\mathcal{A}$ : the objects are morphisms  $\alpha: A \rightarrow B$  in  $\mathcal{A}$  and the morphisms are commutative squares in  $\mathcal{A}$ , that is, of the form  $(f, g): \alpha \rightarrow \alpha'$  where  $f: A \rightarrow A'$  and  $g: B \rightarrow B'$  are morphisms in  $\mathcal{A}$  such that  $\alpha' \circ f = g \circ \alpha$  (compare [3, p.101]). For an object  $\alpha: A \rightarrow B$  in  $\text{Mor}(\mathcal{A})$ , we write  $s(\alpha) = A$  and  $t(\alpha) = B$ , which are called the *source* and *target* of  $\alpha$ , respectively. Note that  $\text{Mor}(\mathcal{A})$  is an abelian category such that a sequence  $\alpha' \rightarrow \alpha \rightarrow \alpha''$  is exact if and only if the induced sequences of sources and targets are exact in  $\mathcal{A}$  ([17, Corollary 1.2]).

Recall that an *exact category* in the sense of Quillen is an additive category together with an *exact structure*, that is, a distinguished class of ker-coker sequences, which are called *conflations*, subject to certain axioms. Recall that a full additive subcategory of an abelian category which is closed under extensions has a natural exact structure such that conflations are just short exact sequences with terms in the subcategory ([22, Appendix A] and [23, Section 4]). Moreover, there is a notion of Frobenius exact category and the associated stable category modulo projective objects is still triangulated; compare [18, p.10-11], [22, subsection 1.2 b)] and [23, Section 6].

Recall that our main concern is the *monomorphism category*  $\text{Mon}(\mathcal{A})$ , which is the full subcategory of  $\text{Mor}(\mathcal{A})$  consisting of monomorphisms in  $\mathcal{A}$ . We will consider the following two functors: the first functor  $i_1: \mathcal{A} \rightarrow \text{Mon}(\mathcal{A})$  is defined such that  $i_1(A) = 0 \rightarrow A$  and  $i_1(f) = (0, f)$  where  $A$  is an object and  $f$  is a morphism in  $\mathcal{A}$ ; the second  $i_2: \mathcal{A} \rightarrow \text{Mon}(\mathcal{A})$  is defined such that  $i_2(A) = A \xrightarrow{\text{Id}_A} A$  and  $i_2(f) = (f, f)$ . We observe that both functors are exact and fully faithful.

**Lemma 2.1.** *Let  $\mathcal{A}$  be an abelian category. Then the monomorphism category  $\text{Mon}(\mathcal{A})$  is an exact category such that conflations are given by sequences  $\alpha' \rightarrow \alpha \rightarrow \alpha''$  with the induced sequences of sources and targets short exact in  $\mathcal{A}$ .*

*Assume further that  $\mathcal{A}$  is Frobenius. Then the exact category  $\text{Mon}(\mathcal{A})$  is Frobenius such that its projective objects are equal to direct summands of objects of the form  $i_1(P) \oplus i_2(P)$  where  $P$  is a projective object in  $\mathcal{A}$ .*

*Proof.* We observe that  $\text{Mon}(\mathcal{A})$  is an additive subcategory of the abelian category  $\text{Mor}(\mathcal{A})$  which is closed under extensions by Snake Lemma. Then it is an exact category with conflations induced by short exact sequences in  $\text{Mor}(\mathcal{A})$ ; see Example 4.1 in [23].

Assume now that the abelian category  $\mathcal{A}$  is Frobenius. We will show first that objects of the form  $i_1(P)$  and  $i_2(P)$  are projective and injective. Recall that for an object  $\alpha$  in  $\text{Mon}(\mathcal{A})$  we denote by  $s(\alpha)$  and  $t(\alpha)$  the source and target of  $\alpha$ , respectively. We have the following natural isomorphisms

$$\text{Hom}_{\text{Mon}(\mathcal{A})}(i_1(P), \alpha) \simeq \text{Hom}_{\mathcal{A}}(P, t(\alpha)) \text{ and } \text{Hom}_{\text{Mon}(\mathcal{A})}(i_2(P), \alpha) \simeq \text{Hom}_{\mathcal{A}}(P, s(\alpha)).$$

These isomorphisms show that the objects  $i_1(P)$  and  $i_2(P)$  are projective. Similarly, we have the following natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Mon}(\mathcal{A})}(\alpha, i_1(P)) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathrm{Cok} \alpha, P)$$

and

$$\mathrm{Hom}_{\mathrm{Mon}(\mathcal{A})}(\alpha, i_2(P)) \simeq \mathrm{Hom}_{\mathcal{A}}(t(\alpha), P).$$

These isomorphisms show that the objects  $i_1(P)$  and  $i_2(P)$  are injective; here, we use that the functor  $\mathrm{Cok}$  of taking the cokernels is exact on  $\mathrm{Mon}(\mathcal{A})$  by Snake Lemma.

Let  $\alpha$  be an object in  $\mathrm{Mon}(\mathcal{A})$ . Take epimorphisms  $P \rightarrow s(\alpha)$  and  $P \rightarrow t(\alpha)$  with  $P$  projective in  $\mathcal{A}$ . Then we have an epimorphism  $i_1(P) \oplus i_2(P) \rightarrow \alpha$  whose kernel lies in  $\mathrm{Mon}(\mathcal{A})$ . This shows that the exact category  $\mathrm{Mon}(\mathcal{A})$  has enough projective objects. On the other hand, for the object  $\alpha$ , take monomorphisms  $a: t(\alpha) \rightarrow P$  and  $b': \mathrm{Cok} \alpha \rightarrow P$  with  $P$  projective in  $\mathcal{A}$ . Denote by  $b$  the composite  $t(\alpha) \rightarrow \mathrm{Cok} \alpha \xrightarrow{b'} P$  where the first morphism is the natural projection. Consider the following morphism in  $\mathrm{Mor}(\mathcal{A})$

$$\left( \begin{pmatrix} a \circ \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right): \alpha \longrightarrow i_2(P) \oplus i_1(P).$$

It is a monomorphism and by a diagram-chasing its cokernel lies in  $\mathrm{Mon}(\mathcal{A})$ . Then it becomes a conflation in  $\mathrm{Mon}(\mathcal{A})$ . This shows that the exact category  $\mathrm{Mon}(\mathcal{A})$  has enough injective objects. From the argument above, it is direct to conclude that in the exact category  $\mathrm{Mon}(\mathcal{A})$  the class of projective objects coincides with the class of injective objects, and projective objects are direct summands of objects of the form  $i_1(P) \oplus i_2(P)$  where  $P$  is a projective object in  $\mathcal{A}$ .  $\square$

**Remark 2.2.** With a slightly modified proof as above, one can show that a similar result holds if the category  $\mathcal{A}$  is an exact category. In this case, one replaces  $\mathrm{Mon}(\mathcal{A})$  by the *inflation category* of  $\mathcal{A}$ ; compare [22, Section 5] and [21].  $\square$

For a Frobenius abelian category  $\mathcal{A}$ , we denote by  $\underline{\mathrm{Mon}}(\mathcal{A})$  the stable category of  $\mathrm{Mon}(\mathcal{A})$  modulo projective objects; it is a triangulated category. We will call it the *stable monomorphism category* of  $\mathcal{A}$ .

We observe that both the functors  $i_1$  and  $i_2$  are fully faithful and send projective objects to projective objects. Then they induce fully faithful triangle functors  $i_1: \underline{\mathcal{A}} \rightarrow \underline{\mathrm{Mon}}(\mathcal{A})$  and  $i_2: \underline{\mathcal{A}} \rightarrow \underline{\mathrm{Mon}}(\mathcal{A})$  ([18, p.23, Lemma 2.8]).

### 3. Tilting Objects in Stable Monomorphism Category

In this section, we will show that for a Frobenius abelian category  $\mathcal{A}$ , a tilting object in the stable category  $\underline{\mathcal{A}}$  induces naturally a tilting object in the stable monomorphism category  $\underline{\mathrm{Mon}}(\mathcal{A})$ . We characterize the stable category of Ringel-Schmidmeier as the bounded derived category of a finite dimensional algebra.

Following Keller, we recall that a triangulated category is *algebraical* provided that it is triangle equivalent to the stable category of a Frobenius exact category ([24,

subsection 8.7]). One has a well-behaved notion of tilting object in an algebraical triangulated category.

Let  $\mathcal{T}$  be an algebraical triangulated category. Denote by  $[1]$  the shift functor and by  $[n]$  its  $n$ -th power for each  $n \in \mathbb{Z}$ . An object  $T$  in  $\mathcal{T}$  is a *tilting object* if the following conditions are satisfied:

- (T1)  $\text{Hom}_{\mathcal{T}}(T, T[n]) = 0$  for  $n \neq 0$ ;
- (T2) the smallest *thick* triangulated subcategory of  $\mathcal{T}$  containing  $T$  is  $\mathcal{T}$  itself;
- (T3)  $\text{End}_{\mathcal{T}}(T)$  is an artin algebra having finite global dimension.

Here, we recall that a triangulated subcategory of  $\mathcal{T}$  is called *thick* if it is closed under taking direct summands. We point out that the notion of tilting object presented here is slightly different from, however closely related to, the ones in [18] and [24].

Recall that an additive category is said to be *idempotent-split* provided that each idempotent  $e: X \rightarrow X$  admits a factorization  $X \xrightarrow{u} Y \xrightarrow{v} X$  such that  $u \circ v = \text{Id}_Y$  ([18, Chapter I, 3.2]). Recall that for an artin algebra  $A$  having finite global dimension, the bounded derived category  $\mathbf{D}^b(\text{mod } A)$  is algebraical and idempotent-split (see the proof of [18, Chapter I, Corollary 4.9]), and it has  $A_A$  as its tilting object.

The following remarkable result due to Keller claims that the converse holds true (compare [9, Theorem 1]).

**Lemma 3.1.** (Keller) *Let  $\mathcal{T}$  be an idempotent-split algebraical triangulated category with a tilting object  $T$ . Then there is a triangle equivalence*

$$\mathcal{T} \simeq \mathbf{D}^b(\text{mod } \text{End}_{\mathcal{T}}(T)).$$

*Proof.* Set  $A = \text{End}_{\mathcal{T}}(T)$ . By [24, Theorem 8.51 a)] there is a triangle functor  $F': \mathcal{T} \rightarrow \mathbf{D}(A')$  sending  $T$  to  $A'$ , where  $A'$  is a differential graded algebra with the only nonzero cohomology  $H^0(A') \simeq A$  and  $\mathbf{D}(A')$  is the (unbounded) derived category of differential graded (right) modules on  $A'$ . By [24, subsection 8.4] there is a triangle equivalence  $\mathbf{D}(A') \simeq \mathbf{D}(\text{Mod } A)$  identifying  $A'$  with  $A_A$ , where  $\text{Mod } A$  is the category of (not necessarily finitely generated) right  $A$ -modules. Consequently, there is a triangle functor  $F: \mathcal{T} \rightarrow \mathbf{D}(\text{Mod } A)$  sending  $T$  to  $A$ . Using (T1) and (T2) and applying Beilinson Lemma ([18, p.72, Lemma 3.4]), the triangle functor  $F$  is fully faithful. Then we may view  $\mathcal{T}$  as a triangulated subcategory of  $\mathbf{D}(\text{Mod } A)$ ; moreover, since  $\mathcal{T}$  is idempotent-split, it is necessarily a thick subcategory of  $\mathbf{D}(\text{Mod } A)$ . By (T3) the artin algebra  $A$  has finite global dimension, and then the smallest thick triangulated subcategory of  $\mathbf{D}(\text{Mod } A)$  containing  $A_A$  is  $\mathbf{D}^b(\text{mod } A)$ . From this we conclude that the essential image of  $F$  is  $\mathbf{D}^b(\text{mod } A)$ . Therefore  $F$  induces the required equivalence.  $\square$

Our first observation states that a tilting object in the stable category  $\underline{\mathcal{A}}$  induces naturally a tilting object in the stable monomorphism category  $\underline{\text{Mon}}(\mathcal{A})$ . Recall that for an artin algebra  $A$ ,  $T_2(A) = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$  is the  $2 \times 2$  upper triangular matrix algebra ([3, Chapter III, Section 2]).

**Theorem 3.2.** *Let  $\mathcal{A}$  be a Frobenius abelian category such that  $T$  is a tilting object in its stable category  $\underline{\mathcal{A}}$ . Then  $T' = i_1(T) \oplus i_2(T)$  is a tilting object in  $\underline{\text{Mon}}(\mathcal{A})$ ; moreover, we have an isomorphism  $\text{End}_{\underline{\text{Mon}}(\mathcal{A})}(T') \simeq T_2(\text{End}_{\underline{\mathcal{A}}}(T))$  of algebras.*

*Proof.* Recall that  $i_1: \underline{\mathcal{A}} \rightarrow \underline{\text{Mon}}(\mathcal{A})$  and  $i_2: \underline{\mathcal{A}} \rightarrow \underline{\text{Mon}}(\mathcal{A})$  are fully faithful triangle functors. Observe that for objects  $A$  and  $B$  in  $\mathcal{A}$ ,  $\text{Hom}_{\underline{\text{Mon}}(\mathcal{A})}(i_2(A), i_1(B)) = 0$ . So to check the condition (T1) for  $T'$ , it suffices to show that  $\text{Hom}_{\underline{\text{Mon}}(\mathcal{A})}(i_1(T), i_2(T)[n]) = 0$  for  $n \neq 0$ . For this end, note that since  $i_2$  is a triangle functor, we have

$$i_2(T)[n] \simeq i_2(T[n]) = T[n] \xrightarrow{\text{Id}_{T[n]}} T[n].$$

Thus a morphism in  $\text{Hom}_{\underline{\text{Mon}}(\mathcal{A})}(i_1(T), i_2(T)[n])$  is of the form  $(0, f)$ , where  $f: T \rightarrow T[n]$  is a morphism in  $\mathcal{A}$ . By the condition (T1) for  $T$ ,  $f$  factors through a projective object  $P$  in  $\mathcal{A}$ . Therefore the morphism  $(0, f)$  factors through  $i_1(P)$ , which is projective in  $\text{Mon}(\mathcal{A})$ ; see Lemma 2.1. Hence  $(0, f) = 0$  in the stable monomorphism category  $\underline{\text{Mon}}(\mathcal{A})$ .

To check (T2) for  $T'$ , recall that each object  $\alpha$  fits into a conflation

$$i_2(s(\alpha)) \longrightarrow \alpha \longrightarrow i_1(\text{Cok } \alpha)$$

and thus into a triangle

$$i_2(s(\alpha)) \longrightarrow \alpha \longrightarrow i_1(\text{Cok } \alpha) \longrightarrow i_2(s(\alpha))[1].$$

Here as in Section 2,  $s(\alpha)$  denotes the source of  $\alpha$ . Hence the smallest triangulated subcategory of  $\underline{\text{Mon}}(\mathcal{A})$  containing  $i_1(\underline{\mathcal{A}})$  and  $i_2(\underline{\mathcal{A}})$  is  $\underline{\text{Mon}}(\mathcal{A})$  itself. Now applying the condition (T2) of  $T$ , we infer that (T2) holds for  $T'$ .

Finally to see the condition (T3) for  $T'$ , it is direct to check that  $\text{End}_{\underline{\text{Mon}}(\mathcal{A})}(T') \simeq T_2(\text{End}_{\underline{\mathcal{A}}}(T))$ . Recall that the algebra  $\text{End}_{\underline{\mathcal{A}}}(T)$  has finite global dimension. Then by [3, Chapter III, Proposition 2.6] we infer that  $\text{End}_{\underline{\text{Mon}}(\mathcal{A})}(T')$  has finite global dimension.  $\square$

We will give an application of Theorem 3.2. Let  $A = \bigoplus_{n \geq 0} A_n$  be a positively graded artin algebra. Denote by  $c$  the maximal integer such that  $A_c \neq 0$ . Consider the following upper triangular matrix algebra

$$b(A) = \begin{pmatrix} A_0 & A_1 & \cdots & A_{c-2} & A_{c-1} \\ 0 & A_0 & \cdots & A_{c-3} & A_{c-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 & A_1 \\ 0 & 0 & \cdots & 0 & A_0 \end{pmatrix}.$$

Here the multiplication of  $b(A)$  is induced from the one of  $A$ . This algebra is called the *Beilinson algebra* of  $A$  in [12].

Denote by  $\text{mod}^{\mathbb{Z}} A$  the category of finitely generated  $\mathbb{Z}$ -graded  $A$ -modules with homomorphisms preserving degrees. We say that  $A$  is *graded self-injective* provided that  $\text{mod}^{\mathbb{Z}} A$  is a Frobenius category. In fact, this is equivalent to that as a ungraded algebra  $A$  is self-injective ([15, 12]). In this case, we denote by  $\underline{\text{mod}}^{\mathbb{Z}} A$  the stable category of  $\text{mod}^{\mathbb{Z}} A$  modulo projective modules; it is a triangulated category.

We say that a graded algebra  $A$  is *right well-graded*, provided that  $A_c$ , as a right  $A_0$ -module, is sincere in the sense of [3, p.317]. In fact, for a graded self-injective algebra  $A$ , it is right well-graded if and only if it is left well-graded; see [12, Lemma 2.2]. In this case we will simply say that the graded algebra  $A$  is *well-graded*.

**Corollary 3.3.** *Let  $A = \bigoplus_{n \geq 0} A_n$  be a positively graded self-injective artin algebra which is well-graded. Suppose that  $A_0$  has finite global dimension. Then there is a triangle equivalence*

$$\underline{\text{Mon}}(\text{mod}^{\mathbb{Z}} A) \simeq \mathbf{D}^b(\text{mod } T_2(\text{b}(A))).$$

*Proof.* By [3, Chapter III, Proposition 2.6], the Beilinson algebra  $\text{b}(A)$  and then  $T_2(\text{b}(A))$  has finite global dimension. By [12, Corollary 1.2] there is a triangle equivalence  $\underline{\text{mod}}^{\mathbb{Z}} A \simeq \mathbf{D}^b(\text{mod } \text{b}(A))$ . In particular, there is a tilting object  $T$  in  $\underline{\text{mod}}^{\mathbb{Z}} A$  with endomorphism algebra  $\text{b}(A)$ . We apply Theorem 3.2 to get a tilting object  $T'$  in  $\underline{\text{Mon}}(\text{mod}^{\mathbb{Z}} A)$  whose endomorphism algebra is isomorphic to  $T_2(\text{b}(A))$ . Note that the stable monomorphism category  $\underline{\text{Mon}}(\text{mod}^{\mathbb{Z}} A)$  is idempotent-split; in fact, it is even a Krull-Schmidt category. Then the result follows immediately from Lemma 3.1.  $\square$

In what follows, we will apply the obtained results to the stable category of Ringel-Schmidmeier.

Let  $k$  be a field and let  $p \geq 2$  be an integer. Consider the truncated polynomial algebra  $A = k[t]/(t^p)$  with  $t$  an indeterminate; it is positively graded such that  $\deg t = 1$ . Observe that  $A$  is graded self-injective and moreover it is well-graded. In particular, the category  $\text{mod}^{\mathbb{Z}} A$  of finitely generated graded  $A$ -modules is Frobenius. Following [36, subsection 0.4], we denote by  $\mathcal{S}(\tilde{p})$  the category of pairs  $(V, U)$ , where  $V$  is a graded module over  $A$  and  $U \subseteq V$  is a graded submodule, and the morphisms in this category are given by morphisms in the graded module category which respect the inclusion. There is a natural identification  $\mathcal{S}(\tilde{p}) = \text{Mon}(\text{mod}^{\mathbb{Z}} A)$  and then by Lemma 2.1 it is a Frobenius exact category. Hence its stable category  $\underline{\mathcal{S}}(\tilde{p})$  modulo projective objects is triangulated. This triangulated category will be called the *stable category of Ringel-Schmidmeier*.

We note that the Beilinson algebra  $\text{b}(A)$  of the graded algebra  $A$  is isomorphic to the path algebra  $k\mathbb{A}_{p-1}$  of the linear quiver  $\mathbb{A}_{p-1}$  with  $p - 1$  vertices and linear orientation (compare [33, Example 2.9]). Then the  $2 \times 2$  upper triangular matrix algebra  $T_2(\text{b}(A))$  is given by the following quiver with  $2p - 2$  vertices subject to the commutativity relation



We observe that  $T_2(\text{b}(A)) \simeq k\mathbb{A}_2 \otimes_k k\mathbb{A}_{p-1}$ . Let us mention that these diagrams and algebras are studied in [28].

Then we have the following immediate consequence of Corollary 3.3.

**Corollary 3.4.** *Use the notation above. Then there is a triangle equivalence*

$$\underline{\mathcal{S}}(\tilde{p}) \simeq \mathbf{D}^b(\text{mod } k\mathbb{A}_2 \otimes_k k\mathbb{A}_{p-1}).$$

**Remark 3.5.** Let us remark that taking into account of the results obtained in [26] and [28, Corollary 1.2], one may find a close relation between Corollary 3.4 and some results in [27].

Recall that  $T = \bigoplus_{i=0}^{p-2} (A/(t^{p-i-1}))(i)$  is a tilting object in  $\underline{\text{mod}}^{\mathbb{Z}} A$ , where  $(i)$  denote the degree-shift functors ([31] and [15]). This assertion can be obtained from the proof of [33, Corollary 2.8] or [12, Corollary 1.2]. We apply Theorem 3.2 to deduce that  $T' = i_1(T) \oplus i_2(T)$  is a tilting object in  $\underline{\mathcal{S}}(\tilde{p})$ , which yields the triangle equivalence in Corollary 3.4. We point out that this explicit tilting object is also obtained in [27, Lemma 4.7] via a different method.

#### 4. Stable Monomorphism Category as Singularity Category

In this section, we will relate the stable monomorphism category of the (graded) module category of a (graded) self-injective algebra to the (graded) singularity category of the associated (graded)  $2 \times 2$  upper triangular matrix algebra. We characterize the stable category of Ringel-Schmidmeier as the graded singularity category of a finite dimensional graded algebra.

Let  $A$  be an artin algebra. Recall that the bounded homotopy category  $\mathbf{K}^b(\text{proj } A)$  of projective modules is viewed naturally as a triangulated subcategory of  $\mathbf{D}^b(\text{mod } A)$ . Following [32, 33], we call the Verdier quotient triangulated category

$$\mathbf{D}_{\text{sg}}(A) = \mathbf{D}^b(\text{mod } A) / \mathbf{K}^b(\text{proj } A)$$

the *singularity category* of  $A$ ; compare [10] and [19].

Recall that for an artin algebra  $A$ ,  $T_2(A)$  is the  $2 \times 2$  upper triangular matrix algebra of  $A$ . We consider the following composite functor

$$G_A: \text{Mon}(\text{mod } A) \hookrightarrow \text{mod } T_2(A) \longrightarrow \mathbf{D}^b(\text{mod } T_2(A)) \longrightarrow \mathbf{D}_{\text{sg}}(T_2(A)).$$

Here, the first inclusion is obtained by regarding morphisms in  $\text{mod } A$  as (right)  $T_2(A)$ -modules ([3, Chapter III, Proposition 2.2]), the middle functor identifies modules with stalk complexes concentrated at degree zero ([20, p.40, Proposition 4.3]), and the last functor is the quotient functor.

Our second observation is as follows.

**Theorem 4.1.** *Let  $A$  be a self-injective algebra. Then the functor  $G_A$  induces a triangle equivalence*

$$\underline{\text{Mon}}(\text{mod } A) \simeq \mathbf{D}_{\text{sg}}(T_2(A)).$$

Before giving the proof, we recall several notions. Let  $A$  be an artin algebra. Following [5, p.400], an acyclic complex  $P^\bullet$  of projective  $A$ -modules is called *totally acyclic* if the Hom complex  $\text{Hom}_A(P^\bullet, A)$  is still acyclic (also see [25, Section 7]). An  $A$ -module  $M$  is said to be *Gorenstein-projective* if there is a totally acyclic complex  $P^\bullet$  such that its zeroth cocycle  $Z^0(P^\bullet)$  is isomorphic to  $M$  ([16, Chapter 10]). Recall that a module



$M$  is Gorenstein-projective if and only if  $\text{Ext}_A^i(M, A) = 0$ ,  $\text{Ext}_{A^{\text{op}}}^i(\text{Hom}_A(M, A), A) = 0$  for  $i \geq 1$  and the natural map  $M \rightarrow \text{Hom}_{A^{\text{op}}}(\text{Hom}_A(M, A), A)$  is an isomorphism (compare [14, Definition (1.1.2)]).

We denote by  $\text{Gproj } A$  the full subcategory of  $\text{mod } A$  consisting of Gorenstein-projective modules. Observe that projective modules are Gorenstein-projective and thus  $\text{proj } A \subseteq \text{Gproj } A$ . Moreover, by [2, Proposition 5.1] the subcategory  $\text{Gproj } A$  is closed under extensions and taking direct summands (also see [16]), and then it is direct to see that  $\text{Gproj } A$  is a Frobenius exact category such that its projective objects are equal to projective  $A$ -modules ([6, Proposition 3.8(i)] and [13, Proposition 3.1(1)]). Denote by  $\underline{\text{Gproj}} A$  its stable category modulo projective  $A$ -modules; it is a triangulated category.

Recall that an artin algebra  $A$  is said to be *Gorenstein* if the regular modules  ${}_A A$  and  $A_A$  have finite injective dimensions ([19]). In this case the two dimensions are equal and the common value is denoted by  $\text{G.dim } A$ . We say that the Gorenstein algebra  $A$  is *1-Gorenstein* provided that  $\text{G.dim } A \leq 1$ .

For an artin algebra  $A$ , denote by  $\text{sub } A$  the full subcategory of  $\text{mod } A$  consisting of submodules of projective modules; these modules are called *torsionless modules*. We remark that homological properties of torsionless modules are studied in [4].

The following result is well known.

**Lemma 4.2.** *Let  $A$  be a 1-Gorenstein algebra. Then we have  $\text{Gproj } A = \text{sub } A$ .*

*Proof.* The inclusion  $\text{Gproj } A \subseteq \text{sub } A$  is easy. On the other hand, assume that  $M$  is a torsionless module. Consider a short exact sequence  $0 \rightarrow M \rightarrow P \rightarrow M' \rightarrow 0$  with  $P$  projective. Since the regular module  $A_A$  has injective dimension at most one, using dimension shift, we infer that  $\text{Ext}_A^i(M, A) = 0$  for  $i \geq 1$ . Then by [16, Corollary 11.5.3] (see also [13, Lemma 3.7] and [25, Proposition 7.13]),  $M$  is Gorenstein-projective.  $\square$

The next observation is essentially due to Li and Zhang ([30, Theorem 1.1]; also see [7, Example 4.17] and [21, Proposition 3.6]). Recall that for an artin algebra  $A$ , a morphism of (right)  $A$ -modules is identified with a (right) module over  $T_2(A)$ ; in fact, this yields an equivalence  $\text{Mor}(\text{mod } A) \simeq \text{mod } T_2(A)$  of categories; see [3, Chapter III, Proposition 2.2].

**Lemma 4.3.** *Let  $A$  be a self-injective algebra. Then we have an equivalence of categories*

$$\text{Mon}(\text{mod } A) \simeq \text{sub } T_2(A).$$

*Proof.* Recall the equivalence  $\text{Mor}(\text{mod } A) \simeq \text{mod } T_2(A)$ . Observe that the regular module  $T_2(A)_{T_2(A)}$  corresponds to the monomorphism  $\begin{pmatrix} 0 \\ \text{Id}_A \end{pmatrix}: A \rightarrow A \oplus A$ . From this one infers that torsionless  $T_2(A)$ -modules correspond to monomorphisms in  $\text{mod } A$ . On the other hand, the third paragraph of the proof of Lemma 2.1 already shows that for a monomorphism  $\alpha$ , there is a short exact sequence  $0 \rightarrow \alpha \rightarrow \begin{pmatrix} 0 \\ \text{Id}_P \end{pmatrix} \rightarrow \alpha' \rightarrow 0$  in  $\text{Mor}(\text{mod } A)$  such that  $P$  is a projective  $A$ -module. Observe that the monomorphism  $\begin{pmatrix} 0 \\ \text{Id}_P \end{pmatrix}$  corresponds to a projective  $T_2(A)$ -module. Therefore the monomorphism  $\alpha$  corresponds to a torsionless  $T_2(A)$ -module. This completes the proof.  $\square$

We will recall the last ingredient in our proof. Let  $A$  be an artin algebra. Consider the following composite of functors

$$F_A: \text{Gproj } A \hookrightarrow \text{mod } A \longrightarrow \mathbf{D}^b(\text{mod } A) \longrightarrow \mathbf{D}_{\text{sg}}(A)$$

where from the left side, the first functor is the inclusion, the second identifies modules with stalk complexes concentrated in degree zero ([20, p.40, Proposition 4.3]) and the last is the quotient functor. Observe that the additive functor  $F_A$  vanishes on projective modules and then induces uniquely an additive functor  $\underline{\text{Gproj}} A \rightarrow \mathbf{D}_{\text{sg}}(A)$ , which is still denoted by  $F_A$ .

The following important result is due to Buchweitz ([10, Theorem 4.4.1]) and independently due to Happel ([19, Theorem 4.6]); also see [13, Proposition 3.5 and Theorem 3.8].

**Lemma 4.4.** (Buchweitz-Happel) *Let  $A$  be an artin algebra. Then the functor  $F_A: \underline{\text{Gproj}} A \rightarrow \mathbf{D}_{\text{sg}}(A)$  is a fully faithful triangle functor. Moreover, if  $A$  is Gorenstein, then the functor  $F_A$  is dense and thus a triangle equivalence.*

**Proof of Theorem 4.1.** We observe that by [11, Remark 3.5] (also see [17, 19]) the algebra  $T_2(A)$  is 1-Gorenstein and then we can apply Lemma 4.2. Then Lemma 4.3 yields an equivalence of categories  $\text{Mon}(\text{mod } A) \simeq \text{Gproj } T_2(A)$ . We observe that this equivalence preserves the exact structures, that is, the equivalence and its quasi-inverse preserve short exact sequences in  $\text{Mon}(\text{mod } A)$  and  $\text{Gproj } T_2(A)$ . Therefore, this equivalence is an equivalence of Frobenius exact categories. Consequently, we have an induced equivalence of triangulated categories

$$\underline{\text{Mon}}(\text{mod } A) \simeq \underline{\text{Gproj}} T_2(A).$$

Then the result follows directly from Lemma 4.4.  $\square$

We will need a graded version of Theorem 4.1. Let  $A = \bigoplus_{n \geq 0} A_n$  be a positively graded artin algebra. Denote by  $\text{proj}^{\mathbb{Z}} A$  the full subcategory of  $\text{mod}^{\mathbb{Z}} A$  consisting of projective objects. Following [33], one has the *graded singularity category* of  $A$  defined by

$$\mathbf{D}_{\text{sg}}^{\mathbb{Z}}(A) = \mathbf{D}^b(\text{mod}^{\mathbb{Z}} A) / \mathbf{K}^b(\text{proj}^{\mathbb{Z}} A).$$

For a graded module  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  and an integer  $d \in \mathbb{Z}$ , its *shifted module*  $M(d)$  has the same module structure as  $M$  while it is graded such that  $M(d)_i = M_{d+i}$  for all  $i \in \mathbb{Z}$ . This defines automorphisms  $(d): \text{mod}^{\mathbb{Z}} A \rightarrow \text{mod}^{\mathbb{Z}} A$ , which are called *degree-shift functors*. For graded modules  $M, N$ , we write  $\text{HOM}_A(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{mod}^{\mathbb{Z}} A}(M, N(i))$  and set  $\text{EXT}_A^n(-, -)$  to be the  $n$ -th right derived functors ([31] and [15]).

An acyclic complex  $P^\bullet$  in  $\text{proj}^{\mathbb{Z}} A$  is *totally acyclic* if the complex  $\text{HOM}_A(P^\bullet, A)$  in  $\text{proj}^{\mathbb{Z}} A^{\text{op}}$  is acyclic. A graded  $A$ -module is called *graded Gorenstein-projective* provided that it is the zeroth cocycle of a totally acyclic complex. Thus we have a full subcategory  $\text{Gproj}^{\mathbb{Z}} A$  of  $\text{mod}^{\mathbb{Z}} A$  consisting of graded Gorenstein-projective modules and evidently  $\text{proj}^{\mathbb{Z}} A \subseteq \text{Gproj}^{\mathbb{Z}} A$ . As in the ungraded case, the category  $\text{Gproj}^{\mathbb{Z}} A$  is a Frobenius exact category with its projective objects equal to graded projective  $A$ -modules.

Recall that a graded artin algebra  $A$  is said to be *graded Gorenstein* if the graded regular modules  ${}_A A$  and  $A_A$  have finite injective dimensions in  $\text{mod}^{\mathbb{Z}} A$  and  $\text{mod}^{\mathbb{Z}} A^{\text{op}}$ , respectively. In this case the two dimensions are the same, which will be denoted by  $\text{G.dim}^{\mathbb{Z}} A$ .

We observe the following fact, which guarantees in principle that most results in Gorenstein homological algebra hold true in the graded situation.

**Lemma 4.5.** *Let  $A$  be a positively graded artin algebra, and let  $M$  be a graded  $A$ -module. Then we have*

- (1) *the module  $M$  is graded Gorenstein-projective if and only if it is Gorenstein-projective as a ungraded module;*
- (2) *the algebra  $A$  is graded Gorenstein if and only if it is Gorenstein as a ungraded algebra; in this case, we have  $\text{G.dim}^{\mathbb{Z}} A = \text{G.dim} A$ .*

*Proof.* For (1), it suffices to recall that a graded module  $M$  is graded Gorenstein-projective if and only if  $\text{EXT}_A^i(M, A) = 0$ ,  $\text{EXT}_{A^{\text{op}}}^i(\text{HOM}_A(M, A), A) = 0$  for  $i \geq 1$  and the natural map  $M \rightarrow \text{HOM}_{A^{\text{op}}}(\text{HOM}_A(M, A), A)$  is an isomorphism of graded modules; moreover, for graded modules  $M$  and  $N$  we have for each  $i$  a natural identification  $\text{EXT}_A^i(M, N) = \text{Ext}_A^i(M, N)$  ([31, Corollary 2.4.7]). For (2), we observe that a graded module  $M$  has finite injective dimension in  $\text{mod}^{\mathbb{Z}} A$  if and only if it has finite injective dimension as a ungraded module; moreover, the two dimensions are the same ([31, Theorem 2.8.7]). □

One can show the graded analogues of Lemmas 4.2, 4.3 and 4.4. Using these, we have the following graded analogue of Theorem 4.1.

**Proposition 4.6.** *Let  $A = \bigoplus_{n \geq 0} A_n$  be a positively graded self-injective artin algebra. Denote by  $T_2(A)$  the  $2 \times 2$  upper triangular matrix algebra of  $A$  which is graded such that  $T_2(A)_n = T_2(A_n)$  for  $n \geq 0$ . Then we have a triangle equivalence*

$$\underline{\text{Mon}}(\text{mod}^{\mathbb{Z}} A) \simeq \mathbf{D}_{\text{sg}}^{\mathbb{Z}}(T_2(A)).$$

We apply Proposition 4.6 to the stable category of Ringel-Schmidmeier.

Let  $k$  be a field and  $p \geq 2$  be an integer. Recall from Section 3 that  $A = k[t]/(t^p)$  with  $\text{deg } t = 1$ , which is graded self-injective. We observe that  $T_2(A)$  is isomorphic, as a graded algebra, to  $T_2(k)[t]/(t^p)$ , while the latter is graded such that  $\text{deg } T_2(k) = 0$  and  $\text{deg } t = 1$ .

Recall that the category  $\mathcal{S}(\tilde{p})$  is identified with  $\text{Mon}(\text{mod}^{\mathbb{Z}} A)$ , and then the stable category  $\underline{\mathcal{S}}(\tilde{p})$  of Ringel-Schmidmeier is identified with  $\underline{\text{Mon}}(\text{mod}^{\mathbb{Z}} A)$ . Then the following is an immediate consequence of Proposition 4.6.

**Corollary 4.7.** *Use the notation above. Then there is a triangle equivalence*

$$\underline{\mathcal{S}}(\tilde{p}) \simeq \mathbf{D}_{\text{sg}}^{\mathbb{Z}}(T_2(k)[t]/(t^p)).$$

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