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ERRATUM: A PRESENTATION FOR HILDEN'S SUBGROUP OF THE BRAID GROUP

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1. Introduction

After publication of [1] Allen Hatcher found a gap in the proof that the complex $\overline{\mathbf{X}}_n$ is simply connected. This complex is defined in terms of isotopy classes of discs, but the argument uses representatives of the isotopy classes. There was an implicit assumption that for an edge path in the complex there exists sufficiently nice representatives of each isotopy class. In Definition 2 of this paper the properties of these representatives will be made explicit. It is clear that such representatives exist for a path, the problem is that for a loop it is not obvious that the representative at the beginning and end can be chosen to coincide. This problem is addressed in Lemma 3. Section 2 of this paper contains the complete proof that $\overline{\mathbf{X}}_n$ is simply connected, incorporating all of the necessary changes.

There were also small errors in Figure 7 and Figure 8 and the correct versions of these are included in Section 3.

2. The complex $\overline{\mathbf{X}}_n$

An embedded disc $D \subseteq H^3$ is said to *cut out* a_i if the interior of D is disjoint from a_*, the arc a_i is contained in the boundary of D and the boundary of D lies in the union of the arc a_i and the boundary of half-space, i.e. $a_i \subset \partial D$ and $\partial D \subset a_i \cup \partial H^3$. A *cut system for* a_* is the isotopy class of n pairwise disjoint discs $\langle D_1, D_2, \ldots D_n\rangle$ where each D_i cuts out the arc a_i. Say that two cut systems differ by a simple i-move if there exist representatives $\langle D_1, D_2, \ldots, D_n\rangle$ and $\langle E_1, E_2, \ldots, E_n\rangle$ such that $D_i \cap E_i = a_i$ and $D_j = E_j$ for all $j \neq i$. If this is the case we will suppress the non-changing discs and write $\langle D_i\rangle \to \langle E_i\rangle$.

Definition 1. Define the cut system complex $\overline{\mathbf{X}}_n$ as follows. The set of all cut systems for a_* forms the vertex set $\overline{\mathbf{X}}_n^0$. Two vertices are connected by a single edge iff they differ by a simple move. Finally, glue faces into every loop of the following form, giving triangular and rectangular faces.

$$\begin{tikzcd} \langle D_i\rangle \arrow[rr, dash] \arrow[dr, dash] & & \langle D_i'\rangle \arrow[dl, dash] \\ & \langle D_i''\rangle & \end{tikzcd} \qquad \begin{tikzcd} \langle D_i, D_j\rangle \arrow[r, dash] \arrow[d, dash] & \langle D_i', D_j\rangle \arrow[d, dash] \\ \langle D_i, D_j'\rangle \arrow[r, dash] & \langle D_i', D_j'\rangle \end{tikzcd}$$

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Define the basepoint to be $v_0 = \langle d_1, d_2, \ldots, d_n \rangle$ where the d_i are vertical discs below the a_i, see Figure 1. Sometimes it is convenient to think of the a_i and d_i rotated by a quarter turn.

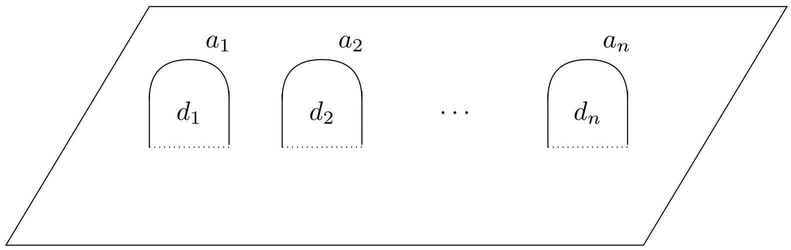


FIGURE 1. The arcs a_i and the discs d_i

Definition 2. Given an edge path $P = (v_1, v_2, \ldots, v_N)$ in $\overline{\mathbf{X}}_n$ we will say that a collection of discs $\{D_{i,k}\}$ form a *realisation* of P if for each vertex v_k of P we have

$$v_k = \langle D_{1,k}, D_{2,k}, \ldots, D_{n,k} \rangle$$

and if for each k there exists an i such that $D_{j,k} = D_{j,k+1}$ for all $j \neq i$ and either $D_{i,k} = D_{i,k+1}$ or $D_{i,k} \cap D_{i,k+1} = a_i$. In other words at most one disc changes as we go from k to $k+1$.

Lemma 3. *Every edge path $P = (v_1, \ldots, v_N)$ admits a realisation and any loop (i.e. whenever $v_1 = v_N$) can be extended with stationary edges to*

$$P' = (v_1, v_2, \ldots, v_N, v_N, \ldots, v_N)$$

so that P' admits a realisation for which the representatives of the first and last vertex are the same.

Proof. Suppose that we have the required discs for the first k vertices. If (v_k, v_{k+1}) is a stationary edge, i.e. $v_k = v_{k+1}$, then let $D_{i,k+1} = D_{i,k}$ for each i. Otherwise, if (v_k, v_{k+1}) is an i–move then there exist discs $D_1, \ldots, D_n, E$ such that $v_k = \langle D_1, D_2, \ldots D_n \rangle$ and $v_{k+1} = \langle D_1, \ldots, E, \ldots, D_n \rangle$. Now as $\langle D_1, D_2, \ldots D_n \rangle = \langle D_{1,k}, D_{2,k}, \ldots, D_{n,k} \rangle$ there exists an isotopy F_t such that F_0 is the identity and $F_1(D_j) = D_{j,k}$ for each j. So let $D_{j,k+1} = D_{j,k}$ for $j \neq i$ and $D_{i,k+1} = F_1(E)$.

Continuing in this way we can pick the required representative of each vertex, but if P is a loop the representatives of v_1 and v_N may differ.

Suppose that the representatives of the first and last vertex are as follows.

$$v_1 = \langle D_1, D_2, \ldots, D_n \rangle \qquad v_N = \langle E_1, E_2, \ldots, E_n \rangle$$

As $v_1 = v_n$ there exists an isotopy $F \colon H^3 \times I \to H^3$ with $F_0 = \mathrm{Id}_{H^3}$, F_t the identity on a_i for all $t \in I$ and for each i, and $F_1(E_i) = D_i$ for each i.

Now, for each $t \in I$ there exists an open neighbourhood U of t such that there exist closed neighbourhoods ϕ_i of each disc $F_t(D_i)$ which satisfy the following. Each ϕ_i is homeomorphic to a ball and intersects the boundary of half-space in a disc. These neighbourhoods are pairwise disjoint, i.e. $\phi_i \cap \phi_j = \varnothing$ for $i \neq j$. And as we move $F_s(E_i)$, for $s \in U$, the discs remains in ϕ_i, i.e. $F(E_i \times U) \subset \phi_i$.

Therefore, there exists a partition $0 = t_0 < t_1 < \cdots < t_K = 1$ such that for each k we have the following. Each disc $F_{t_k}(D_i)$ has a closed neighbourhood ϕ_i homeomorphic to a ball. Each of these balls intersect the boundary of half-space in a disc. These neighbourhoods are pairwise disjoint. And each disc remains in the same ball as we move from t_k to t_{k+1}, i.e. $F(E_i \times [t_k, t_{k+1}]) \subset \phi_i$ for each i.

Within each ϕ_i there is only one isotopy class of discs cutting out a_i.

We may assume that $F_{t_k}(E_i)$ and $F_{t_{k+1}}(E_i)$ intersect transversely and so intersect in a collection of arcs and circles. If this intersection is not just a_i then we can carry out the following.

Each arc or circle of this intersection separates $F_{t_{k+1}}(E_i)$ into two pieces. Say that one of these pieces is minimal if it contains no arcs or circles of this intersection. Now pick a minimal piece $A \subset F_{t_{k+1}}(E_i)$ which comes from an arc or circle α of the intersection. Similarly each arc or circle cuts $F_{t_k}(E_i)$ into two pieces. Hence we can cut $F_{t_k}(E_i)$ along α giving two pieces B_1 and B_2. So we have that $F_{t_k}(E_i) = B_1 \cup B_2$ and $\alpha = B_1 \cap B_2$.

For one and only one of the B_p we have $B_p \cap a_i = A \cap a_i$. So we can perform the following surgery on $F_{t_k}(E_i)$: discard B_p, glue in A and push off slightly. This gives a new disc that will have at least one fewer arc or circle of intersection with $F_{t_{k+1}}(E_i)$ and which only intersects $F_{t_k}(E_i)$ along a_i.

Repeating in this way gives a sequence of discs from $F_{t_k}(E_i)$ to $F_{t_{k+1}}(E_i)$ each intersecting the next only along the arc a_i and, as each one is contained in ϕ_i, each isotopic to $F_{t_k}(E_i)$ rel $H^3 \setminus \phi_i$. Therefore, by changing one disc at a time, we have a realisation of a stationary path from v_N to v_1. □

From now on we will assume we have already chosen a realisation for each path. Furthermore we will assume that for loops sufficiently many stationary edges have been added at the end and that the representatives of the first and last vertex are the same.

Definition 4. A triple (v, D, D^*), where $v = \langle D_1, D_2, \dots D_n \rangle$ is a vertex of $\overline{\mathbf{X}}_n$ with a choice of representative discs, D and D^* are two discs cutting out the arc a_i with $D \cap D^* = a_i$, forms a *substitution* if we can replace any occurrence of D with D^*. Note that we include the possibility that $D \neq D_i$. In other words, either $D = D_i$ and for all $j \neq i$ we have that $D_j \cap D^* = \varnothing$ or $D \neq D_i$.

So if (v, D, D^*) is a substitution then there is a (possibly stationary) edge (v, v^*) where

$$v^* = \begin{cases} v & \text{if } D_i \neq D, \\ \langle D^* \rangle & \text{if } D_i = D. \end{cases}$$

Similarly, for any edge path $P = (v_1, v_2, \dots, v_N)$ with a choice of realisation by discs, we say (P, D, D^*) forms a *substitution* iff for each vertex v of P the tuple (v, D, D^*) forms a substitution and for each edge (u, v) of P there is a (possibly stationary) edge (u^*, v^*).

If (P, D, D^*) forms a substitution then we can replace each vertex v with v^*, giving a new path $P^* = (v_1^*, v_2^*, \dots, v_N^*)$ whose realisation is given by replacing each occurrence of the disc D with the disc D^*.

Lemma 5. *Suppose P is a path with a given realisation. If (P, D, D^*) forms a substitution, where $P = (v_1, \ldots, v_N)$, then the loop*

$$\begin{tikzcd} v_1 \arrow[r, dash, "P"] \arrow[d, dash] & v_N \arrow[d, dash] \\ v_1^* \arrow[r, dash, "P^*"] & v_N^* \end{tikzcd}$$

is null homotopic. Moreover, if P is a loop then so is P^* and they are homotopic as loops.

Proof. For each edge (u, v) in P we have the following rectangle where some edges could be degenerate stationary edges.

$$\begin{tikzcd} u \arrow[r, dash] \arrow[d, dash] & v \arrow[d, dash] \\ u^* \arrow[r, dash] & v^* \end{tikzcd}$$

If one edge is degenerate then this loop is contained in the boundary of a triangular face. If more than one edge is degenerate then this loop is contained within an edge.

So suppose that none of the edges are degenerate and that u and v have representatives as follows.

$$u = \langle D_1, D_2, \ldots, D_n \rangle \qquad v = \langle D_1', D_2', \ldots, D_n' \rangle$$

With $D_i \neq D_i'$ and $D_j = D_j'$ for all $j \neq i$. Suppose that $D \cap D^* = a_k$. In other words that u and v differ by a simple i–move and u and u^* differ by a simple k–move.

As $v \neq v^*$ we must have that D is one of the discs representing v. Similarly, as $u \neq u^*$ we must have that D is one of the discs representing u. Hence $k \neq i$ and $D_k = D_k' = D$. Therefore this loop is the boundary of the following rectangular face.

$$\begin{tikzcd} \langle D, D_i \rangle \arrow[r, dash] \arrow[d, dash] & \langle D, D_i' \rangle \arrow[d, dash] \\ \langle D^*, D_i \rangle \arrow[r, dash] & \langle D^*, D_i' \rangle \end{tikzcd}$$

□

Theorem 6. *The complex $\overline{\mathbf{X}}_n$ is connected and simply connected.*

Proof. It suffices to show that any loop is homotopic to the constant loop at v_0. Given a loop in $\overline{\mathbf{X}}_n$ it is homotopic to an edge path P. Now by Lemma 3, after adding sufficiently many stationary points, we can pick a collection of discs realising P. We can make this choice such that the intersection of any two representative discs and between any representative disc and any of the d_i are transverse. We shall write $D \in P$ if D is one of the discs chosen as a representative of some vertex of P.

Claim. *The path P is homotopic to a path with a realisation whose discs intersect the discs $d_1, d_2, \ldots, d_n$ only in the arcs $a_1, a_2, \ldots, a_n$.*

Assuming that the intersection of the discs $D \in P$ with $d_1 \cup d_2 \cup \ldots \cup d_n$ isn't only $a_1, a_2, \ldots, a_n$ we can carry out the following procedure.

For some i the union of the discs in P intersects d_i in a non-trivial collection of arcs and circles. Each arc or circle of this intersection separates d_i into two pieces. Say that one of these pieces is minimal if it contains no other complete arc or circle of this intersection.

Now pick a minimal piece $A \subset d_i$ which comes from an arc or circle α on its boundary. The arc α is a subset of some $D \in P$, i.e. $\alpha \subset D \cap d_i$. We can cut D along α giving two pieces B_1 and B_2. So we have that $D = B_1 \cup B_2$ and $\alpha = B_1 \cap B_2$.

For one and only one of the B_k we have $B_k \cap a_i = A \cap a_i$. So we can perform the following surgery on D: discard B_k, glue in A and push off slightly. This gives a new disc D^*. The new disc will intersect d_i in at least one less arc or circle.

Any disc $E \in P$ for which $E \cap D = a_j$ or $\varnothing$ also has $E \cap D^* = a_j$ or $\varnothing$ respectively; if not E must intersect D^* in the section parallel to d_i and this contradicts the condition that A contains no complete arc or circle of $E \cap d_i$. Therefore the triple (P, D, D^*) forms a substitution and, by Lemma 5, we can replace D with D^* to get a new homotopic loop P^*.

We now have a homotopic loop P^* that has fewer intersections with $d_1 \cup d_2 \cup \ldots \cup d_n$. So by induction on the number of intersections we have proved the claim.

So we may assume that the representatives of P meets $d_1, d_2, \ldots, d_n$ only in the arcs $a_1, a_2, \ldots, a_n$. Therefore, for each $D \in P$ cutting out the arc a_i, the triple (P, D, d_i) forms a substitution and so by in turn replacing each $D \in P$ with the corresponding d_i we see that P is homotopic to the constant path (v_0). The connectedness of $\overline{\mathbf{X}}_n$ follows by taking P to be a constant loop. $\square$

3. Other errors

Unfortunately Figure 7 and Figure 8 were incorrect in [1]. In Figure 7 the diagram for s_i is the same as the one for p_i. This is clearly incorrect; two of the crossings need to be changed. In Figure 8 the diagrams for the generators are the same as for their inverses, and this is also incorrect. The correct versions of these figures are as follows.

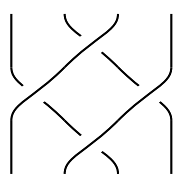
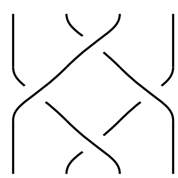
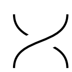
$s_i =$  $p_i =$  $t_i =$ 

FIGURE 7. Generators of $\mathbf{H}_{2n}$

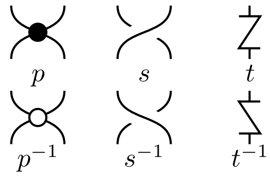


FIGURE 8. Pictorial representation of the p, s, t and their inverses

References

[1] S. Tawn, *A presentation for Hilden's subgroup of the braid group*, Math. Res. Lett. **15** (2008), no. 6, 1277–1293.

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