

LINKED HOM SPACES

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ABSTRACT. In this note, we describe a theory of linked Hom spaces which complements that of linked Grassmannians. Given two chains of vector bundles linked by maps in both directions, we give conditions for the space of homomorphisms from one chain to the other to be itself represented by a vector bundle. We apply this to present a more transparent version of an earlier construction of limit linear series spaces out of linked Grassmannians.

1. Introduction

In [5], spaces of linked Grassmannians are introduced in order to use them in a new construction of limit linear series schemes. Given a base scheme S , integers $r < d$, and a chain of vector bundles $\mathcal{E}_1, \dots, \mathcal{E}_n$ on S of rank d , with homomorphisms $f_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$ and $g_i : \mathcal{E}_{i+1} \rightarrow \mathcal{E}_i$ satisfying certain natural conditions, the associated linked Grassmannian parametrizes tuples of subbundles $\mathcal{F}_i \subseteq \mathcal{E}_i$ of rank r , which are all mapped into one another under the f_i and g_i . These schemes behave like flat degenerations of the classical Grassmannian $G(r, d)$, and indeed according to [2], whenever the f_i and g_i are generically isomorphisms, the linked Grassmannian does in fact yield a flat degeneration of $G(r, d)$.

In this note, we study a Hom version of the linked Grassmannian construction, which arises naturally in the construction of limit linear series spaces out of linked Grassmannians. Given chains of vector bundles $\mathcal{F}_1, \dots, \mathcal{F}_n$ and $\mathcal{G}_1, \dots, \mathcal{G}_n$ on S , with the \mathcal{F}_i of rank r and the \mathcal{G}_i of rank m , and homomorphisms $f_i : \mathcal{F}_i \rightarrow \mathcal{F}_{i+1}$, $f^i : \mathcal{F}_{i+1} \rightarrow \mathcal{F}_i$, $g_i : \mathcal{G}_i \rightarrow \mathcal{G}_{i+1}$, $g^i : \mathcal{G}_{i+1} \rightarrow \mathcal{G}_i$, we determine natural conditions (see Definition 2.3 below) so that the resulting linked Hom functor $\mathcal{LH}(\{\mathcal{F}_i\}, \{\mathcal{G}_i\}, \{f_i, f^i\}, \{g_i, g^i\})$ parametrizing tuples of morphisms $\varphi_i : \mathcal{F}_i \rightarrow \mathcal{G}_i$ which commute with all f_i, f^i, g_i, g^i is well behaved. Specifically, we show:

Theorem 1.1. *A linked Hom functor $\mathcal{LH}(\{\mathcal{F}_i\}, \{\mathcal{G}_i\}, \{f_i, f^i\}, \{g_i, g^i\})$ is represented by a vector bundle LH on S of rank rm .*

Using this result, we are able to give a less ad hoc and more symmetric version of the construction of limit linear series spaces out of linked Grassmannians, which in particular reduces the need for sections in the smoothing families. Although the resulting shift in perspective is relatively minor, we believe it will be important for a new application to higher-rank Brill-Noether theory. Specifically, Bertram, Feinberg and Mukai observed that in the case of rank 2 vector bundles of canonical determinant, there are symmetries which result in the spaces of bundles with prescribed numbers of sections having dimension higher than previously expected, when they are non-empty. One would like to show such spaces (and generalizations, as suggested in

Received by the editors August 31, 2010.

[4]) are nonempty via degeneration techniques, but the limit linear series construction introduced by Eisenbud and Harris in [1] and generalized to higher rank by Teixidor i Bigas in [6] does not appear superficially to preserve the necessary symmetries introduced by the canonical determinant condition. The simpler nature of our construction should lead to a proof that the dimensions of spaces of limit linear series with special determinants satisfy the desired modified lower bounds.

2. Linked Hom spaces

We work throughout over a fixed base scheme S .

The basic definition is as follows:

Definition 2.1. Let r, m, n be integers. Suppose that $\mathcal{F}_1, \dots, \mathcal{F}_n$ are vector bundles of rank r on S and $\mathcal{G}_1, \dots, \mathcal{G}_n$ are vector bundles of rank m on S . Suppose also that we have homomorphisms

$$f_i : \mathcal{F}_i \rightarrow \mathcal{F}_{i+1}, \quad f^i : \mathcal{F}_{i+1} \rightarrow \mathcal{F}_i, \quad g_i : \mathcal{G}_i \rightarrow \mathcal{G}_{i+1}, \quad g^i : \mathcal{G}_{i+1} \rightarrow \mathcal{G}_i$$

for each $i = 1, \dots, n-1$. Then the functor

$$\mathcal{LH}(\{\mathcal{F}_i\}, \{\mathcal{G}_i\}, \{f_i, f^i\}, \{g_i, g^i\})$$

associates to an S -scheme T the tuples of homomorphisms $\varphi_i : \mathcal{F}_i|_T \rightarrow \mathcal{G}_i|_T$ such that $\varphi_{i+1} \circ f_i = g_i \circ \varphi_i$ and $\varphi_i \circ f^i = g^i \circ \varphi_{i+1}$ for $i = 1, \dots, n-1$.

Lemma 2.2. *In the situation of Definition 2.1, the functor*

$$\mathcal{LH}(\{\mathcal{F}_i\}, \{\mathcal{G}_i\}, \{f_i, f^i\}, \{g_i, g^i\})$$

is represented by an affine scheme LH of finite presentation over S .

Note that the functor naturally takes values in \mathcal{O}_T -modules, and as a result LH has the structure of a (not necessarily quasi-coherent) \mathcal{O} -module over the big Zariski (or étale, or fppf) site on S .

Proof. It is clear that our functor is a subfunctor of $\bigoplus_{i=1}^n \mathcal{H}om(\mathcal{F}_i, \mathcal{G}_i)$, which is represented by a vector bundle over S , because $\mathcal{H}om$ commutes with base change for locally free sheaves. Moreover, it is cut out by the conditions $\varphi_{j+1} \circ f_j - g_j \circ \varphi_j = 0$ and $\varphi_j \circ f^j - g^j \circ \varphi_{j+1} = 0$, which can be viewed as the preimage of the zero section under morphisms

$$\bigoplus_{i=1}^n \mathcal{H}om(\mathcal{F}_i, \mathcal{G}_i) \rightarrow \mathcal{H}om(\mathcal{F}_j, \mathcal{G}_{j+1})$$

and

$$\bigoplus_{i=1}^n \mathcal{H}om(\mathcal{F}_i, \mathcal{G}_i) \rightarrow \mathcal{H}om(\mathcal{F}_{j+1}, \mathcal{G}_j)$$

respectively, and thus give closed (and finitely generated) conditions. We thus conclude that LH exists, with the desired properties. \square

We now describe the additional conditions we will impose in order to obtain good behavior for LH . Rather than literally considering a $\mathcal{H}om$ version of linked Grassmannians (that is, imposing the same conditions on both the \mathcal{F}_i and the \mathcal{G}_i as were imposed on the \mathcal{E}_i in a linked Grassmannian), we will impose somewhat stronger conditions on the \mathcal{G}_i and weaker ones on the \mathcal{F}_i . This not only leads to good behavior,

but is precisely the situation which arises in the application to construction of limit linear series spaces. See Remark 2.6 below for further discussion.

Definition 2.3. In the situation of Definition 2.1 and Lemma 2.2, we say that LH is a **linked Hom space** if the following conditions are satisfied:

(I) There exists some $s \in \Gamma(S, \mathcal{O}_S)$ such that

$$f^i f_i = f_{i-1} f^{i-1} = s \text{ id}$$

and

$$g^i g_i = g_{i-1} g^{i-1} = s \text{ id}$$

for all suitable i .

(II) For all $x \in S$ with $s = 0$ in $\kappa(x)$, and all $i = 1, \dots, n-1$, the kernel of g_i restricted to $\mathcal{G}_i \otimes \kappa(x)$ is precisely the image of g^i , and vice versa.

(III) For all $x \in S$ with $s = 0$ in $\kappa(x)$, and all $i = 1, \dots, n-2$, we have $\text{im } g_i$ complementary to $\ker g_{i+1}$, and $\text{im } g^{i+1}$ complementary to $\ker g^i$ in $\mathcal{G}_{i+1} \otimes \kappa(x)$.

The following notation will be convenient:

Notation 2.4. In the situation of Definition 2.3, given $i < j$, denote by $f_{i,j}$, $f^{j,i}$, $g_{i,j}$ and $g^{j,i}$ the compositions

$$\begin{aligned} f_{j-1} \circ f_{j-2} \circ \dots \circ f_{i+1} \circ f_i, \\ f^i \circ f^{i+1} \circ \dots \circ f^{j-2} \circ f^{j-1}, \\ g_{j-1} \circ g_{j-2} \circ \dots \circ g_{i+1} \circ g_i, \end{aligned}$$

and

$$g^i \circ g^{i+1} \circ \dots \circ g^{j-2} \circ g^{j-1},$$

respectively. We use the convention that $f_{i,i}$, $f^{i,i}$, $g_{i,i}$ and $g^{i,i}$ all denote the identity map.

Our hypotheses on the chain \mathcal{G}_i yield the following simple structure:

Lemma 2.5. *In the situation of Definition 2.1, suppose that LH is a linked Hom space. Then locally on S , we have decompositions*

$$\mathcal{G}_i \cong \mathcal{G}'_i \oplus \mathcal{G}''_i$$

with \mathcal{G}'_i and \mathcal{G}''_i free \mathcal{O}_S -modules, satisfying the following conditions for all $i = 1, \dots, n-1$:

- (i) g_i maps \mathcal{G}'_i to \mathcal{G}'_{i+1} and \mathcal{G}''_i to \mathcal{G}''_{i+1} , and similarly for g^i .
- (ii) The induced maps $(g_i)' : \mathcal{G}'_i \rightarrow \mathcal{G}'_{i+1}$ and $(g^i)'' : \mathcal{G}''_{i+1} \rightarrow \mathcal{G}''_i$ are isomorphisms.

Proof. Given $x \in S$, if $s \neq 0$ in $\kappa(x)$, then there exists an open neighborhood U of x on which s is a unit, and by condition (I) of a linked Hom space, the g_i and g^i are isomorphisms. We may thus set $\mathcal{G}'_i = \mathcal{G}_i$ and $\mathcal{G}''_i = 0$ for all i .

On the other hand, if $s = 0$ in $\kappa(x)$, choose subspaces $\bar{\mathcal{G}}'_1 \subseteq \mathcal{G}_1 \otimes \kappa(x)$ and $\bar{\mathcal{G}}''_n \subseteq \mathcal{G}_n \otimes \kappa(x)$, complementary to $\ker g_1$ and $\ker g^{n-1}$ respectively. Putting together conditions (II) and (III) for a linked Hom space, we see that for all i , we have $g_{1,i}$ injective on $\bar{\mathcal{G}}'_1$, and $g^{n,i}$ injective on $\bar{\mathcal{G}}''_n$, and furthermore setting $\bar{\mathcal{G}}'_i = g_{1,i} \bar{\mathcal{G}}'_1$ and $\bar{\mathcal{G}}''_i = g^{n,i} \bar{\mathcal{G}}''_n$ we

have that \mathcal{G}'_i and \mathcal{G}''_i are complementary in $\mathcal{G}_i \otimes \kappa(x)$. We thus obtain a decomposition of the desired form over $\kappa(x)$. In particular, if $\dim \mathcal{G}'_1 = \ell$, then $\dim \mathcal{G}''_n = m - \ell$.

Now, choose U any affine open neighborhood of x on which the \mathcal{G}_i become free. Multiplying by units in $\mathcal{O}_{S,x}$ as necessary, we may choose bases for \mathcal{G}'_1 and \mathcal{G}''_n which can be lifted to some $v_1, \dots, v_\ell \in \mathcal{G}_1(U)$ and $w_1, \dots, w_{m-\ell} \in \mathcal{G}_n(U)$. Let \mathcal{G}'_1 and \mathcal{G}''_n be the submodules of $\mathcal{G}_1|_U$ and $\mathcal{G}_n|_U$ spanned by the v_i and w_i , respectively. Restrict U as necessary to the complement of the closed subschemes on which the maps

$$\begin{aligned} \mathcal{O}_U^{\oplus \ell} &\xrightarrow{v_i} \mathcal{G}_1 \\ \mathcal{O}_U^{\oplus m-\ell} &\xrightarrow{w_i} \mathcal{G}_n \end{aligned}$$

and for each $i = 1, \dots, n$

$$g_{1,i} \oplus g^{n,i} : \mathcal{G}'_1 \oplus \mathcal{G}''_n \rightarrow \mathcal{G}_i$$

do not have full rank. To see that the last condition makes sense, we observe that under the hypothesis that the first map has full rank, it must in fact be an isomorphism onto its image \mathcal{G}'_1 , and similarly for \mathcal{G}''_n . It suffices to check this on stalks, and the full rank condition implies that on each stalk, the fiber of \mathcal{G}'_1 has dimension ℓ . Thus Nakayama's lemma implies that \mathcal{G}'_1 is free of rank ℓ on stalks, and thus the map induced by the v_i is an isomorphism on stalks, as desired. The same argument goes through for \mathcal{G}''_n .

As before, we set $\mathcal{G}'_i = g_{1,i}\mathcal{G}'_1$ and $\mathcal{G}''_i = g^{n,i}\mathcal{G}''_n$, and we claim that we obtain the desired decomposition. Our first claim is that $g_{1,i} : \mathcal{G}'_1 \rightarrow \mathcal{G}'_i$ is an isomorphism for each i , and similarly for $g^{n,i}$. In particular, \mathcal{G}'_i and \mathcal{G}''_i are free \mathcal{O}_U -modules, and condition (ii) is satisfied. Arguing as before, this follows from Nakayama's lemma and the full rank hypotheses. Next, for each i , we have the natural morphism

$$\mathcal{G}'_i \oplus \mathcal{G}''_i \rightarrow \mathcal{G}_i$$

induced by the inclusions, which we wish to show is an isomorphism. Since both modules are free of rank m , to obtain the desired decomposition it suffices to check surjectivity, which holds on fibers by the full rank hypothesis, and hence on stalks by Nakayama's lemma.

Since we have already checked condition (ii), it remains to check that (i) is satisfied. We see that g_i maps \mathcal{G}'_i to \mathcal{G}'_{i+1} by construction. On the other hand, $g_i\mathcal{G}''_i = g_i g^i \mathcal{G}''_{i+1} = s \mathcal{G}''_{i+1}$ by condition (I) of a linked Hom space. Thus, g_i preserves the decomposition, and we see similarly that the maps g^i preserve the decomposition, giving us condition (i), as desired. \square

We can now prove our main theorem:

Proof of Theorem 1.1. The question being local on S , we may assume without loss of generality that the \mathcal{G}_i decompose as in Lemma 2.5. Denote by m_1 the rank of \mathcal{G}'_1 and by m_2 the rank of \mathcal{G}''_n . The existence of the isomorphisms $(g_i)'$ and $(g^i)''$ imply that \mathcal{G}'_i has rank m_1 for all i , and \mathcal{G}''_i has rank m_2 for all i . In particular, $m_1 + m_2 = m$. We then claim that in fact LH represents

$$\mathcal{H} := \mathcal{H}om(\mathcal{F}_1, \mathcal{G}'_1) \oplus \mathcal{H}om(\mathcal{F}_n, \mathcal{G}'_n)$$

and is thus a vector bundle of rank $rm_1 + rm_2 = rm$, as desired.

For $i < j$, denote by $(g_{i,j})'$ and $(g^{j,i})''$ the compositions

$$(g_{j-1})' \circ (g_{j-2})' \circ \cdots \circ (g_{i+1})' \circ (g_i)'$$

and

$$(g^i)'' \circ (g^{i+1})'' \circ \cdots \circ (g^{j-2})'' \circ (g^{j-1})'',$$

respectively. We use the convention that $(g_{i,i})'$ and $(g^{i,i})''$ denote the identity map.

Clearly, there is a natural forgetful map from h_{LH} to \mathcal{H} . To construct the inverse map, if we are given $\varphi_1'' \in \text{Hom}(\mathcal{F}_1, \mathcal{G}_1'')_T$ and $\varphi_n' \in \text{Hom}(\mathcal{F}_n, \mathcal{G}_n')_T$, we can construct $\varphi_i \in \text{Hom}(\mathcal{F}_i, \mathcal{G}_i)_T$ for all i by setting

$$\varphi_i = ((g_{i,n})')^{-1} \circ \varphi_n' \circ f_{i,n} \oplus ((g^{i,1})'')^{-1} \circ \varphi_1'' \circ f^{i,1}.$$

We check the linkage condition directly: we have

$$\begin{aligned} \varphi_{i+1} \circ f_i &= ((g_{i+1,n})')^{-1} \circ \varphi_n' \circ f_{i+1,n} \circ f_i \oplus ((g^{i+1,1})'')^{-1} \circ \varphi_1'' \circ f^{i+1,1} \circ f_i \\ &= ((g_{i+1,n})')^{-1} \circ \varphi_n' \circ f_{i,n} \oplus s((g^{i+1,1})'')^{-1} \circ \varphi_1'' \circ f^{i,1}, \end{aligned}$$

while

$$\begin{aligned} g_i \circ \varphi_i &= g_i \circ ((g_{i,n})')^{-1} \circ \varphi_n' \circ f_{i,n} \oplus g_i \circ ((g^{i,1})'')^{-1} \circ \varphi_1'' \circ f^{i,1} \\ &= ((g_{i+1,n})')^{-1} \circ \varphi_n' \circ f_{i,n} \oplus g_i \circ (g^i)'' \circ ((g^{i+1,1})'')^{-1} \circ \varphi_1'' \circ f^{i,1}, \end{aligned}$$

so we conclude equality from the fact that $g_i \circ (g^i)'' = \text{id}$. Similarly, we have $\varphi_i \circ f^i = g^i \circ \varphi_{i+1}$, so the linkage condition holds. Moreover, the fact that $(g_i)'$ and $(g^i)''$ are isomorphisms together with the linkage condition imply that the φ_i are uniquely determined by φ_1'' and φ_n' , so we obtain the desired isomorphism of functors and conclude the theorem. \square

Remark 2.6. Note that the conditions on the f_i, f^i are quite minimal, and the only additional conditions we impose on the g_i, g^i beyond that for a linked Grassmannian are that $\text{im } g_i$ actually be complementary to $\ker g_{i+1}$ and similarly for the g^i , while for linked Grassmannians we required only that $\text{im } g_i \cap \ker g_{i+1} = (0)$ and $\text{im } g^{i+1} \cap \ker g^i = (0)$.

We see moreover that this mild strengthening is indeed necessary in order for Theorem 1.1 to hold: consider the case that $r = 1$, $m = 3$, and $n = 3$, with $s = t^2$, all modules free, and in terms of chosen bases, maps given as follows:

$$\begin{aligned} f_1 &= f_2 = f^1 = f^2 = t, \\ g_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix}, \quad g^1 = \begin{bmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ g_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}, \quad g^2 = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

First suppose that S is a point, and that $t = 0$. In this case, the space LH parametrizes triples of vectors in $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ which are in the kernels of all relevant maps, and this has dimension 4, which is larger than $rm = 3$.

If instead S is the spectrum of a DVR, and t is a uniformizer, then we have that LH has dimension 3 over the generic point, but dimension 4 over the closed point, so LH is not even flat over S .

3. Application to limit linear series

We conclude with a brief explanation of the application of Theorem 1.1 to the construction of spaces of limit linear series. We follow the notation of [5]. We begin with a family $\pi : X \rightarrow B$ of curves of genus g satisfying the conditions of a smoothing family, as specified in Definition 3.1 of *loc. cit.* In particular, over the locus Δ of B over which X is not smooth, X is nodal, consisting of two smooth components Y and Z glued along a section Δ' of π . Although [5] also treats the case in which π is smooth, we will only be interested in the other two cases, where $\Delta = B$ or where Δ is a Cartier divisor in B . Given integers r, d , Definition 4.5 of *loc. cit.* then describes a limit linear series functor which can be summarized roughly as follows.

In the case that $\Delta = B$, to a B -scheme T , the functor associates the set of line bundles \mathcal{L} on $X|_T$ of degree d on $Y|_T$ and degree 0 on $Z|_T$, together with rank- $(r+1)$ subbundles V_i of $\pi_*\mathcal{L}_i$, where \mathcal{L}_i is obtained from \mathcal{L} by gluing together the line bundles $\mathcal{L}|_Y(-i\Delta')$ and $\mathcal{L}|_Z(i\Delta')$. We further require that each V_i map into V_{i+1} under the maps which are zero on Y and the natural inclusion on Z , and vice versa.

In the case that Δ is a Cartier divisor in B , to a B -scheme T the functor associates the set of line bundles \mathcal{L} on $X|_T$ of degree d , having degree d on $Y|_T$ and degree 0 on $Z|_T$, together with rank- $(r+1)$ subbundles V_i of $\pi_*\mathcal{L}_i$, where $\mathcal{L}_i := \mathcal{L} \otimes \mathcal{O}_X(iY)|_T$. We further require that the V_i map into one another under the maps which are the natural inclusions in one direction, and which are obtained from a choice of isomorphism $\mathcal{O}_X(-Z) \cong \mathcal{O}_X(Y)$ in the other.

In fact, the definition also allows for imposed ramification along sections, but this part of the construction is unaffected by Theorem 1.1, so for the sake of simplicity we ignore it. The construction of the scheme representing the functor (as given in Theorem 5.3 of [5]) then proceeds as follows. Let P be the relative Picard scheme parametrizing line bundles of degree d on Y and degree 0 on Z , and let \mathcal{L} be the universal line bundle on $X \times_B P$. Let \mathcal{L}_i be obtained from \mathcal{L} as in the definition of the functor, depending on which case we are in. Then the \mathcal{L}_i also have maps in both directions as described above. Choose D a sufficiently ample divisor on X , and consider the linked Grassmannian LG parametrizing tuples of rank- $(r+1)$ subbundles V_i of $p_{2*}\mathcal{L}_i(p_1^*D)$ which map into one another under the given maps. This space has dimension $\dim B + g + (r+1)(d + \deg D - g - r)$. We can cut out the scheme representing our functor by requiring that the V_i are in fact contained in $p_{2*}\mathcal{L}_i$ for each i , or equivalently that the induced maps $V_i \rightarrow p_{2*}(\mathcal{L}_i(p_1^*D)|_{p_1^*D})$ are all zero. However, the trick is to give the appropriate bounds on the dimension. In [5], this was accomplished in an *ad hoc* manner by writing $D = D_Y + D_Z$ for divisors D_Y, D_Z supported entirely on Y and Z respectively, and checking that imposing vanishing along p_1^*D for all i is equivalent to imposing vanishing on $p_1^*D_Y$ for V_0 , and vanishing on $p_1^*D_Z$ for V_d .

With Theorem 1.1, we can proceed more directly. We can rephrase the above discussion as follows: the space LG constructed above carries the universal chains of bundles $\{V_i\}$ and $\{p_{2*}(\mathcal{L}_i(p_1^*D)|_{p_1^*D})\}$, of ranks $r+1$ and $\deg D$ respectively, together with a linked homomorphism between them, and the desired limit linear series space is cut out precisely by the condition that this linked homomorphism be 0 for all i . Now, it is easy to see that our chains of bundles satisfy the conditions for a linked

Hom space. Indeed, condition (I) is inherited from the $p_{2*}\mathcal{L}_i(p_1^*D)$. Next, we have that the maps between the $p_{2*}(\mathcal{L}_i(p_1^*D)|_{p_1^*D})$ fail to be isomorphisms precisely over Δ . Over Δ , we have that a section of $\mathcal{L}_i(p_1^*D)|_{p_1^*D}$ is in the image of $\mathcal{L}_{i-1}(p_1^*D)|_{p_1^*D}$ if and only if it vanishes along $p_1^*(D \cap Y)$, which is to say if it is in the kernel of the map back to $\mathcal{L}_{i-1}(p_1^*D)|_{p_1^*D}$. Thus, condition (II) is satisfied. On the other hand, a section is in the kernel of the map to $\mathcal{L}_{i+1}(p_1^*D)|_{p_1^*D}$ if and only if it vanishes along $p_1^*(D \cap Z)$. However, $\mathcal{L}_i(p_1^*D)|_{p_1^*D}$ decomposes as a direct sum of the spaces of sections vanishing on $p_1^*(D \cap Y)$ and on $p_1^*(D \cap Z)$, so we obtain condition (III) as well. Because the limit linear series space is the preimage of the zero section under the given map to the linked Hom space, we conclude from Theorem 1.1 that each component has codimension at most $(r+1)\deg D$ inside LG , and thus that each component has dimension at least

$$\dim B + g + (r+1)(d-g-r) = \dim B + (r+1)(d-r) - rg,$$

agreeing with the formula obtained in Theorem 5.3 of [5].

Remark 3.1. Because we remove the necessity of writing the divisor D as $D_Y + D_Z$, we can also weaken the hypotheses on the smoothing families for which we construct limit linear series schemes. Specifically, we can replace the hypothesis that we have sections through each component of the degenerate fibers by the hypothesis that we have a single smooth section. This section, together with the canonical twisting line bundles, is all that is needed to construct the Picard schemes of given multidegrees, or to construct relatively very ample divisors. This should substantially simplify the construction of universal limit linear series spaces over (the complement of a codimension 2 subset of) $\overline{\mathcal{M}}_{g,1}$, as required for instance by Khosla in [3].

Acknowledgements. I would like to thank Montserrat Teixidor i Bigas for helpful comments.

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