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GROUP WIDTH

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Dedicated to the memory of John Stallings.

Abstract. There are many "minimax" complexity functions in mathematics: width of a tree or a link, Heegaard genus of a 3-manifold, the Cheeger constant of a Riemannian manifold. We define such a function w, "width", on countable (or finite) groups and show $w(\mathbb{Z}^k) = k - 1$.

Let K be a countable (or finite) simplicial complex and give the real line $\mathbb{R}$ the cell structure with the integers $\mathbb{Z} \subset \mathbb{R}$ the vertices. Abusing the usual terminology, we call any simplicial map $f : K \to \mathbb{R}$ "morse."

Definition 1. *Connected width rank*, $cwr(K) := \min_{\mathrm{f}\ ,\mathrm{morse}} \max_{\mathrm{i} \in \mathbb{Z}} \mathrm{rank}(inc_\#(\pi_1 C))$, where C is some connected component of $f^{-1}[i, i+1]$, and rank means the smallest number of generators of a given group. The inclusion is $C \subset K$, and $inc_\#(\pi_1 C)$ is a subgroup of $\pi_1 K$. $cwr(K)$ can assume values: $0, 1, 2, \ldots, \infty$.

Definition 2. Given a countable group G, its *width*, $w(G)$, is the minimum of $cwr(K)$ over all K with $\pi_1(K) \cong G$.

Clearly free groups (and only free groups) have width zero. Let's work out $w(\mathbb{Z}^k)$, the width of the free abelian group. Consider $f : K \to \mathbb{R}$ with $\pi_1(K) \cong \mathbb{Z}^k$. Define the quotient graph Q_f by taking an edge for each connected component of $f^{-1}(i)$ and a vertex for each component of $f^{-1}[i, i+1]$, $i \in \mathbb{Z}$, and gluing the latter to the former according to inclusion.

There is an induced map $\theta : K \to Q_f$ admitting a right inverse. The map θ determines an epimorphism of groups:

$$\pi_1 K \twoheadrightarrow \pi_1 Q_f,$$

so in our case: $\pi_1 K \cong \mathbb{Z}^k$, and Q_f is either contractible (a tree) or $Q_f \simeq S^1$, is homotopy equivalent to the circle.

First consider the case $Q_f \simeq pt$. Take a minimal connected subgraph $p \subset Q_f$ so that $H_1(\theta^{-1}(p); Q) \to H_1(K, Q)$ is onto, where Q denotes the rationals. To define "minimal" we order subgraphs by inclusion. We show that any such p is a single vertex. Suppose that p is minimal but larger than a single vertex. Cut p at the midpoint of some edge e to obtain the complementary subtrees $p_1, p_2 \subset p$. Let the inverse images under θ be $P_1, P_2 \subset P$. Applying the Q-homology Mayer-Vietoris sequence to the inclusions (and using connectivity of $P_1 \cap P_2$), we find that there are

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classes $b_1 \in H_1(P_1; Q)$ and $b_2 \in H_1(P_2; Q)$ so that image(b_1) and image($H_1(P_2; Q)$) are rationally independent in $H_1(K; Q)$ and image(b_2) and image($H_1(P_1; Q)$) are also independent in $H_1(K; Q)$. Let $\beta_1(\beta_2) \subset P_1(P_2)$ be corresponding loops carrying $b_1(b_2)$. The commutation of β_1 and β_2 in $\pi_1(K)$ conflicts usefully with the following lemma. Let T^+ be the 2-torus $S^1 \times S^1$ with "flanges" glued to the factor circles:

$$T^+ = S^1 \times S^1 \bigcup_{x \times * \equiv x \times * \times 0} S^1 \times * \times [0,1] \bigcup_{* \times x \equiv * \times x \times 0} * \times S^1 \times [0,1],$$

Denote $S^1 \times * \times 1$ by α_1 and $* \times S^1 \times 1$ by α_2.

Lemma 3. It is not possible to cover T^+ by open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ with $\alpha_1 \in \mathcal{U}_1$ and $\alpha_2 \in \mathcal{U}_2$ with image($H_1(\mathcal{U}_1, Q)$) $\subset H_1(T^+, Q)$ and image($H_1(\mathcal{U}_2, Q)$) $\subset H_1(T^+, Q)$, each rank one.

Proof. This is an exercise in Lusternik-Schnirelmann category. Consider the cup-product diagram:

$$\begin{array}{ccccc}
H^1(T^+, U_1; Q) & \times & H^1(T^+, U_2; Q) & \longrightarrow & H^2(T^+, T^+; Q) \cong 0 \\
\downarrow i_1 & & \downarrow i_2 & & \downarrow \\
H^1(T^+; Q) & \times & H^1(T^+; Q) & \longrightarrow & H^2(T^+; Q) \\
\cup & & \cup & & \cup \\
\hat{\alpha_1} & \times & \hat{\alpha_2} & \longmapsto & 1
\end{array}$$

Using the exact sequence of pairs, Image(i_1) = Span($\hat{\alpha_1}$) and Image(i_2) = Span($\hat{\alpha_2}$), where $\hat{\alpha}_i$ denotes the Poincaré dual to the loops α_i, $i = 1, 2$. The factoring of the cup product $\hat{\alpha_1} \smile \hat{\alpha_2} = 1$ through zero is a contradiction. □

Since $\pi_1(K)$ is abelian, there is a map $g : T^+ \to K$ carrying α_1 to β_1 and α_2 to β_2. Taking θ^{-1}, cutting the edge e divides K into K_1 and K_2, containing P_1 and P_2 (resp.). Let K_1^+ and K_2^+ be open sets containing and homotopy equivalent to K_1 and K_2 (resp.), $K_1^+ \simeq K_1$ and $K_2^+ \simeq K_2$. Setting $\mathcal{U}_i = g^{-1}(K_i^+)$, $i = 1, 2$, contradicts the lemma, showing P consists of a single vertex v. Thus if Q_f is a tree, a finite index subgroup L of $\pi_1(K) \cong \mathbb{Z}^k$ must be generated by $\theta^{-1}(v)$, and rank$(L) = k$.

Next consider the case $Q_f \simeq S^1$. Again take a minimal $p \subset Q_f$. The subgraph p must contain an imbedded circle $\gamma \subset Q_f$, otherwise the epimorphism $H_1(K, Q) \overset{\theta_*}{\twoheadrightarrow} H_1(Q_f; Q)$ would factor through a trivial $H_1(p; Q)$. By the preceeding argument, $p = \gamma$; we may trim off leaves of Q_f by arguing they cannot increase the image in the rationalized fundamental group, $H_1(K; Q)$.

Let $v_n = v_0, v_1, ..., v_{n-1}$ be the vertices of p and $V_n = V_0, V_1, ..., V_{n-1}$ their θ-preimages: $V_i = \theta^{-1}(v_i)$. We claim that for $0 \leq i, j \leq n-1$, image($H_1(V_i; \mathbb{Z})$) = image($H_1(V_j; \mathbb{Z})$) $\subset H_1(K)$. To see this, note that for any loop $\delta \subset V_i$, there is a map of a torus $h : S^1 \times S^1 \to K$ with $\theta h(S^1 \times *)$ parameterizing γ and $h(* \times S^1)$ parameterizing δ. Using transversality, we may arrange that corresponding to the center point $\widehat{e_1}, ..., \widehat{e_n}$ of each edge in p, $h^{-1}(\widehat{e_k})$ is a 1-manifold in $S^1 \times S^1$ meeting $S^1 \times *$ transversely in a single point. These 1-manifolds all (up to sign) represent the same class $inc_*[\delta] \in H_1(K; \mathbb{Z})$ since they are homologous on $S^1 \times S^1$.

Using the connectivity of $\theta^{-1}(v_k)$, $k = 0, .., n-1$, and again, the Mayer-Vietoris sequence, we see that $inc_* H_1(V_0 \cup \tilde{\gamma}; Q) = H_1(K; Q)$ where $\tilde{\gamma}$ is some lift of γ, $\theta(\tilde{\gamma}) = \gamma$. Similarly, for all V_k, $1 \le k \le n-1$. It is also clear that $inc_*[\tilde{\gamma}]$ and $inc_* H_1(V_0; Q)$ must be indepedent in $H_1(K; Q)$, otherwise a homotopy, in K, of a multiple of $\tilde{\gamma}$ into V_0, would, under θ, constitute a null homotopy in Q_f of a multiple of the essential cycle γ. Thus $inc_* H_1(V_0; Q)$ has rank$= k-1$ in $H_1(K; Q)$.

We have shown that if $Q_f \simeq *$, then some component C carries all of $\pi_1(K) \otimes Q \cong H_1(K; Q)$ and if $Q_f \simeq S^1$ then some component C carries a rank $k-1$ subgroup. Since the obvious Morse function on the k-torus T^k has all levels (even critical levels) carrying a rank $k-1$ subspace of $H_1(T^k, \mathbb{Z})$ (and has $Q_f \equiv S^1$), we conclude that $w(\mathbb{Z}^k) = k-1$.

Extensions

For a finite abelian group A, $w(A) = \text{rank}(A)$. The proof is similar to the computation of $w(\mathbb{Z}^k)$ except for two modifications. First, Q_f is now certainly a tree so only that case requires generalization. Second, in all computations, the rationals Q should be replaced with the field $\mathbb{Z}/q\mathbb{Z}$, where q is a prime contained in the factorization of order(A) at least as often as any other prime.

Combining the arguments for both free and torsion cases, one finds that for a finitely generated, but infinite, abelian group B, that $w(B) = \text{rank}(B) - 1$.

It is easy to say a little more about finite groups: a finite group F of width one is cyclic. To prove this, assume $\pi_1(K) \cong F$ and $f : K \to \mathbb{R}$ exhibits the width of F to be one. Choose a maximal subtree $p \subset Q_f$ with respect to the property that $\pi_1(P)$ has cyclic image X in $\pi_1(K)$, where $P = \theta^{-1}(p)$. Let p^+ be p union an adjacent edge e of Q_f and let $P^+ = \theta^{-1}(p^+)$. Write $P^+ = P \cup C$, where $C = \theta^{-1}(e)$. Note image$(\pi_1(P^+)) = H \subset \pi_1(K)$ is not cyclic, but image$(\pi_1(C)) =: Y \subset F$ is cyclic. Let $Z \subset F$ be the cyclic group $Z = X \cap Y \subset F$ and let $G := X \underset{Z}{*} Y$ be the abstract free product with amalgamation. There is an epimorphism $\gamma : G \twoheadrightarrow H$. Since G is infinite and H is finite, there must be a nontrivial relation $R \in \ker \gamma$. Since $Z \cap \ker \gamma = \{id\}$, R can be written as a cyclically reduced word alternating "letters" from $X \setminus Z$ and $Y \setminus Z$. Think of R as a map $R : D^2 \to K$, which on the boundary maps to a wedge of circles $S^1 \bigvee S^1$, the first summand lying in P and the second summand in C. Make R transverse to $P \cap C$ and consider an innermost arc $\omega \subset D^2$, $\omega \subset R^{-1}(P \cap C)$. The subdisk $\Delta \subset D^2$ between ω and ∂D^2 determines a (pointed) homotopy of some letter of R into $P \cap C$. Since image$(\pi_1(P \cap C)) \subset Z \subset F$, this contradicts the form of R, i.e. that its letters lie in $(X \setminus Z) \amalg (Y \setminus Z)$. It follows that $p = Q_f$ and $F = \pi_1(K)$ is cyclic.

Formal properties of width include:

$$w(G_1 \times G_2) \le w(G_1) + \text{rank}(G_2)$$

and

$$w(G_1 * G_2) = \max\{w(G_1), w(G_2)\}.$$

To prove the latter, given $f : K \to \mathbb{R}$ with $\pi_1(K) \cong G_1 * G_2$, one may precompose with the covering $\delta_j : K_j \to K$, $\pi_1(K_j) \cong G_j$, to obtain $f_j = f \circ \delta_j$, $j = 1$ or 2. For any connected component C_j of $f_j^{-1}[i, i+1]$, the image H_j of $\pi_1(C_j)$ in $\pi_1(K)$ is a free factor of the corresponding image H of $\pi_1(C)$ in $\pi_1(K)$, where $C =$

$\delta_j(C_j)$. Consequently, Grusko's theorem implies $\text{rank}(H_j) \leq \text{rank}(H)$, establishing $w(G_1 * G_2) \geq \max\{w(G_1), w(G_2)\}$. The opposite inequality is immediate.

Applications

The computation $w(\mathbb{Z}^k) = k-1$ immediately gives negative answers to two MathOverflow questions: mathoverflow.net/questions/30567/ and mathoverflow.net/questions/42629/. More specifically, for dimension $d \geq 4$ consider a smooth closed d-manifold M with $\pi_1(M) = \mathbb{Z}^k$. Any Morse function on M must have some connected component C of some level with first Betti number $b_1(C) \geq k-1$. It requires only a little thought to see that this estimate also applies to generic levels. The "complexity" of connected levels is thus seen to increase with k.

Secondly, if M is divided up into connected "blocks" along codimension$=1$ manifold faces, at least one block must have $b_1(\text{block}) \geq k-1$ (blocks will map to vertices of Q_f, their faces to edges of Q_f). Product collars can be added along faces to build a simplicial Morse function $M \to \mathbb{R}$ as in Definition 1, with all blocks corresponding to components C. Thus general d manifolds cannot be cut into simple pieces, comprising only a finite number n_d of diffeomorphism types, with purely $(d-1)$-manifold cuts, as was asked. If the (second) question, instead, permitted gluing along codimension one *and* two faces, it would not be touched by this group theoretic method and appears open. Also, restricting the question to simply connected manifolds would require a different method. This question looks difficult in the case of simply connected (smooth) 4-manifolds.

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References

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