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A REMARK ON ZAK'S THEOREM ON TANGENCIES

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Abstract. We present a slightly different formulation of Zak's theorem on tangencies as well as some applications. In particular, we obtain a better bound on the dimension of the dual variety of a manifold and we classify extremal and next-to-extremal cases when its secant variety does not fill up the ambient projective space.

1. Introduction

In this note, we state Zak's theorem on tangencies ([15, Theorem 0], see also [16, Ch. I, Corollary 1.8]) for non-singular complex varieties in the following way:

Theorem 1.1 (Reformulation of Zak's theorem on tangencies)**.** *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension n, and let $L \subset \mathbb{P}^N$ be a linear subspace of dimension m which is tangent to X along a closed subvariety $Y \subset X$ of dimension r. Then*

$$r \leq \min\{m - n, \dim SX - 1 - n\}.$$

The former statement appears to have some advantages. The new inequality $r \leq \dim SX - 1 - n$ is vacuous if the secant variety of X (denoted by SX) fills up $\mathbb{P}^N$, but it is significant when $SX \neq \mathbb{P}^N$. For instance, this is always the case if $m = N - 1$. In this setting, Zak's theorem on tangencies has several consequences (see [10] for an account) that can be sharpened, thanks to Theorem 1.1. Let $s := \dim SX$ and $c := N - s$.

Corollary 1.1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension n, and let $X^* \subset \mathbb{P}^{N*}$ denote its dual variety (of dimension n^*). The following holds:*

(i) *The twisted normal bundle $N_{X/\mathbb{P}^N}(-1)$ is k-ample for $k \geq s - 1 - n$ (cf.* [10, Example 6.3.7]).
(ii) $n^* \geq n + c$ (*cf.* [10, Corollary 3.4.20]). *In particular, if $SX \neq \mathbb{P}^N$ then X^* is a singular variety.*
(iii) *If $s \leq 2n - 1$ (resp. $2n - 2$) then every hyperplane section of X is reduced (resp. normal)* (*cf.* [10, Corollary 3.4.19]).

Some examples of manifolds satisfying the equality $n^* = n$ (and hence $SX = \mathbb{P}^N$) are given by hypersurfaces in $\mathbb{P}^{n+1}$, Segre embeddings $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$, the Grassmannian $\mathbb{G}(1, 4) \subset \mathbb{P}^9$ and the 10-dimensional spinor variety $S_4 \subset \mathbb{P}^{15}$. Moreover, these are the only examples under the additional assumption $3n \leq 2N$ by Ein [2, Theorem 4.5] (cf. Remark 2.3). On the other hand, if $SX \neq \mathbb{P}^N$ we show that furthermore $n^* \geq n + c + 1$ and manifolds satisfying the equality are classified, giving a new characterization of the Veronese surface:

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Theorem 1.2. *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension n. If $SX \neq \mathbb{P}^N$ then $n^* \geq n+c+1$, with equality if and only if X is either a curve or the Veronese surface in $\mathbb{P}^5$.*

Going one step further, let us consider the next-to-extremal case when $SX \neq \mathbb{P}^N$. If $n \leq 3$ it is easy to see (cf. Remark 2.2) that $n^* = n+c+2$ if and only if X is either a surface, or a dual defective threefold (i.e., a scroll over a curve), or a secant defective threefold (see [3] for the classification). On the other hand, for $n \geq 4$ we get the following:

Theorem 1.3. *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension $n \geq 4$. If $SX \neq \mathbb{P}^N$ and $n^* = n+c+2$, then X is a scroll over a manifold W and $\dim W \leq 2$.*

If $\dim W = 2$ we will prove in Theorem 3.1 that the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^{n-2}$ is the only scroll as in Theorem 1.3, so we actually get the following refinement:

Theorem 1.4. *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension $n \geq 4$ such that $SX \neq \mathbb{P}^N$. Then $n^* = n+c+2$ if and only if X is either a scroll over a curve, or (an isomorphic projection of) the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{3n-4}$.*

2. Proofs

Theorem 1.1 is a consequence of the following application of the Fulton–Hansen connectedness theorem [5]. First, we recall the definition of the relative tangent (resp. secant) variety. Given a subvariety $Y \subset X$, we define $T(Y,X) := \cup_{y \in Y} T_y X$, where $T_y X \subset \mathbb{P}^N$ denotes the embedded tangent space to X at $y \in Y$, and $S(Y,X) \subset \mathbb{P}^N$ as the closure of $\{z \in \mathbb{P}^N \mid \exists y \in Y, \exists x \in X \text{ with } z \in \langle y, x \rangle\}$.

Theorem 2.1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate manifold of dimension n, and let $Y \subset X$ be a closed subvariety of dimension r. Then either $\dim T(Y,X) = r+n$ and $\dim S(Y,X) = r+n+1$, or else $T(Y,X) = S(Y,X)$.*

Proof. See [16, Ch. I, Theorem 1.4]. □

We can now prove our results:

Proof of Theorem 1.1. Let $L \subset \mathbb{P}^N$ be a linear subspace of dimension m which is tangent to X along Y. Then $T(Y,X) \subset L$, but $S(Y,X) \not\subset L$ as $X \subset \mathbb{P}^N$ is non-degenerate. Therefore $T(Y,X) \neq S(Y,X)$, and hence $r+n = \dim T(Y,X) \leq \dim L = m$ by Theorem 2.1. But Theorem 2.1 also yields $r+n+1 = \dim S(Y,X) \leq \dim SX$, so $r \leq \min\{m-n, \dim SX - 1 - n\}$. □

Remark 2.1. If $SX \neq \mathbb{P}^N$ and $m = N-1$, the case we are more interested in, the new bound $r \leq s-1-n$ is sharp. For example, equality holds for Severi varieties [16, Ch. IV] when $Y \subset X$ is an $n/2$-dimensional quadric.

Proof of Corollary 1.1. According to Theorem 1.1, the dimension of the fibres of the second projection of the conormal variety $\mathcal{P}_X := \{(x,H) \mid T_x X \subset H\} \subset X \times \mathbb{P}^{N^*}$ is bounded by $s-1-n$. So $N_{X/\mathbb{P}^N}(-1)$ is k-ample for $k \geq s-1-n$. This proves (i). Since $\dim \mathcal{P}_X = N-1$, we get $n^* \geq n+c$. Assume now that X^* is smooth. Then $\dim (X^*)^* \geq n^* \geq n+c$, and Segre's reflexivity theorem $(X^*)^* = X$ ([13], see also [9] for a detailed account) yields $c = 0$, whence $SX = \mathbb{P}^N$ proving (ii). Part (iii) is an immediate consequence of Theorem 1.1. □

The main ingredients of the proof of Theorem 1.2 are Zak's classification of Severi varieties and Ein's bound on the defect of subcanonical manifolds [2, Theorem 4.4]:

Proof of Theorem 1.2. Assume $n \geq 2$ and $n^* \leq n+c+1$. Let $\operatorname{def}(X)$ and $\delta(X)$ denote the dual and secant defect of $X \subset \mathbb{P}^N$, respectively. As $\operatorname{def}(X) := N-1-n^*$ and $\delta(X) := 2n+1-s$, the inequality $n^* \leq n+c+1$ is equivalent to the inequality $\operatorname{def}(X)+\delta(X) \geq n-1$. Since we assume $SX \neq \mathbb{P}^N$, we get $\delta(X) \leq n/2$ by Zak's theorem on linear normality [16, Ch. II, Corollary 2.11] and equality holds if and only if $X \subset \mathbb{P}^N$ is a Severi variety. Assume first that the Picard group of X is cyclic. Then $\operatorname{def}(X) \leq (n-2)/2$ by Ein [2, Theorem 4.4], and hence

$$n-1 \leq \operatorname{def}(X)+\delta(X) \leq \frac{n-2}{2}+\frac{n}{2} = n-1$$

implies that $X \subset \mathbb{P}^N$ is a Severi variety with $\operatorname{def}(X) = (n-2)/2$. So X is the Veronese surface in $\mathbb{P}^5$. Assume now that the Picard group of X is not cyclic. Then $\delta(X) \leq 2$ by the Barth–Larsen theorem, as otherwise $X \subset \mathbb{P}^N$ could be isomorphically projected into $\mathbb{P}^{2n-2}$ (see for instance [10, Corollary 3.2.3]). Therefore $\operatorname{def}(X) \geq n-3$, whence $\operatorname{def}(X) = n-2$ by Landman's parity theorem (unpublished, see [2, Theorem 2.4]) and $X \subset \mathbb{P}^N$ is a scroll over a curve by Ein [2, Theorem 3.2]. This yields $1 \leq \delta(X) \leq 2$, contradicting Lemma 3.2. □

Remark 2.2. (i) The bound $n^* \geq n+c$ given in Theorem 1.1 is equivalent to the bound $\operatorname{def}(X)+\delta(X) \leq n$. Furthermore, if $SX \neq \mathbb{P}^N$ then $\operatorname{def}(X)+\delta(X) \leq n-1$ by Theorem 1.2. To the best of the author's knowledge, these relations involving both dual and secant defects appear to be new.

(ii) For $n \geq 3$ the bound obtained in Theorem 1.2 is equivalent to the bound $\operatorname{def}(X)+\delta(X) \leq n-2$. This can be seen as a refinement of the inequality $\operatorname{def}(X) \leq n-2$ of Landman and Zak when $SX \neq \mathbb{P}^N$ (cf. [16, Ch. I, Remark 2.7]).

(iii) Besides curves and the Veronese surface, the bound $n^* \geq n+c+2$ (or equivalently $\operatorname{def}(X)+\delta(X) \leq n-2$) is sharp. Equality holds for surfaces with $\delta(X)=0$, threefolds with $\delta(X)=1$, scrolls over curves with $\delta(X)=0$ and the Segre embeddings $\mathbb{P}^2 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{3n-4}$ with $n \geq 4$. We will prove in the sequel that these are actually the only ones.

The key of the proof of Theorem 1.3 is a recent characterization of scrolls among dual defective manifolds obtained by Ionescu and Russo in [6]:

Proof of Theorem 1.3. Let $n^* = n+c+2$, that is, $\operatorname{def}(X)+\delta(X) = n-2$. If X is a scroll over a manifold W then $\delta(X) \leq 2$ by the Barth–Larsen theorem. Thus $n-4 \leq \operatorname{def}(X) = n-2\dim W$, so $\dim W \leq 2$. If X is not a scroll, we can assume $\operatorname{def}(X) \leq (n+1)/3$ by Ionescu and Russo [6, Corollary 3.7] and $\delta(X) \leq (n-1)/2$ by the classification of Severi varieties. Therefore,

$$n-2 = \operatorname{def}(X)+\delta(X) \leq \frac{n+1}{3}+\frac{n-1}{2}$$

yields $n \leq 11$ and, in view of Landman's parity theorem, we get $(n, \operatorname{def}(X), \delta(X)) \in \{(5,1,2),(6,2,2),(9,3,4)\}$. The first two cases are excluded by Ein [1, Theorems 5.1 and 5.2]. In the third case, X is a Fano manifold of dimension 9 with cyclic Picard group generated by the hyperplane section and index $(n+\operatorname{def}(X)+2)/2 = 7$ (see [2, Lemma 4.2]), so it is ruled out by Mukai's classification of Fano manifolds of coindex 3 (see [11]). □

Remark 2.3. (i) In a similar way, we can prove that a non-degenerate n-fold $X \subset \mathbb{P}^N$ with $n^* = n$ is either the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$, or else $4n+5 \geq 3N$ and equality holds if and only if X is the 10-dimensional spinor variety $S_4 \subset \mathbb{P}^{15}$. Since $n^* = n$ we deduce $SX = \mathbb{P}^N$ by Corollary 1.1, and hence $\delta(X) = 2n+1-N$. We point out that $n^* = n$ is equivalent to $\mathrm{def}(X) + \delta(X) = n$. If X is a scroll then $\delta(X) \leq 2$, whence $\mathrm{def}(X) \geq n-2$. Therefore $\mathrm{def}(X) = n-2$, $\delta(X) = 2$ and X is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$ by Proposition 3.1. On the other hand, if X is not a scroll then $\mathrm{def}(X) \leq (n+2)/3$ and equality holds if and only if X is the 10-dimensional spinor variety by Ionescu and Russo [6, Corollary 3.7]. Thus $\delta(X) \geq (2n-2)/3$, and hence $4n+5 \geq 3N$, with equality if and only if X is the 10-dimensional spinor variety $S_4 \subset \mathbb{P}^{15}$.

(ii) If $n^* = n$ and one furthermore assumes that $3n \leq 2N$ (cf. [2, Theorem 4.5]) then one also gets hypersurfaces and the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$, as in [6, Corollary 3.9].

Proof of Theorem 1.4. In Theorem 1.3, if $\dim W = 2$ then $\mathrm{def}(X) = n-4$ and hence $\delta(X) = 2$, since $\mathrm{def}(X) + \delta(X) = n-2$. So we conclude in view of Theorem 3.1. □

3. A result on secant defective scrolls over surfaces

In this section we prove the results on scrolls quoted in Section 2. We say that $X_W \subset \mathbb{P}^N$ (or simply X) is a *scroll* if there exists a vector bundle $\mathcal{E}$ over a manifold W such that $X_W \cong \mathbb{P}_W(\mathcal{E})$ and the fibres of the map $\pi : X_W \to W$, that we denote by F_w for $w \in W$, are linearly embedded in $\mathbb{P}^N$. An equivalent definition of scroll is the following. Let $\mathbb{G}(k,N)$ denote the Grassmannian of k-planes in $\mathbb{P}^N$. Consider the incidence correspondence $\mathcal{U} := \{(\mathbb{P}^k, p) \mid p \in \mathbb{P}^k\}$ with projection maps $\pi_1 : \mathcal{U} \to \mathbb{G}(k,N)$ and $\pi_2 : \mathcal{U} \to \mathbb{P}^N$. For every subvariety $W \subset \mathbb{G}(k,N)$, we denote $\mathcal{U}_W := \pi_1^{-1}(W)$ and $X_W := \pi_2(\mathcal{U}_W)$. Then $X_W \subset \mathbb{P}^N$ is a scroll if and only if W is smooth and $\pi_2 : \mathcal{U}_W \to X_W$ is an isomorphism. The following consequence of Terracini's lemma [14] will be useful. Let $\Sigma_z \subset X$ denote the *entry locus* of $z \in SX$, that is, the closure of the set $\{x \in X \mid \exists x' \in X \text{ with } z \in \langle x, x' \rangle\}$. We recall that $\dim(\Sigma_z) = \delta(X)$ for general $z \in SX$.

Lemma 3.1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate scroll over W and let $z \in SX$ be a smooth point. If $\Sigma_z \cap F_w \neq \emptyset$ for every $w \in W$ then $SX = \mathbb{P}^N$.*

Proof. Let $T_z SX \subset \mathbb{P}^N$ be the embedded tangent space to SX at z. For every $w \in W$ there exists some $x \in \Sigma_z \cap F_w$, so $F_w \subset T_x X$. Then it follows from Terracini's lemma that $X = \cup_{w \in W} F_w \subset \cup_{x \in \Sigma_z} T_x X \subset T_z SX$. Since $X \subset \mathbb{P}^N$ is non-degenerate we deduce $T_z SX = \mathbb{P}^N$, and hence $SX = \mathbb{P}^N$. □

The following lemma is well known. We include a short proof based on Lemma 3.1:

Lemma 3.2. *Let $X \subset \mathbb{P}^N$ be a non-degenerate scroll over a curve. If $\delta(X) > 0$ then $SX = \mathbb{P}^N$.*

Proof. Since $\dim \Sigma_z = \delta(X) > 0$ for general $z \in SX$ and $\dim W = 1$, we deduce that $\Sigma_z \cap F_w \neq \emptyset$ for every $w \in W$. Therefore $SX = \mathbb{P}^N$ by Lemma 3.1. $\square$

Let $X_W \subset \mathbb{P}^N$ be an n-dimensional scroll. It follows from the Barth–Larsen theorem that $\delta(X) \leq 2$. From now on, we will focus on the extremal case $\delta(X) = 2$. On the one hand, if W is a curve then $X_W \subset \mathbb{P}^N$ is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$ (see [8, pp. 307–308]). We prove this result in a more geometric and elementary way. The idea of the proof is essentially due to Fyodor Zak:

Proposition 3.1. *The only n-dimensional scroll over a curve in $\mathbb{P}^{2n-1}$ is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$.*

Proof. For every $w \in W$, let $\sigma_w := \{g \in \mathbb{G}(n-1, 2n-1) \mid \mathbb{P}^{n-1}_g \cap F_w \neq \emptyset\}$. Then σ_w is a hyperplane section of $\mathbb{G}(n-1, 2n-1)$, embedded by Plücker, and $w \in \sigma_w$ is a singular point of multiplicity n. The intersection of W and σ_w is supported at w since $\pi_2 : \mathcal{U}_W \to X_W$ is injective. Moreover, W and σ_w meet transversally at w since $d\pi_2$ is injective. So we deduce that $\deg X_W = \deg W = m_w(\sigma_w) \cdot m_w(W) = n$, where $m_w(\sigma_w)$ and $m_w(W)$ denote the multiplicity of σ_w and W at w, respectively (see [4, Corollary 12.4]). Therefore, for every $w \in W$ there exists a hyperplane section σ_w in the Plücker embedding of W such that the intersection product $\sigma_w \cdot W = nw$. This property characterizes the rational normal curve of degree n, so $X_W \subset \mathbb{P}^{2n-1}$ is a non-degenerate (otherwise $F_w \cap F_{w'} \neq \emptyset$ for every $w' \in W$) rational normal scroll of degree n. Consequently, X_W is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$. $\square$

Remark 3.1. The hypothesis of Proposition 3.1 can be weakened. Arguing with a general $w \in W$, the same proof works if $W \subset \mathbb{G}(n-1, 2n-1)$ is an integral curve and $\pi_2 : \mathcal{U}_W \to X_W$ is an isomorphism (or even if $\pi_2 : \mathcal{U}_W \to X_W$ has finitely many double points).

On the other hand, if W is a surface there exists a complete classification of scrolls with $\delta(X) = 2$ only for $n = 3$ (see [7, Proposition 4; 12]). We now prove that (an isomorphic projection of) the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{3n-4}$ with $n \geq 4$ is the only scroll over a surface whose secant variety does not fill up the ambient space. The main idea of the proof is to show that $X \subset \mathbb{P}^N$ is swept out by a two-dimensional family of Segre embeddings $\mathbb{P}^1 \times \mathbb{P}^{n-2}$. More precisely, we prove that any two fibres of the scroll, F_w and $F_{w'}$, determine a Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-2}$ in the linear span $\langle F_w, F_{w'} \rangle =: \mathbb{P}^{2n-3}_{ww'}$ that they define.

Theorem 3.1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate scroll of dimension n over a surface. If $\delta(X) = 2$ and $SX \neq \mathbb{P}^N$ then X is (an isomorphic projection of) the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{3n-4}$.*

Proof. We claim that $\dim S(F_w, X) = 2n-2$ for general $w \in W$. Since $\langle F_w, F_{w'} \rangle = S(F_w, F_{w'}) \subsetneq S(F_w, X)$ for every $w' \in W$, we get $\dim S(F_w, X) \geq 2n-2$. Fix a general $z \in SX$. We deduce from Lemma 3.1 that $\Sigma_z \cap F_w = \emptyset$ for general $w \in W$, and hence $z \notin S(F_w, X)$. Therefore $S(F_w, X) \subsetneq SX$ proving the claim, as $\dim SX = 2n-1$. For every $w \in W$, consider the subvariety $\mathcal{G}_w := \{\langle F_w, F_{w'} \rangle \mid w' \in W\} \subset \mathbb{G}(2n-3, N)$. If $\dim \mathcal{G}_w = 2$ for general $w \in W$ then $S(F_w, X) \subset \mathbb{P}^N$ is a $(2n-2)$-dimensional subvariety swept out by a two-dimensional family of $(2n-3)$-dimensional linear subspaces, so $S(F_w, X) \subsetneq \mathbb{P}^N$ itself is a linear subspace. This contradicts the non-degeneracy of $X \subset \mathbb{P}^N$. Thus $\dim \mathcal{G}_w = 1$ for general (and hence every) $w \in W$. In particular, for every $w, w' \in W$ there exists an integral curve $T_{ww'} \subset W$ such that $\langle F_w, F_{w'} \rangle = \langle F_w, F_{w''} \rangle$ for every $w'' \in T_{ww'}$. So $X_{T_{ww'}} = \mathbb{P}^1 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{2n-3}_{ww'}$ by Remark 3.1. Consequently, for every $w \in W$ and every $x \in F_w$ there exists a one-dimensional family of lines each of them meeting F_w at x and giving a two-dimensional cone $C_x \subset X$. Since F_w and C_x are contained in $T_xX = \mathbb{P}^n$ we deduce $\deg(C_x) = C_x \cdot F_w$. We claim that $C_x \cdot F_{w'} = 1$ for every $w' \in W$, and hence $\deg(C_x) = C_x \cdot F_w = C_x \cdot F_{w'} = 1$. Let us prove the claim. If $C_x \cdot F_{w'} \geq 2$ then $T_xX = \langle F_w, C_x \cap F_{w'} \rangle \subset \langle F_w, F_{w'} \rangle$. Therefore $T(F_w, X) \subset \langle F_w, F_{w'} \rangle$, contradicting Theorem 1.1. Since $\deg(C_x) = C_x \cdot F_w = 1$, we deduce that $C_x = \mathbb{P}^2$ for every $x \in F_w$ and that C_x is a section of $\pi : X_W \to W$ (in particular, $W \cong \mathbb{P}^2$). Thus X is a scroll over $\mathbb{P}^2$ and $\mathbb{P}^{2n-2}$, respectively. So $X \subset \mathbb{P}^N$ is an isomorphic projection of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{3n-4}$. □

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