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NON-LEFT-COMPLETE DERIVED CATEGORIES

Amnon Neeman

Abstract. We give some examples of abelian categories $\mathcal{A}$ for which the derived category $\mathbf{D}(\mathcal{A})$ is not left-complete. Perhaps the most natural of these is where $\mathcal{A}$ is the category of representations of the additive group $\mathbb{G}_a$ over a field k of characteristic $p > 0$.

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0. Assumed background

In this article, we assume the reader is familiar with derived categories and with t–structures on them. See Verdier [5] for the theory of derived categories, and Beilinson *et al.* [1, Chapter 1] for an introduction to t–structures.

1. The counterexample

Suppose $\mathcal{A}$ is an abelian category and $\mathbf{D}(\mathcal{A})$ is its derived category. For any object $x \in \mathbf{D}(\mathcal{A})$, we write $x^{\geq n}$ for the truncation of x with respect to the standard t–structure. We have canonical maps $x^{\geq n} \longrightarrow x^{\geq n+1}$, and a (non-canonical) map

$$\varphi_x \ : \ x \longrightarrow \underleftarrow{\mathrm{Holim}} \ x^{\geq n}.$$

The category $\mathbf{D}(\mathcal{A})$ is said to be *left-complete* if, for every object $x \in \mathbf{D}(\mathcal{A})$, any map φ_x as above is an isomorphism. Even though the map φ_x is not canonical, it can be shown that, for given x, if one φ_x is an isomorphism then they all are.

The reader can find much more about left-complete categories in Lurie [3, Section 7] or [4, Subsection 1.2.1, more precisely starting from Proposition 1.2.1.17]. See also Drinfeld and Gaitsgory [2].

In this note we will see how to produce many $\mathcal{A}$ for which $\mathbf{D}(\mathcal{A})$ is not left-complete. Our counterexamples will be of a very special form, which allows us to easily compute the homotopy inverse limit $\underleftarrow{\mathrm{Holim}}\ x^{\geq n}$. Let us now sketch what we will do.

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We will suppose that the abelian category $\mathcal{A}$ satisfies the axiom [AB4], that is, coproducts are exact; this makes it easy to compute coproducts in the derived category $\mathbf{D}(\mathcal{A})$, just form the coproduct as complexes. Suppose A is an object in our [AB4] abelian category $\mathcal{A}$, and let

$$x = \coprod_{i=0}^{\infty} A[i].$$

It is clear that, for $n > 0$, we have

$$x^{\geq -n} = \coprod_{i=0}^{n} A[i] = \prod_{i=0}^{n} A[i],$$

where the last equality is because finite coproducts agree with finite products. Now the homotopy inverse limit of the products is a genuine inverse limit, and we have

$$\underleftarrow{\text{Holim}}\ x^{\geq n} = \prod_{i=0}^{\infty} A[i].$$

Thus, our problem reduces to deciding whether the map

$$\coprod_{i=0}^{\infty} A[i] \xrightarrow{\ \varphi\ } \prod_{i=0}^{\infty} A[i]$$

is an isomorphism. Note that in this case the map is canonical; our homotopy inverse limit happens to be a genuine inverse limit, removing the arbitrariness. The left-hand side is easy to work with; its cohomology is A in each degree $n \leq 0$. What we will show is how to produce examples where the right-hand side has a lot of more cohomology. More precisely, we have

$$\prod_{i=0}^{\infty} A[i] = A[0] \oplus \left(\prod_{i=1}^{\infty} A[i] \right)$$

and the expectation would be for the second term to have a vanishing H^0; what we will show is how to produce non-zero classes in

$$H^0 \left(\prod_{i=1}^{\infty} A[i] \right).$$

It is time to disclose what will be our choice for the category $\mathcal{A}$ and for the object $A \in \mathcal{A}$.

Construction 1.1. Let k be a field, let R_1 be a finitely generated k algebra, and let $\mathfrak{m}$ be a k–point of $\text{Spec}(R_1)$. In other words, $\mathfrak{m} \subset R$ is a maximal ideal with $R_1/\mathfrak{m} \cong k$. We make a string of definitions:

(i) $R_n = \otimes_{i=1}^{n} R_1$, where the tensor is over the field k.

(ii) The inclusion $R_n \longrightarrow R_{n+1}$ is the inclusion of the tensor product of the first n terms.

(iii) $R = \underrightarrow{\text{colim}} R_n$.

(iv) The map $\Phi_i : R_1 \longrightarrow R$ is the inclusion of the ith factor.

(v) The category $\mathcal{A}$ will be the category of all those R–modules, on which $\Phi_i(\mathfrak{m})$ acts trivially for all but finitely many i.

The object $A \in \mathcal{A}$ will be the colimit over n of the R_n–modules $k = \otimes_{i=1}^{n}[R_1/\mathfrak{m}]$.

The main result is

Theorem 1.1. *Assume that $k = R_1/\mathfrak{m}$ is not projective over the localization $(R_1)_{\mathfrak{m}}$ of the ring R_1 at the maximal ideal $\mathfrak{m}$. With the category $\mathcal{A}$ and the object $A \in \mathcal{A}$ as in Construction 1.1, there is a non-zero element in*

$$H^0\left(\prod_{i=1}^{\infty} A\right).$$

Remark 1.1. The case where $R_1 = k[x]/(x^p)$ is of particular interest. If the field k is of characteristic p then the category $\mathcal{A}$ happens to be the category of representations of the additive group $\mathbb{G}_a$, and we learn that its derived category is not left-complete.

Remark 1.2. We trivially have

$$\prod_{i=1}^{\infty} A[i] = \left(\prod_{i=1}^{n} A[i]\right) \oplus \left(\prod_{i=n+1}^{\infty} A[i]\right),$$

and hence

$$H^0\left(\prod_{i=1}^{\infty} A[i]\right) = H^0\left(\prod_{i=1}^{n} A[i]\right) \oplus H^0\left(\prod_{i=n+1}^{\infty} A[i]\right).$$

On the other hand, with the finite product we have no problem computing

$$H^0\left(\prod_{i=1}^{n} A[i]\right) = H^0\left(\coprod_{i=1}^{n} A[i]\right) = 0,$$

and Theorem 1.1 now allows us to deduce that

$$H^0\left(\prod_{i=n+1}^{\infty} A[i]\right) \neq 0.$$

Translating, we have

$$H^n\left(\prod_{i=1}^{\infty} A[i]\right) \neq 0$$

for all $n \geq 0$. The complexes $A[i]$, $i > 0$ all belong to $\mathbf{D}(\mathcal{A})^{<0}$, but the product $\prod_{i=1}^{\infty} A[i]$ is not bounded above.

2. The proof

We begin with a little lemma.

Lemma 2.1. *Let k be a field, and let R and S be finitely generated k–algebras. Suppose further that we are given k–points of* $\operatorname{Spec}(R)$ *and* $\operatorname{Spec}(S)$*; that is $\mathfrak{m} \subset R$ and $\mathfrak{n} \subset S$ are maximal ideals, with*

$$R/\mathfrak{m} \cong k \cong S/\mathfrak{n}.$$

Let E be an injective envelope of $k = R/\mathfrak{m}$ over the ring R, and F an injective envelope of $k = S/\mathfrak{n}$ over the ring S. Then $E \otimes_k F$ is an injective envelope of k over the ring $R \otimes_k S$.

Proof. We will first prove the case where R and S are polynomial rings.

Let $R' = k[x_1, x_2, \ldots, x_m]$ be a polynomial ring, and let $\mathfrak{m}$ be the maximal ideal generated by $\{x_1, x_2, \ldots, x_m\}$. Then we know the injective envelope E' of $k = R'/\mathfrak{m}$ explicitly: it is the quotient of $S = k[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_m, x_m^{-1}]$ by the R'–submodule generated by all monomials $x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$ with at least one of the $i_j > 0$. As a k–vector space $E' = k[x_1^{-1}, x_2^{-1}, \ldots, x_m^{-1}]$, and the R'-module structure is obvious when we declare $x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} = 0$ if some $i_j > 0$. If $S' = k[y_1, y_2, \ldots, y_n]$ and $\mathfrak{n} \subset S'$ is the ideal generated by $\{y_1, y_2, \ldots, y_n\}$, then the fact that

$$E' \otimes_k F' = k[x_1^{-1}, x_2^{-1}, \ldots, x_m^{-1}] \otimes_k k[y_1^{-1}, y_2^{-1}, \ldots, y_n^{-1}]$$

is the injective hull of k over $R' \otimes S'$ is by inspection.

Now for the general case: assume $R = R'/I$ and $S = S'/J$ where R' and S' are polynomial rings, and $I \subset R'$ and $J \subset S'$ are ideals contained in the $\mathfrak{m}$ and $\mathfrak{n}$ above. Then the injective hull E of $k = R/\mathfrak{m}$ over the ring R is the largest R–submodule of the R'–module E', that is, the R'–submodule $E \subset E'$ of all elements annihilated by the ideal I. The lemma therefore comes down to the fact that the submodule of $E' \otimes_k F'$ annihilated by the ideal $I \otimes_k S' + R' \otimes_k J$ is precisely $E \otimes_k F$. □

Proof of Theorem 1.1. Let $\overline{R}$ be the localization of R_1 at the maximal ideal $\mathfrak{m}$. We are assuming that k is not projective over $\overline{R}$, that is the projective dimension of k is at least one. Choose and fix a minimal free resolution of $k = \overline{R}/\mathfrak{m}\overline{R}$ as an $\overline{R}$–module. Let us write this resolution as

$$\longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0 .$$

Then the modules P_i are all finite and free over the ring $\overline{R}$, the differentials are all matrices over $\overline{R}$, and the minimality guarantees that the entries in these matrices all belong to the ideal $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R} \subset \overline{R}$. Now let E be the $\overline{R}$–injective envelope of the module k; applying the functor $\mathrm{Hom}_{\overline{R}}(-, E)$ to the projective resolution above, we produce an injective resolution I^* of k, which we write as

$$0 \longrightarrow k \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow$$

We know that each $I^j = \mathrm{Hom}(P_j, E)$ is a finite coproduct of copies of E, and that the differentials $I^j \longrightarrow I^{j+1}$ are matrices whose entries belong to the ideal $\overline{\mathfrak{m}}$. The fact that the projective dimension of k is at least one tells us that $P_1 \neq 0$, and therefore $I^1 \neq 0$. Note that an injective envelope E of k over the localized ring $\overline{R} = (R_1)_{\mathfrak{m}}$ is also an injective envelope of k over the ring R_1, and hence we have produced an injective resolution of k over R_1. Next we

(i) Choose a non-zero element a in the image of the map $k \longrightarrow I^0$.
(ii) Choose a non-zero element $b \in I^1$, with $\mathfrak{m}b = 0$.

If we view k as a module over the ring $R_n = \otimes_{i=1}^n R_1$, then the tensor product $J_n^* = \otimes_{i=1}^n I^*$ is certainly a resolution of k as an R_n module, and Lemma 2.1 guarantees further that

(iii) Each J_n^i is injective as a module over R_n.
(iv) Let the inclusion $J_n^* \longrightarrow J_{n+1}^*$ be the map taking $x \in J_n^*$ to

$$x \otimes a \in J_n^* \otimes I^0 \subset J_n^* \otimes I^* = J_{n+1}^* ,$$

where $a \in I^0$ is as in (i) above. We define J^* to be

$$J^* = \underrightarrow{\mathrm{colim}} J^*_n \ ;$$

then J^* is an injective resolution of k in the category $\mathcal{A}$.

To prove the theorem we need to find a non-zero element in $H^0\left(\prod_{i>0} k[i]\right)$, and our next observation is that the product in the derived category $\prod_{i>0} k[i]$ is obtained as the the ordinary product of injective resolutions. The complex $J^*[i]$ is an injective resolution of $k[i]$, and hence the derived product $\prod_{i>0} k[i]$ is just the usual product $\prod_{i>0} J^*[i]$. Now for every $i \geq 1$ let

$$S_i = \{i^2+1, \dots, i^2+i\},$$

and observe that the sets S_i are disjoint. In the injective R_{i^2+i}–module

$$J^i_{i^2+i} = \coprod_{\sum \ell_m = i} I^{\ell_1} \otimes I^{\ell_2} \otimes \cdots \otimes I^{\ell_{i^2+i}}$$

or more specifically in the summand

$$\left(I^0\right)^{\otimes i^2} \otimes \left(I^1\right)^{\otimes i}$$

we take the term

$$\lambda_i = a^{\otimes i^2} \otimes b^{\otimes i},$$

where $a \in I^0$ and $b \in I^1$ are as in (i) and (ii) above. The embedding $J^*_{i^2+i} \longrightarrow J^*$ of (iv) gives us an element which we will denote $\lambda_i \in J^i$. The elements λ_i have the properties

(v) Each λ_i is a cycle; the differential $J^i \longrightarrow J^{i+1}$ kills λ_i.
(vi) $\Phi_j(\mathfrak{m})\lambda_i = 0$ for all i and j.

We are assuming $i > 0$, so each λ_i must be a boundary because $H^i(J^*) = 0$. But if $\mu_i \in J^{i-1}$ maps to λ_i, then there must exist an integer $j \in S_i$ so that $\Phi_j(\mathfrak{m})$ does not kill μ_i. Now form the element

$$\prod_{i=1}^{\infty} \lambda_i \in \prod_{i=1}^{\infty} J^i,$$

where the product is in the category of all R–modules.

Caution 2.1. The reader is reminded that the category $\mathcal{A}$ is a subcategory of the category of R-modules. Both categories have infinite products; the products in the category of R-modules are just the usual cartesian products, while the products in $\mathcal{A}$ are subtler. To form the product in $\mathcal{A}$ of a bunch of objects in $\mathcal{A}$, one first forms the usual cartesian product, and then considers inside it the largest object belonging to $\mathcal{A}$, that is the collection of all elements satisfying part (v) of Construction 1.1.

The element $\prod_{i=1}^{\infty} \lambda_i$ is a degree 0 cycle in the complex $\prod_{i\geq 1} J^*[i]$, and it is annihilated by $\Phi_j(\mathfrak{m})$ for all j. By Caution 2.1 we have that $\prod_{i=1}^{\infty} \lambda_i$ belongs to $\prod_{i=1}^{\infty} J^i$ even when the product is understood in $\mathcal{A}$. However, it is not a boundary in $\mathcal{A}$. If we try to express $\prod_{i=1}^{\infty} \lambda_i$ as the boundary of

$$\prod_{i=1}^{\infty} \mu_i \in \prod_{i=1}^{\infty} J^{i-1},$$

then we discover that each μ_i fails to be annihilated by some $\Phi_j(\mathfrak{m})$ with $j \in S_i$. As the S_i are disjoint, this produces infinitely many $\Phi_j(\mathfrak{m})$ not annihilating $\prod_{i=1}^{\infty} \mu_i$, meaning that it does not belong to $\mathcal{A}$. $\square$

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Centre for Mathematics and its Applications, Mathematical Sciences Institute, John Dedman Building, The Australian National University, Canberra, ACT 0200, AUSTRALIA
E-mail address: Amnon.Neeman@anu.edu.au