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A MAASS LIFTING OF Θ^3 AND CLASS NUMBERS OF REAL AND IMAGINARY QUADRATIC FIELDS

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Abstract. We give an explicit construct of a harmonic weak Maass form F_Θ that is a "lift" of Θ^3, where Θ is the classical Jacobi theta function. Just as the Fourier coefficients of Θ^3 are related to class numbers of imaginary quadratic fields, the Fourier coefficients of the "holomorphic part" of F_Θ are associated to class numbers of real quadratic fields.

1. Introduction and Statement of Results

Ramanujan's mock theta functions proved mysterious for more than 80 years. They are q-hypergeometric series such as

$$f(\tau) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}, \tag{1.1}$$

with $q := \mathrm{e}^{2\pi i\tau}$ ($\tau \in \mathbb{H}$), that have "nearly modular" properties, but fail to be fully modular. Modularity, explained by Zwegers in his Ph.D. thesis [22, 23], is obtained if one adds to the mock theta function a certain non-holomorphic integral

$$P_g(\tau) := \int_{-\overline{\tau}}^{i\infty} \frac{g(z)}{\sqrt{-i(\tau+z)}}\, dz.$$

Here g is a weight 3/2 unary theta function, which in the case of the mock theta function f is given by

$$g(\tau) := \frac{1}{6} \sum_{n\equiv 1 \pmod 6} n q^{\frac{n^2}{24}}$$

(see [22]). The resulting function, $\mathcal{M}_f(\tau) := q^{-\frac{1}{24}} f(\tau) + 2i\sqrt{3}P_g(\tau)$, is a harmonic weak Maass form of weight 1/2 (see Section 2 for the definition). Given $\mathcal{M}_f$ we may recover g by applying a differential operator $\xi_{\frac12}$ (also see Section 2). Following Zagier [21], we call the image of $\mathcal{M}_f$ under $\xi_{\frac12}$ the *shadow* of the mock theta function f. In this case $\xi_{\frac12}\mathcal{M}_f$ is a unary theta function. In general, the shadow is a modular form. In view of this, we define a *mock modular form* to be the holomorphic part (see Section 2) of a harmonic weak Maass form.

Conversely, one may begin with a non-holomorphic integral of a weight 3/2 unary theta function and produce a mock theta function. The resulting mock theta functions may be written as Lerch sums [23] and in many cases may also be written as q-series similar to that in (1.1). A similar construction exists for the analogous

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non-holomorphic integral of a weight 1/2 unary theta function [2, 3]. However, no such construction exists for nonunary theta functions.

This raises the question whether a nonunary theta function may appear as the shadow of a mock modular form. Let F be a harmonic weak Maass form of weight 1/2. In this paper, we give an explicit construction of such an example, namely Θ^3, where

$$\Theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}.$$

From work of Bruinier and Funke [4] we know the existence of such a mock modular form, but not its explicit form. In fact, Bruinier and Funke [4] implies the existence of a mock modular form with shadow equal to any holomorphic modular form of positive weight.

In recent work, Duke *et al.* [11] construct mock modular forms that have certain weight 3/2 weakly holomorphic forms as their shadows. Their work and work of Knopp [16] suggests that such a mock modular form can be constructed from a weight 1/2 non-holomorphic Poincaré series. Earlier work of Kubota [15] also gives insight into the construction of such a form. Along these lines, we give an explicit construction of a harmonic weak Maass form with shadow equal to Θ^3 and prove that the coefficients of the associated harmonic weak Maass form are related to class numbers of real and imaginary quadratic fields.

The theory of mock modular forms has exploded in recent years and with its development many questions have arisen concerning the arithmetic nature of the Fourier coefficients of mock modular forms (see, for example, [6, 7, 11, 25]). For instance, one might ask: When are the coefficients rational or algebraic? It is believed that the Fourier coefficients of a mock modular form are rational when the shadow is a modular form with complex multiplication. For instance, as a result of the constructions for unary theta functions, it is clear that the coefficients of the associated mock modular form are integral (see, for example, [24]).

This paper demonstrates that the coefficients of lifts of nonunary theta function are essentially given by special values of Dirichlet L-functions associated to quadratic fields. In our specific case, the coefficients of the mock modular form are related to the logarithms of fundamental units of quadratic fields. Despite not being integral, the coefficients of the mock modular forms associated with theta functions still carry arithmetic data. For more examples of mock modular forms whose coefficients are not rational but still encode arithmetic information see, for instance, [6, 7, 11].

Returning to our example, let us start by recalling what is known about the relationship between Θ^3 and L-functions of quadratic fields. Recall that the Fourier coefficients of Θ^3, which we denote by $r(n)$, themselves encode class numbers. To be more precise, for $N > 0$ and $N \equiv 0, 3 \pmod 4$, we write $H(-N)$ for the Hurwitz class number, i.e., the number of equivalence classes of quadratic forms of discriminant $-N$, where each class C is counted with multiplicity $1/\mathrm{Aut}(C)$. Then

$$r(n) = \begin{cases} 12H(-4n) & \text{if } n \equiv 1, 2 \pmod 4, \\ r(n/4) & \text{if } n \equiv 0 \pmod 4, \\ 24H(-n) & \text{if } n \equiv 3 \pmod 8, \\ 0 & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

The Hurwitz class number $H(-N)$ itself is related to the class number $h(-N)$ of the ring of integers of $\mathbb{Q}(\sqrt{-N})$. To state this relationship accurately, write $-N = -\Delta_N f^2$, where $-\Delta_N$ is a negative fundamental discriminant. Then we have that [10]

$$H(-N) = \frac{2h(-\Delta_N)}{\omega_N} T_1^{\psi_{-N}}(f),$$

where ω_N denotes the number of units in $\mathbb{Q}(\sqrt{-N})$ and for $n \neq 0$, $\psi_n(\cdot) := \left(\frac{D}{\cdot}\right)$ with D the discriminant of $\mathbb{Q}(\sqrt{n})$. Moreover, for a character χ, T_s^χ is the multiplicative function defined by

$$T_s^\chi(w) := \sum_{a|w} \mu(a)\chi(a)a^{s-1}\sigma_{2s-1}\left(\frac{w}{a}\right),$$

where μ is the Möbius function and σ_ℓ denotes the ℓth divisor sum. We also write $T_1 := T_1^{\psi_1}$.

Finally, the relationship between $r(n)$ and special values of L-functions of quadratic fields is given by Dirichlet's class number formula, which for ψ_n with $n < 0$ states

$$L(1, \psi_n) = \frac{2\pi}{w_n\sqrt{-n}} h(n).$$

Here, $L(s, \chi)$ is the Dirichlet L-function associated to character χ.

In conclusion we may note that the Fourier coefficients $r(n)$ are related to the value of Dirichlet L-functions associated to imaginary quadratic fields at $s = 1$. In our work we show that the complementary values for real quadratic fields appear as coefficients in the mock modular form having shadow Θ^3. To state our results accurately define $F_\Theta : \mathbb{H} \to \mathbb{C}$ by the following Fourier expansion:

$$F_\Theta(\tau) := \sum_{n=0}^{\infty} c^+(n)q^n + 2y^{\frac{1}{2}} + \sum_{n=1}^{\infty} c^-(n)\Gamma(\tfrac{1}{2}; 4\pi ny)q^{-n},$$

where $\Gamma(a; x) := \int_x^\infty \mathrm{e}^{-t}t^{a-1}\,dt$ denotes the incomplete gamma-function, and the coefficients $c^+(n)$ and $c^-(n)$ are given by

$$\begin{aligned} c^+(n) :=& \pi \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}} \overline{Z_{-n}}, \\ c^-(n) :=& \sqrt{\pi}\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}} \overline{Z_n}. \end{aligned}$$

Remark. In Section 2 it is shown that

$$c^-(n) = -\frac{1}{2\sqrt{\pi n}} r(n).$$

To define the values Z_n, we write $n \neq 0$ as $n = f^2 d$ with d squarefree and $f = 2^q w$ with w odd, and let

$$c_n := \begin{cases} 2 - \psi_{-n}(2) & \text{if } n \equiv 1, 2 \pmod 4, \\ 2^{-Q}(1 - \psi_{-n}(2)) & \text{otherwise,} \end{cases}$$

where

$$Q := \begin{cases} q & \text{if } d \equiv 3 \pmod 4, \\ q - 1 & \text{if } d \equiv 1, 2 \pmod 4. \end{cases}$$

Then we define

$$Z_n := \begin{cases} \mathrm{e}^{\frac{3\pi \mathrm{i}}{4}} \frac{6}{\pi^2} \log(2) & \text{if } n = 0, \\ \mathrm{e}^{\frac{3\pi \mathrm{i}}{4}} \frac{6}{\pi^2} \log(2) \frac{T_1(w)}{w} & \text{if } n \text{ is a square}, \\ \mathrm{e}^{\frac{3\pi \mathrm{i}}{4}} \frac{6}{\pi^2} L(1, \psi_{-n}) \frac{T_1^{\psi_{-n}}(w)}{w} \cdot c_n & \text{otherwise.} \end{cases}$$

Let

$$F_\Theta^+(\tau) := \sum_{n=0}^{\infty} c^+(n) q^n.$$

Dirichlet's class number formula for real quadratic fields, that is for $n > 0$, states

$$L(1, \psi_n) = \frac{\log(\epsilon_n)}{\sqrt{n}} h(n),$$

where ϵ_n is the fundamental unit in the field $\mathbb{Q}(\sqrt{n})$. Therefore, the coefficients F_Θ^+ may be written as simple expressions in terms of class numbers.

Theorem 1.1. *The function F_Θ^+ is a mock modular form of weight $\frac{1}{2}$ with respect to $\Gamma_0(4)$ with shadow Θ^3. Furthermore, the harmonic weak Maass form F_Θ is a Hecke eigenform.*

Remark. Nonunary theta functions are closely related to Eisenstein series of half-integral weight. Such series typically have Whittaker–Fourier coefficients equal to a quotient of Hecke L-functions, often associated to imaginary quadratic fields. In general, a mock modular form with shadow equal to a nonunary theta function will have Fourier coefficients of the same shape, often associated to real quadratic fields.

See Section 4 for further discussion on the work of Duke, Imamoḡlu, Tóth and other works dealing with the arithmetic nature of the Fourier coefficients of harmonic weak Maass forms.

In Section 2, we construct a Maass–Poincaré series of weight 1/2 related to Θ^3. In Section 3, we compute its Fourier expansion, resulting in the relations to L-series and proving Theorem 1.1.

2. A Maass–Poincaré series representation for F_Θ

In this section we write F_Θ as a Poincaré series. We begin by recalling the definition of a harmonic weak Maass form. With Γ a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $\nu : \Gamma \to \mathbb{C}$ a multiplier, a *harmonic weak Maass form* of weight k with respect to Γ is a smooth function $F : \mathbb{H} \to \mathbb{C}$ with the following properties

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have $F(A\tau) = \nu(A)(c\tau + d)^k F(\tau)$.

(2) We have that $\Delta_k F = 0$, where for $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$, the weight k *hyperbolic Laplacian* is given by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + \mathrm{i} \frac{\partial}{\partial y} \right). \tag{2.1}$$

(3) F has at most linear exponential growth toward each cusp of $\Gamma \backslash \mathbb{H}$.

The *shadow* of a harmonic weak Maass form F is a weakly holomorphic modular form of weight $2-k$ equal to $\xi_k(F)$ where $\xi_k := 2\mathrm{i}y^k\overline{\frac{\partial}{\partial\overline{\tau}}}$.

Every weight k harmonic weak Maass form $F(z)$ has a Fourier expansion of the form

$$F(\tau) = \sum_{n\gg-\infty} c_F^+(n)q^n + Cy^{1-k} + \sum_{n\ll+\infty, n\neq 0} c_F^-(n)\Gamma(1-k,-4\pi ny)q^n, \tag{2.2}$$

As (2.2) reveals, $F(z)$ naturally decomposes into two summands

$$F^+(\tau) := \sum_{n\gg-\infty} c_F^+(n)q^n, \tag{2.3}$$

$$F^-(\tau) := Cy^{1-k} + \sum_{n\ll+\infty, n\neq 0} c_F^-(n)\Gamma(1-k,-4\pi ny)q^n. \tag{2.4}$$

A direct computation shows that $\xi_k(F)$ is given simply in terms of $F^-(z)$, the *non-holomorphic part* of F. The *holomorphic part* of F is $F^+(z)$.

Next we recall a series representation for $r(n)$. For this, we define for $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_\Theta := \left\langle \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)\right\rangle$ the theta multiplier [17]

$$\nu_\Theta\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right) := \begin{cases} \left(\frac{d}{c}\right)^* \mathrm{e}^{-\frac{\pi\mathrm{i}c}{4}} & \text{if } b\equiv c\equiv 1 \pmod 2, a\equiv d\equiv 0 \pmod 2, \\ \left(\frac{c}{d}\right)_* \mathrm{e}^{\frac{\pi\mathrm{i}(d-1)}{4}} & \text{if } b\equiv c\equiv 0 \pmod 2, a\equiv d\equiv 1 \pmod 2. \end{cases}$$

Here, for $c\neq 0$, we define, using the usual Jacobi symbol,

$$\left(\frac{c}{d}\right)^* := \left(\frac{c}{|d|}\right),$$
$$\left(\frac{c}{d}\right)_* := \left(\frac{c}{|d|}\right)(-1)^{\frac{\mathrm{sgn(c)}-1}{2}\frac{\mathrm{sgn(d)}-1}{2}}.$$

Moreover, we set

$$\left(\frac{0}{\pm 1}\right)^* = \left(\frac{0}{1}\right)_* = -\left(\frac{0}{-1}\right)_* = 1.$$

Remark. ν_Θ is the multiplier for $\Theta(\tau/2)$, a form on Γ_Θ, rather than on $\Gamma_0(4)$.

For $c\in\mathbb{N}$, define the sum of Kloosterman type

$$\mathcal{S}(n;c) := \sum_{d \,(\mathrm{mod}\, 2c)} \overline{\lambda(d,c)}^3 \mathrm{e}^{\frac{\pi\mathrm{i}dn}{c}} \tag{2.5}$$

with

$$\lambda(d,c) := \begin{cases} \mathrm{e}^{-\frac{\pi\mathrm{i}c}{4}}\left(\frac{d}{c}\right) & \text{if } c \text{ is odd, } d \text{ is even,} \\ \mathrm{e}^{\frac{\pi\mathrm{i}(d-1)}{4}}\left(\frac{c}{d}\right) & \text{if } c \text{ is even, } d \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

We require the Kloosterman zeta-function, which is defined for $\mathrm{Re}(s)$ sufficiently large,

$$Z_n(s) := \sum_{c=1}^\infty \frac{\mathcal{S}(n;c)}{c^{s+\frac{1}{2}}}.$$

It is known (see, for example, [16]) that for $n \neq 0$, $Z_n(s)$ has an analytic continuation to $s = 1$. The analytic continuation of $Z_0(s)$ to $s = 1$ is shown in Theorem 3.2. Using the above notation, we can state the following series expansion for $r(n)$ (a proof may, for example, be found in [1]):

$$r(n) = 2\mathrm{e}^{-\frac{3\pi \mathrm{i}}{4}} \pi n^{\frac{1}{2}} Z_n(1). \tag{2.6}$$

To construct the Poincaré series required, we let $\psi(\tau; s) := 2^{-s+\frac{1}{4}} y^{s-\frac{1}{4}}$ and $\Gamma_\infty(2) := \{\pm \left(\begin{smallmatrix} 1 & 2n \\ 0 & 1 \end{smallmatrix}\right); n \in \mathbb{Z}\}$. We formally define the Poincaré series

$$F_\Theta(\tau; s) := \sum_{A \in \Gamma_\infty(2) \backslash \Gamma_\Theta} \psi(A\tau; s) \nu_\Theta^3(A) (c\tau + d)^{-\frac{1}{2}}.$$

One can show that for $s > 3/4$ the function $F_\Theta(\tau; s)$ is absolutely convergent and transforms like an automorphic form of weight $\frac{1}{2}$ with multiplier ν_Θ^{-3} and eigenvalue $\left(s - \frac{1}{4}\right)\left(\frac{3}{4} - s\right)$ under the weight $\frac{1}{2}$ hyperbolic Laplacian $\Delta_{\frac{1}{2}}$. We are interested in the case $s = \frac{3}{4}$ which will be obtained by continuing the Fourier expansion of $F_\Theta(\tau; s)$ analytically.

To state the Fourier expansion of $F_\Theta(\tau; s)$, we define

$$\mathcal{W}_n(y; s) := \begin{cases} |n|^{-\frac{1}{2}} \Gamma\left(s + \frac{\operatorname{sgn}(n)}{4}\right)^{-1} (4\pi |n| yt)^{-\frac{1}{4}} W_{\frac{1}{4}\operatorname{sgn}(n), s-\frac{1}{2}}(4\pi |n| y) & \text{if } n \neq 0, \\ \frac{2^{2s-\frac{1}{2}}}{(2s-1)\Gamma\left(2s-\frac{1}{2}\right)} y^{\frac{3}{4}-s} & \text{if } n = 0, \end{cases}$$

where $W_{\nu,\mu}$ is the usual W-Whittaker function.

Theorem 2.1. *We have the following Fourier expansion*

$$F_\Theta(\tau; s) = \left(\frac{y}{2}\right)^{s-\frac{1}{4}} + \sum_{n \in \mathbb{Z}} a_n(s) \mathcal{W}_n\left(\tfrac{y}{2}; s\right) \mathrm{e}^{\pi i n x},$$

where

$$a_0(s) = 2^{1-4s} \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}} \pi^{\frac{1}{2}} \Gamma(2s) \overline{Z_0\left(2\overline{s} - \frac{1}{2}\right)}$$

and for $n \neq 0$

$$a_n(s) = 2^{\frac{1}{2}-2s} \pi^{s+\frac{1}{4}} |n|^{s-\frac{1}{4}} \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}} \overline{Z_{-n}\left(2\overline{s} - \frac{1}{2}\right)}.$$

Moreover, the series $F_\Theta(\tau; s)$ has an analytic continuation to $s = \frac{3}{4}$ and we have the expansion

$$\begin{aligned} F_\Theta(\tau) &:= 2F_\Theta\left(2\tau; \frac{3}{4}\right) \\ &= 2y^{\frac{1}{2}} + \frac{1}{2}\pi \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}} \overline{Z_0(1)} + \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}} \pi \sum_{n=1}^{\infty} \overline{Z_{-n}(1)} q^n \\ &\quad + \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}} \sqrt{\pi} \sum_{n=1}^{\infty} \overline{Z_n(1)} \Gamma\left(\tfrac{1}{2}; 4\pi n y\right) q^{-n}. \end{aligned}$$

The function F_Θ is a harmonic weak Maass form of weight $\frac{1}{2}$ for $\Gamma_0(4)$ satisfying

$$\xi_{\frac{1}{2}}(F_\Theta) = \Theta^3. \tag{2.7}$$

Proof. Since the proof of the Fourier expansion is quite standard (see [13] for a similar calculation), we do not give it here. The analytic continuation of $F_\Theta(\tau; s)$ to $s = \frac{3}{4}$ follows directly from the analytic continuation of $Z_n(2s - \frac{1}{2})$. The expansion of F_Θ is then obtained by setting $s = \frac{3}{4}$ and using special values of Whittaker functions (see [11], for example). Moreover, it is well known that if $f(\tau)$ transforms like a modular form of weight $\frac{1}{2}$ for Γ_Θ with multiplier ν_Θ^{-3}, then $f(2\tau)$ transforms like a modular form on $\Gamma_0(4)$.

Finally, (2.7) follows by a direct calculation, using the explicit form of $r(n)$ stated in (2.6). More precisely, we have $\xi_{\frac{1}{2}}\left(2y^{\frac{1}{2}}\right) = 1$ and $\xi_{\frac{1}{2}}\left(\Gamma\left(\frac{1}{2}; y\right)\right) = \mathrm{e}^{-y}$. Using the anti-linearity of $\xi_{\frac{1}{2}}$ then easily gives the claim. □

We conclude this section by showing that F_Θ is a Hecke eigenform. Since Θ^3 is a Hecke eigenform with eigenvalue $1 + p$ under the Hecke operator $T(p^2)$ (see [18]), one may easily conclude that

$$F_\Theta | T(p^2) - \left(1 + \frac{1}{p}\right) F_\Theta$$

is a weakly holomorphic modular form of weight $\frac{1}{2}$ on $\Gamma_0(4)$. Moreover, its principal part is constant so it is a holomorphic modular form. By the Serre–Stark basis theorem the space of holomorphic modular forms of on $\Gamma_0(4)$ is known to be one dimensional and spanned by Θ. Computing the action of the Hecke operators explicitly, one sees that its constant term is 0; thus the form must be 0.

3. Relation to L-series

In this section, we will show that $Z_n(1) = Z_n$, where Z_n was defined in the introduction. For this, we will distinguish the cases $n \neq 0$ and $n = 0$.

3.1. Computation of $Z_n(s)$ for $n \neq 0$.

Theorem 3.1. *Let $n = f^2 d \neq 0$ be an integer with d square-free and $f = 2^q w$ with w odd. Then we have that*

$$Z_n(s) = Z_n^{odd}(s) R_n(s)$$

with

$$Z_n^{odd}(s) := \mathrm{e}^{\frac{3\pi i}{4}} \frac{L(s, \psi_{-n})}{\zeta(2s)} w^{1-2s} T_s^{\psi_{-n}}(w) \frac{1 - \psi_{-n}(2) 2^{-s}}{1 - 2^{-2s}}$$

and

$$R_n(s) := 1 + 2^{-s} - 2^{1-s} R_n^*(s).$$

Here

$$R_n^*(s) := \begin{cases} 0 & \text{if } n \equiv 1, 2 \,(\mathrm{mod}\, 4), \\ \frac{1 - 2^{-2s}}{1 - \psi_{-n}(2) 2^{-s}} 2^{Q(1-2s)} T_s^{\psi_{-n}}(2^Q) & \text{otherwise.} \end{cases}$$

Proof. We first relate our functions to certain functions studied by Zagier [19]. For this define for $n \in \mathbb{Z}$

$$\gamma_c(n) := \frac{1}{\sqrt{c}} \sum_{d=1}^{2c} \lambda_Z(d, c)\, \mathrm{e}^{-\frac{\pi i d n}{c}},$$

where

$$\lambda_Z(d,c) := \begin{cases} \mathrm{i}^{\frac{1-c}{2}} \left(\frac{d}{c}\right) & \text{if } c \text{ is odd, } d \text{ is even,} \\ \mathrm{i}^{\frac{d}{2}} \left(\frac{c}{d}\right) & \text{if } c \text{ is even, } d \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to see that

$$\overline{\lambda^3(d,c)} = \mathrm{e}^{\frac{3\pi \mathrm{i}}{4}} (-1)^{c+1} \lambda_Z(d,c)$$

yielding

$$S(n;c) = \mathrm{e}^{\frac{3\pi \mathrm{i}}{4}} (-1)^{c+1} \sqrt{c}\gamma_c(-n).$$

We next split $Z_n(s)$ into an even and into an odd part of c. For this, we write $c = 2^r c'$ with c' odd, $r \in \mathbb{N}$ and by [19] for $r \geq 1$ and $N \neq 0$ we may decompose $\gamma_c(N)$ as

$$\gamma_c(N) = Q_r(N)\gamma_{c'}(N),$$

where

$$Q_r(N) := \begin{cases} 2^{\frac{r}{2}} (-1)^{\frac{m-1}{4}} & \text{if } r \text{ is even, } N = 2^{r-2}m,\ m \equiv 1 \pmod 4, \\ 2^{\frac{r-1}{2}} (-1)^{\frac{m(m-1)}{2}} & \text{if } r \text{ is odd, } N = 2^{r-1}m, \\ 0 & \text{otherwise.} \end{cases}$$

This gives

$$Z_n(s) = \mathrm{e}^{\frac{3\pi \mathrm{i}}{4}} \sum_{c=1}^{\infty} \frac{(-1)^{c+1}\gamma_c(-n)}{c^s} = \mathrm{e}^{\frac{3\pi \mathrm{i}}{4}} \sum_{\substack{c'=1 \\ c' \text{ odd}}}^{\infty} \frac{\gamma_{c'}(-n)}{c'^s} \left(1 - \sum_{r=1}^{\infty} \frac{Q_r(-n)}{2^{rs}}\right).$$

By [19], we know that

$$\begin{aligned} \sum_{\substack{c'=1 \\ c' \text{ odd}}}^{\infty} \frac{\gamma_{c'}(-n)}{c'^s} &= \prod_{p \neq 2} \frac{1 - p^{-2s}}{1 - \psi_{-n}(p)p^{-s}} w^{1-2s} T_s^{\psi_{-n}}(w) \\ &= \frac{1 - \psi_{-n}(2)2^{-s}}{1 - 2^{-2s}} \frac{L(s, \psi_{-n})}{\zeta(2s)} w^{1-2s} T_s^{\psi_{-n}}(w). \end{aligned}$$

To evaluate the second factor, we define

$$\widetilde{R}_N(s) := \frac{1}{2} \left(1 + \sum_{r=1}^{\infty} \frac{Q_r(N)}{(2^{r-1})^s}\right).$$

In [19] it is shown that

$$\widetilde{R}_N(s) = \begin{cases} 0 & \text{if } N \equiv 2, 3 \pmod 4, \\ \frac{1-2^{-2s}}{1-\psi_N(2)2^{-s}} 2^{Q(1-2s)} T_s^{\psi_N}(2^Q) & \text{if } N = F^2 D. \end{cases}$$

Here D is the discriminant of $\mathbb{Q}(\sqrt{N})$ and we write $F = 2^Q r$ with r odd. Now the claim follows from

$$1 - \sum_{r=1}^{\infty} \frac{Q_r(-n)}{2^{rs}} = -2^{1-s}\widetilde{R}_{-n}(s) + 2^{-s} + 1.$$

□

To finish the evaluation of $Z_n(1)$, we distinguish whether $-n$ is a square or not. If $-n$ is not a square, $L(s,\psi_n)$ converges and we may deduce that $Z_n(s)$ converges for $s=1$. We may then evaluate $Z_n(s)$ at $s=1$ by using Theorem 3.1. If $-n$ is a square, then $L(s,\psi_{-n})=\zeta(s)$ and we have by Theorem 3.1 that

$$Z_n(1)=\frac{\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{\zeta(2)}\frac{T_1^{\psi_1}(w)}{w}\lim_{s\to 1}\frac{\zeta(s)R_n(s)}{(1+2^{-s})}.$$

One easily computes

$$\frac{R_n(s)}{(1+2^{-s})}=1-2^{(1-s)}.$$

Using that $\zeta(s)=\frac{1}{s-1}+O(1)$ as $s\to 1$, gives that

$$\lim_{s\to 1}\frac{\zeta(s)R_n(s)}{(1+2^{-s})}=\frac{d}{ds}(1-2^{(1-s)})\bigg|_{s=1}=\log(2).$$

From this we may conclude that $Z_n(1)=Z_n$.

3.2. Computation of $Z_0(1)$. This subsection is devoted to the computation of $Z_0(s)$.

Theorem 3.2. *We have for $s>1$*

$$Z_0(s)=\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\frac{\zeta(2s-1)}{\zeta(2s)}\frac{1-2^{-(2s-1)}-2^{-s}}{1-2^{-2s}}.$$

In particular $Z_0(s)$ has an analytic continuation to $s=1$.

Proof. We first assume that c is odd. Then

$$S(0;c)=\sum_{\substack{d\,(\mathrm{mod}\,2c)\\ d\,\mathrm{even}}}\overline{\lambda(d,c)}^3=\mathrm{e}^{\frac{3\pi\mathrm{i}c}{4}}\sum_{\substack{d\,(\mathrm{mod}\,2c)\\ d\,\mathrm{even}}}\left(\frac{d}{c}\right)=\left(\frac{2}{c}\right)\mathrm{e}^{\frac{3\pi\mathrm{i}c}{4}}\sum_{d\,(\mathrm{mod}\,c)}\left(\frac{d}{c}\right).$$

The last sum vanishes unless c is a square in which case it equals $\phi(c)$; thus in this case

$$S(0;c)=\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\phi(c).$$

Next we assume that c is even. Then

$$S(0;c)=\sum_{\substack{d\,(\mathrm{mod}\,2c)\\ d\,\mathrm{odd}}}\overline{\lambda(d,c)}^3=\sum_{\substack{d\,(\mathrm{mod}\,2c)\\ d\,\mathrm{odd}}}\mathrm{e}^{\frac{\pi\mathrm{i}(d-1)}{4}}\left(\frac{c}{d}\right).$$

We write $c=2^rc'$ with $r\geq 1, c'$ odd, and $d=d_1+2^{r+1}d_2$, where d_1 runs $(\mathrm{mod}\,2^{r+1})$ and d_2 runs $(\mathrm{mod}\,c')$. Then

$$\mathrm{e}^{\frac{\pi\mathrm{i}}{4}(d-1)}\left(\frac{c}{d}\right)=\mathrm{e}^{\frac{\pi\mathrm{i}}{4}(d_1-1)}\left(\frac{2}{d_1}\right)^r\left(\frac{c'}{d_1+2^{r+1}d_2}\right);$$

therefore,

$$S(0;c)=\sum_{d_1\,(\mathrm{mod}\,2^{r+1})}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}(d_1-1)}\left(\frac{2}{d_1}\right)^r\sum_{d_2\,(\mathrm{mod}\,c')}\left(\frac{c'}{d_1+2^{r+1}d_2}\right).$$

The sum on d_2 can be simplified as

$$\sum_{d_2 \pmod{c'}} \left(\frac{c'}{d_2}\right) = \begin{cases} \phi(c') & \text{if } c' \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

We next consider the sum on d_1. Changing $d_1 \mapsto d_1 + 2$ one sees that this sum vanishes if r is even. If r is odd, then the sum easily evaluates to $-2^{r-\frac{1}{2}} \mathrm{e}^{\frac{3\pi i}{4}}$. We deduce that

$$\mathcal{S}(0; c) = \begin{cases} \mathrm{e}^{\frac{3\pi i}{4}} \phi(c) & \text{if } r = 0 \text{ and } c' \text{ is a square,} \\ -2^{r-\frac{1}{2}} \mathrm{e}^{\frac{3\pi i}{4}} \phi(c') & \text{if } c' \text{ is a square and } r \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Combining the above gives

$$Z_0(s) = \sum_{c \text{ odd}} \frac{\mathcal{S}(0; c)}{c^{s+\frac{1}{2}}} + \sum_{r \geq 1} \sum_{c \text{ odd}} \frac{\mathcal{S}(0; 2^r c)}{(2^r c)^{s+\frac{1}{2}}} = \mathrm{e}^{\frac{3\pi i}{4}} \sum_{c \text{ odd}} \frac{\phi(c^2)}{c^{2s+1}} \left(1 - \frac{1}{2^{\frac{1}{2}}} \sum_{r \text{ odd}} 2^{r\left(\frac{1}{2}-s\right)}\right).$$

Using $\frac{\zeta(2s-1)}{\zeta(2s)} = \sum_{c \geq 1} \frac{\phi(c^2)}{c^{2s+1}}$ and geometric summation we conclude the theorem. $\square$

We let $s \to 1$ to obtain the evaluation of $Z_n(1)$. By Theorem 3.2 we have

$$Z_0(1) = \frac{4\mathrm{e}^{\frac{3\pi i}{4}}}{3\zeta(2)} \lim_{s \to 1} (\zeta(2s-1)(1 - 2^{-(2s-1)} - 2^{-s})).$$

Using that $\zeta(2s-1) = \frac{1}{2s-1} + O(1)$ as $s \to 1$, we obtain

$$\lim_{s \to 1} \zeta(2s-1)(1 - 2^{-(2s-1)} - 2^{-s}) = \tfrac{3}{4} \log(2).$$

This easily gives $Z_0(1) = Z_0$.

4. Relationship to other works

Zagier [20] (see also [14]) showed that the generating function for the Hurwitz class numbers, namely

$$\mathcal{H}(\tau) := -\frac{1}{12} + \sum_{\substack{n \geq 1 \\ n \equiv 0,3 \pmod 4}} H(-n) q^n,$$

is a mock modular form with shadow Θ. Recently, Duke *et al.* [11] constructed a generalized mock modular form whose shadow is the harmonic weak Maass form obtained by completing $\mathcal{H}(\tau)$ with a term similar to P_g of the introduction. One may use the function constructed in [11] with the relations between $L(\psi_{-n}, s)$, $r(n)$, and the Hurwitz class numbers to give a different construction of the form F_Θ. Their work does not include the explicit evaluation of the Fourier coefficients for square n; however, one can use the calculations here to compute those terms.

Much of the arithmetic of classical holomorphic modular forms relies on the theory of complex multiplication and the fact that the coefficients of modular forms are associated with the value of a modular function at points determined by data associated to an imaginary quadratic fields. The work of Duke, Imamoḡlu, and Tóth [11, 12] demonstrates a similar phenomenon linking the coefficients of mock modular forms and real quadratic fields. Namely, they show that the coefficients of a family of mock modular forms are associated with the values of modular functions at points corresponding to data from real quadratic fields. Our result may be viewed as an additional example of this phenomenon.

Work of Bruinier and Ono [6] demonstrates the relationship between harmonic Maass forms and special values of derivatives of L-functions. As in our work, their work concerns twists by Dirichlet characters associated with both real and imaginary quadratic fields. That work, as well as the related works of Bruinier *et al.* [5] and Bruinier and Yang [8, 9] yield deep connections between weak Maass forms and the theorems of Waldspurger, Borcherds and Gross-Zagier.

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