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ON CABLED KNOTS, DEHN SURGERY, AND LEFT-ORDERABLE FUNDAMENTAL GROUPS

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Abstract. Previous work of the authors establishes a criterion on the fundamental group of a knot complement that determines when Dehn surgery on the knot will have a fundamental group that is not left-orderable [6]. We provide a refinement of this criterion by introducing the notion of a *decayed* knot; it is shown that Dehn surgery on decayed knots produces surgery manifolds that have non-left-orderable fundamental group for all sufficiently positive surgeries. As an application, we prove that sufficiently positive cables of decayed knots are always decayed knots. These results mirror properties of L-space surgeries in the context of Heegaard Floer homology.

1. Introduction

Definition 1.1. A group G is left-orderable if there exists a partition of the group elements

$$G = \mathcal{P} \sqcup \{1\} \sqcup \mathcal{P}^{-1}$$

satisfying $\mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}$ and $\mathcal{P} \neq \emptyset$. The subset $\mathcal{P}$ is called a positive cone.

This is equivalent to G admitting a left-invariant strict total ordering. For background on left-orderable groups relevant to this paper see [2, 6]; a standard reference for the theory of left-orderable groups is [12]. As established by Boyer *et al.* [2] (compare [11]), the fundamental group $\pi_1(K)$ of the complement of a knot K in S^3 is always left-orderable. Indeed, this follows from the fact that any compact, connected, irreducible, orientable three-manifold with positive first Betti number has left-orderable fundamental group [2, Theorem 1.1]. However, the question of left-orderability for fundamental groups of rational homology three-spheres is considerably more subtle (see [2, 6]) and seems closely tied to certain codimension one structures on the three-manifold (see [2, 3, 17]). Continuing along the lines of [6] this paper focuses on Dehn surgery, an operation on knots that produces rational homology three-spheres. We recall this construction in order to fix notation and conventions.

For any knot K in S^3 there is a preferred generating set for the peripheral subgroup $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(K)$ provided by the knot meridian μ and the Seifert longitude λ. The latter is uniquely determined (up to orientation) by the existence of a Seifert surface for K. We orient μ so that it links positively with K, and orient λ so that $\mu \cdot \lambda = 1$. For any rational number r with reduced form $\frac{p}{q}$ we denote the peripheral element $\mu^p\lambda^q$ by α_r. At the level of the fundamental group, the result of Dehn surgery along α_r is summarized by the short exact sequence

$$1 \to \langle\langle \alpha_r \rangle\rangle \to \pi_1(K) \to \pi_1(S^3_r(K)) \to 1.$$

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Here $\langle\langle \alpha_r \rangle\rangle$ denotes the normal closure of α_r, and $S^3_r(K)$ is the three-manifold obtained by attaching a solid torus to the boundary of $S^3 \smallsetminus \nu(K)$, sending the meridian of the torus to a simple closed curve representing the class

$$[\alpha_r] \in H_1(\partial(S^3 \smallsetminus \nu(K)); \mathbb{Z})/\{\pm 1\}.$$

We will blur the distinction between α_r as an element of the fundamental group or as a primitive class in the (projective) first homology of the boundary, and refer to these peripheral elements as slopes.

While many examples of rational homology three-spheres have left-orderable fundamental group [2], there exist infinite families of knots for which sufficiently positive Dehn surgery (that is, along a slope parameterized by a suitable large rational number) yields a manifold with non-left-orderable fundamental group [6]. To make this precise, consider the set of slopes

$$\mathcal{S}_r = \{\alpha_{r'} | r' \geq r\}$$

for some fixed rational r.

Definition 1.2. A non-trivial knot K in S^3 is called r-decayed if, for any positive cone $\mathcal{P}$ in $\pi_1(K)$, either $\mathcal{P} \cap \mathcal{S}_r = \mathcal{S}_r$ or $\mathcal{P} \cap \mathcal{S}_r = \emptyset$.

The existence of decayed knots is established in [6]. For example, the torus knot $T_{p,q}$ is $(pq-1)$-decayed (for $p, q > 0$), and the $(-2, 3, q)$-pretzel knot is $(10+q)$-decayed for odd $q \geq 5$ (see Theorem 2.1). Our interest in this property stems from the following:

Theorem 1.1. *If K is r-decayed then $\pi_1(S^3_{r'}(K))$ is not left-orderable for all $r' \geq r$.*

As a result, it is not restrictive to assume that r is a positive rational number since $\pi_1(S^3_0(K))$ is always left-orderable [2]. Notice however that it is not immediately clear how Theorem 1.1 might be applied in practice, as there is no obvious method for checking when a knot is r-decayed. For this reason, in Section 2 we describe an equivalent formulation of r-decay whose statement is more technical, but easier to verify, than the definition. Together with the proof of Theorem 1.1, the results of Section 2 provide a useful refinement of the ideas from [6].

Results connecting left-orderability and Dehn surgery may be expected to mirror similar results relating to L-spaces, since there is no known example of an L-space with left-orderable fundamental group, while many L-spaces have fundamental group that is not left-orderable, (see [1, 5, 6, 16, 21]). Recall that an L-space is a rational homology sphere with Heegaard Floer homology that is as simple as possible, in the sense that $rk\widehat{\mathrm{HF}}(Y) = |H_1(Y;\mathbb{Z})|$ (see [15]). Theorem 1.1 mirrors a fundamental property of knots admitting L-space surgeries: if $S^3_n(K)$ is an L-space, then $S^3_r(K)$ is an L-space as well for any $r \geq n$.

In the interest of further investigating left-orderability of fundamental groups of 3-manifolds along the lines of [6], we consider the behaviour of Dehn surgery on cables of r-decayed knots (for necessary background, see Section 3). Denoting the (p,q)-cable of the knot K as $C_{p,q}(K)$, the main theorem of this article is:

Theorem 1.2. *If K is r-decayed then $C_{p,q}(K)$ is pq-decayed whenever $\frac{q}{p} > r$.*

The proof of Theorem 1.2 is contained in Section 3. Notice that combining Theorem 2.1 and Theorem 1.2 provides a rather large class of knots for which sufficiently positive surgery yields a non-left-orderable fundamental group.

Dehn surgery on cabled knots and non-left-orderability of the resulting fundamental groups may again be viewed in the context of Heegaard Floer homology. Referring to knots admitting L-space surgeries as L-space knots, Hedden proves:

Theorem 1.3. [9, Theorem 1.10] *If K is an L-space knot then $C_{p,q}(K)$ is an L-space knot whenever $\frac{q}{p} \geq 2g(K) - 1$.*

Here, the quantity $g(K)$ is the Seifert genus of K. Note that the converse of this statement has been recently established by Hom [10].

In order to assess the strength of Theorem 1.2, it is natural to ask when Dehn surgery on a cable knot yields a manifold that has left-orderable fundamental group. It turns out that, in the case that K is r-decayed, Theorem 1.2 is close to describing all possible non-left-orderable surgeries on a cable knot $C_{p,q}(K)$, in the following sense:

Theorem 1.4. *Suppose that C is the (p, q)-cable of some knot. If $r \in \mathbb{Q}$ satisfies $r < pq - p - q$, then $\pi_1(S^3_r(C))$ is left-orderable.*

This result is a special case of a more general observation pertaining to satellite knots that is discussed in Section 4. Notice that Theorem 6 makes no reference to the original knot being r-decayed. However, restricted to r-decayed knots, Theorem 1.2 and Theorem 1.4 combine to produce an interval of surgery coefficients for which the left-orderability of the associated quotient is not determined. More precisely:

Question 1.1. If K is r-decayed and C is a (p, q)-cable of K with $\frac{q}{p} > r$, can Theorem 1.2 and Theorem 1.4 be sharpened to determine when $\pi_1(S^3_{r'}(C))$ is left-orderable for r' satisfying $pq - p - q < r' < pq$?

2. A practical reformulation of Theorem 1.1

We begin with a reformulation of r-decay that will be essential in connecting this work with the results of [6]. This will require the following lemma:

Lemma 2.1. *Let G be a left-orderable group containing elements g, h. If $g \in \mathcal{P}$ implies $h \in \mathcal{P}$ for every positive cone $\mathcal{P}$, then $g \in \mathcal{P}$ if and only if $h \in \mathcal{P}$.*

Proof. We need only show the converse, namely $h \in \mathcal{P}$ implies $g \in \mathcal{P}$ for every positive cone $\mathcal{P} \subset G$. For a contradiction, suppose this is not the case, so there exists a positive cone such that $h \in \mathcal{P}$ and $g \notin \mathcal{P}$. Consider the positive cone $\mathcal{Q} = \mathcal{P}^{-1}$, defining the reverse ordering of G. This gives $g \in \mathcal{Q}$ and $h \notin \mathcal{Q}$, contradicting our assumption. □

Proposition 2.1. *A knot K is r-decayed if and only if for every positive cone $\mathcal{P} \subset \pi_1(K)$ there exists a strictly increasing sequence of positive rational numbers $\{r_i\}$ with $r_i \to \infty$ satisfying*

(1) *$r = r_0$, and*
(2) *$\alpha_r \in \mathcal{P}$ implies $\alpha_{r_i} \in \mathcal{P}$ for all i.*

Proof. Suppose that K is r-decayed , and let $\mathcal{P}$ be any positive cone. Choose a strictly increasing sequence of rational numbers $\{r_i\}$ with $r_0 = r$ and $r_i \to \infty$. Whenever $\alpha_r = \alpha_{r_0} \in \mathcal{P}$ we have $\mathcal{S}_r \cap \mathcal{P} \neq \emptyset$, so that $\mathcal{S}_r \cap \mathcal{P} = \mathcal{S}_r$ since K is r-decayed. It follows that $\alpha_{r_i} \in \mathcal{S}_r \subset \mathcal{P}$ for all i.

To prove the converse, let $\mathcal{P}$ be a positive cone for $\pi_1(K)$. Fix a strictly increasing sequence $\{r_i\}$ of rational numbers limiting to infinity and satisfying (1) and (2). Suppose that $\alpha_r \in \mathcal{P}$, then by assumption $\alpha_{r_i} \in \mathcal{P}$ for all $i > 0$.

Now suppose that $\mu^m\lambda^n$ is an element of $\mathcal{S}_r$. Choose r_i, r_{i+1} with corresponding reduced forms $\frac{p_i}{q_i}, \frac{p_{i+1}}{q_{i+1}}$ such that $r_i < \frac{m}{n} < r_{i+1}$. By solving

$$q_i a + q_{i+1} b = cn,$$
$$p_i a + p_{i+1} b = cm,$$

we can find positive integers a, b and c such that $(\mu^{p_i}\lambda^{q_i})^a(\mu^{p_{i+1}}\lambda^{q_{i+1}})^b = (\mu^m\lambda^n)^c$. Explicitly, Cramer's rule gives

$$a = \begin{vmatrix} n & q_{i+1} \\ m & p_{i+1} \end{vmatrix}, \quad b = \begin{vmatrix} q_i & n \\ p_i & m \end{vmatrix}, \quad c = \begin{vmatrix} q_i & q_{i+1} \\ p_i & p_{i+1} \end{vmatrix};$$

note that all these quantities are positive because of our restriction $r_i < \frac{m}{n} < r_{i+1}$ (compare [6, Lemma 17]) . This shows that $\mu^m\lambda^n$ is positive, since its cth power is expressed as a product of positive elements. Hence $\mathcal{S}_r \cap \mathcal{P} = \mathcal{S}_r$.

This establishes the implication $\alpha_r \in \mathcal{P} \Rightarrow \mathcal{S}_r \subset \mathcal{P}$ for every positive cone $\mathcal{P}$. By Lemma 2.1, this is equivalent to $\mathcal{S}_r \cap \mathcal{P} = \mathcal{S}_r$ or $\mathcal{S}_r \cap \mathcal{P} = \emptyset$ for every positive cone $\mathcal{P}$, so that K is r-decayed. □

Remark 2.1. In practice, it is often more natural to establish $\alpha_r \in \mathcal{P}$ implies $\alpha_{r_i}^{w_i} \in \mathcal{P}$ for all i, where $w_i \in \mathbb{N}$ (see in particular the proofs of Lemmas 3.2 and 3.3). This situation arises when one constructs (for a given positive cone $\mathcal{P}$) a sequence of *unreduced* rationals $\{r_i\} = \{\frac{p_i}{q_i}\}$ for which $\gcd(p_i, q_i) = w_i \geq 1$, and $\mu^{p_0}\lambda^{q_0} \in \mathcal{P}$ implies $\mu^{p_i}\lambda^{q_i} \in \mathcal{P}$ for all i. Notice that the implication $\alpha_r \in \mathcal{P}$ implies $\alpha_{r_i}^{w_i} \in \mathcal{P}$ still allows us to apply Proposition 2.1, since $\alpha_{r_i}^{w_i} \in \mathcal{P}$ if and only if $\alpha_{r_i} \in \mathcal{P}$ (this simple observation holds in any left-orderable group). Ultimately, this results in more flexibility in selecting the sequence $\{r_i\}$.

The equivalence established in Proposition 2.1 shows that all examples considered in [6] are r-decayed for certain r, as [6, Corollary 11] is a special case of Proposition 2.1.

Theorem 2.1. [6, Theorems 24, 28 and 30]

(1) *The (p, q)-torus knot is $(pq - 1)$-decayed for all positive, relatively prime pairs of integers p, q.*
(2) *The $(-2, 3, q)$-pretzel knot is $(10 + q)$-decayed for all odd $q \geq 5$.*
(3) *The $(3, q)$-torus knot with one positive full twist added along two strands is $(3q + 2)$-decayed, for all positive q congruent to 2 modulo 3.*

Proof. We consider the case of K_q, the $(-2, 3, q)$-pretzel knot with $q \geq 5$ odd, the other cases are similar. Set $r = 10 + q$, and $r_i = r + i$. It is shown in [6] that for every positive cone $\mathcal{P}$ in $\pi_1(K_q)$, the implication $\alpha_r \in \mathcal{P} \Rightarrow \alpha_{r_i} \in \mathcal{P}$ holds for all $i \geq 0$. This means that for every left-ordering of $\pi_1(K_q)$, the integer sequence $\{r_i\}$ satisfies the properties required by Proposition 2.1, and we conclude that K_q is r-decayed. □

Note that the above proof illustrates some particularly special behaviour, as the rational sequences $\{r_i\}$ required by Proposition 2.1 (which a priori may be different for each left-ordering) are replaced by a single integer sequence sufficient for every left-ordering. Thus, Proposition 2.1 provides a more workable method (than used previously) for checking when a knot has surgeries that yield a non-left-orderable fundamental group. Combined with the material established in [6, Section 2], we provide a short proof of Theorem 1.1.

Proof of Theorem 1.1. For contradiction, assume that $\pi_1(S^3_{r'}(K))$ is left-orderable for some $r' \geq r$, and consider the short exact sequence

$$1 \to \langle\langle \alpha_{r'} \rangle\rangle \xrightarrow{i} \pi_1(K) \xrightarrow{f} \pi_1(S^3_{r'}(K)) \to 1,$$

as defined in the introduction. Let $\mu, \lambda \in \pi_1(K)$ denote the meridian and longitude. Since $\pi_1(S^3_{r'}(K))$ is left-orderable, $\langle\langle \alpha_{r'} \rangle\rangle \cap \langle \mu, \lambda \rangle = \langle \alpha_{r'} \rangle$ (see [6, Proof of Proposition 20]). In particular, if we fix an arbitrary rational number $s_0 > r'$, then $f(\alpha_{s_0}) \neq 1$. Thus, we may choose a positive cone $\mathcal{Q}$ in $\pi_1(S^3_{r'}(K))$ that contains $f(\alpha_{s_0})$. Next, choose a positive cone $\mathcal{Q}' \subset \langle\langle \alpha_{r'} \rangle\rangle$ not containing $\alpha_{r'}$, and define a positive cone $\mathcal{P} \subset \pi_1(K)$ by

$$\mathcal{P} = i(\mathcal{Q}') \sqcup f^{-1}(\mathcal{Q}).$$

Note that $\alpha_{r'} \notin \mathcal{P}$, and $\alpha_{s_0} \in \mathcal{P}$.

This is a standard construction for creating a left-ordering of a group using a short exact sequence, here the result is a left-ordering of $\pi_1(K)$ with positive cone $\mathcal{P}$, relative to which the subgroup $\langle\langle \alpha_{r'} \rangle\rangle$ is convex. Because $\langle\langle \alpha_{r'} \rangle\rangle$ is convex, the intersection $\langle\langle \alpha_{r'} \rangle\rangle \cap \langle \mu, \lambda \rangle = \langle \alpha_{r'} \rangle$ is convex in the restriction ordering of $\langle \mu, \lambda \rangle$. Therefore, [6, Proposition 18] shows that all slopes α_s with $s > r'$ must have the same sign. In particular, since α_{s_0} is positive it follows that all slopes α_s with $s > r'$ are positive, so that

$$Q \cap \mathcal{S}_r = \{\alpha_s | s > r'\}.$$

Therefore, K is not r-decayed. □

We remark that there is a more geometric argument establishing Theorem 1.1, that relies upon an understanding of the topology of the space of left-orderings of $\mathbb{Z} \oplus \mathbb{Z}$ (see [18, Section 3] and [4, Chapter 6]). Roughly, every left-ordering of the knot group $\pi_1(K)$ restricts to a left-ordering of the peripheral subgroup that defines a line in $\mathbb{Z} \oplus \mathbb{Z}$, with all positive elements of $\mathbb{Z} \oplus \mathbb{Z}$ on one side of the line, and all the negative elements on the other side. As a result, given two rationals $r_1 < r_2$ corresponding to slopes α_{r_1} and α_{r_2} that have the same sign in every left-ordering, no left-ordering can restrict to an ordering of the peripheral subgroup with corresponding slope s between r_1 and r_2. The proof of Theorem 1.1 then follows from checking that whenever $\pi_1(S^3_{r'}(K))$ is left-orderable, we can define a left-ordering of $\pi_1(K)$ that restricts to yield a line of slope r' in the peripheral subgroup (compare [6, Proof of Theorem 9]).

3. The proof of Theorem 1.2

We recall the construction of a cabled knot in order to fix notation. Consider the (p,q)-torus knot $T_{p,q}$, where $p, q > 0$ are relatively prime. As the closure of a p-strand braid, this knot may be naturally viewed in a solid torus T by removing a tubular neighbourhood of the braid axis. The complement of $T_{p,q}$ in T is referred to as a (p,q)-cable space. Now given any knot K in S^3, the cable knot $C_{p,q}(K)$ is obtained by identifying the boundary of T with the boundary of $S^3 \smallsetminus \nu(K)$, identifying the longitude of T with the longitude λ of K. We will denote this cable knot by C whenever this simplified notation does not cause confusion.

The knot group $\pi_1(C)$ may be calculate via the Seifert–Van Kampen Theorem, by viewing the complement $S^3 \smallsetminus \nu(C)$ as the identification of the boundaries of $S^3 \smallsetminus \nu(K)$ and a solid torus $D^2 \times S^1$ along an essential annulus with core curve given by the slope $\mu^q\lambda^p$. If $\pi_1(D^2 \times S^1) = \langle t \rangle$ then this gives rise to a natural amalgamated product

$$\pi_1(C) \cong \pi_1(K) *_{\mu^q\lambda^p = t^p} \mathbb{Z}.$$

Consulting [6, Section 3], the meridian and longitude for C may be calculated as

$$\mu_C = \mu^u\lambda^v t^{-v} \quad \text{and} \quad \lambda_C = \mu_C^{-pq} t^p,$$

where u and v are positive integers satisfying $pu - qv = 1$ (compare [20, Proof of Theorem 3.1]).

Suppose that the knot K is r-decayed, and choose cabling coefficients p and q so that $q/p > r$. To begin, we choose a positive cone $\mathcal{P} \subset \pi_1(C)$ and assume, without loss of generality, that $\mu_C^{pq}\lambda = t^p$ is positive. This means that $t^p = \mu^q\lambda^p \in \mathcal{P}$, so every element $\mu^m\lambda^n$ is positive whenever $m/n > r$, since K is r-decayed. To see this, note that the inclusion $\pi_1(K) \subset \pi_1(C)$ means that every left-ordering of $\pi_1(C)$ induces a left-ordering of $\pi_1(K)$ by restriction. In particular, if two elements of the subgroup $\langle \mu, \lambda \rangle \subset \pi_1(C)$ have opposite signs in a left-ordering of $\pi_1(C)$, then they will also have opposite signs in the induced left-ordering of $\pi_1(K)$. As a result we can use r-decay of the subgroup $\pi_1(K)$ as a property of left-orderings of the larger group $\pi_1(C)$.

Our method of proof will be to check that the cable is pq-decayed by using the equivalence from Proposition 2.1. In particular, we will show that for the given positive cone $\mathcal{P} \subset \pi_1(C)$ there exists an unbounded sequence of increasing rationals $\{r_i\}$ with $r_0 = pq$, such that our assumption $\alpha_{pq} = \mu_C^{pq}\lambda_C \in \mathcal{P}$ implies $\alpha_{r_i} \in \mathcal{P}$ for all $i > 0$.

First consider the case when μ_C is positive in the left-ordering defined by $\mathcal{P}$. Here, $\mu_C^{pq+N}\lambda_C$ is positive for $N \geq 0$, as it is a product of positive elements. Therefore in this case it suffices to choose $r_i = pq + i$ for all $i \geq 0$.

For the remainder of the proof, we assume that μ_C is negative. For repeated use below, we also observe the crucial identity

$$(t^{-v})^p(\mu^u\lambda^v)^p = (t^p)^{-v}\mu^{up}\lambda^{vp} = \mu^{-qv}\lambda^{-pv}\mu^{up}\lambda^{vp} = \mu^{pv-qu} = \mu,$$

and recall that t^p commutes with μ, λ, μ_C, and λ_C. Therefore, we also have

$$(\mu^u\lambda^v)^p(t^{-v})^p = (t^{-v})^p(\mu^u\lambda^v)^p = \mu.$$

Let k be an arbitrary non-negative integer, and consider the element

$$\mu^{-k}(t^{-v}\mu^u\lambda^v)\mu^k.$$

If this element is positive for some k, then the required sequence is provided by Lemma 3.2 (proved below). Therefore, we may assume that

$$\mu^{-k}(t^{-v}\mu^u\lambda^v)\mu^k \notin \mathcal{P}, \tag{3.1}$$

for all k.

Similarly, for k a non-negative integer, we consider

$$(\mu^{-k}t^{-v}\mu^{k})^{p-1}(\mu^{u}\lambda^{v})^{p-1}.$$

If this element is positive for some non-negative k, then we can create the required sequence using Lemma 3.3 (proved below). Therefore, we may assume that

$$(\mu^{-k}t^{-v}\mu^{k})^{p-1}(\mu^{u}\lambda^{v})^{p-1} \notin \mathcal{P}, \tag{3.2}$$

for all $k \geq 0$.

Observe that

$$(\mu^{-k}t^{-v}\mu^{k})^{p-1}(\mu^{u}\lambda^{v})^{p-1} = (\mu^{-k}t^{v}\mu^{k})(\mu^{-k}t^{-vp}\mu^{k})(\mu^{up-u}\lambda^{vp-v}),$$

which, recalling that t^p commutes with the elements μ and λ, simplifies to give

$$(\mu^{-k}t^{v}\mu^{k})(\mu^{-u}\lambda^{-v})t^{-vp}\mu^{up}\lambda^{vp} = (\mu^{-k}t^{v}\mu^{k})(\mu^{-u}\lambda^{-v})\mu = \mu^{-k}t^{v}\lambda^{-v}\mu^{-u}\mu^{k+1} \notin \mathcal{P}$$

for all $k \geq 0$. Taking inverses yields

$$\mu^{-k-1}\mu^{u}\lambda^{v}t^{-v}\mu^{k} = \mu^{-k-1}\mu_C\mu^{k} \in \mathcal{P}.$$

For the following lemmas, let $>$ denote the left-ordering defined by the positive cone $\mathcal{P}$, so that $h > g$ whenever $g^{-1}h \in \mathcal{P}$. We can then calculate:

Lemma 3.1. *If $\mu^{-k-1}\mu_C\mu^{k} \in \mathcal{P}$ holds for all $k \geq 0$, then $\mu^{N+q}\lambda^{p}$ must be positive for all $N \geq 0$.*

Proof. Since $\mu^{-k-1}\mu_C\mu^{k} > 1$, left-multiplying by μ^{k+1} gives $\mu_C\mu^{k} > \mu^{k+1}$ for all $k \geq 0$. Setting $k = 0$ we obtain $\mu_C > \mu$, so that left-multiplying by μ_C gives rise to

$$\mu_C^2 > \mu_C\mu.$$

By setting $k = 1$, we get

$$\mu_C\mu > \mu^2,$$

which combines with the previous expression to give $\mu_C^2 > \mu^2$. Continuing in this manner, we obtain $\mu_C^N > \mu^N$ for all $N \geq 0$. Left-multiplying by t^p, it follows that

$$\mu_C^{N+pq}\lambda_C = \mu_C^{N}t^{p} > \mu^{N}t^{p} = \mu^{N+q}\lambda^{p} > 1,$$

where the final inequality follows from the fact that $(N+q)/p > q/p > r$ and K is r-decayed. ☐

As a result, when (3.1) and (3.2) hold we may choose the sequence of rationals $r_i = pq + i$ for all $i \geq 0$, and the requirements of Proposition 2.1 are met.

To conclude the proof, we establish Lemmas 3.2 and 3.3.

Lemma 3.2. *If $\mu^{-k}(t^{-v}\mu^{u}\lambda^{v})\mu^{k} \in \mathcal{P}$ for some $k \geq 0$, then there exists a sequence of rationals $\{r_i\}$ with $r_0 = pq$ such that $\alpha_{r_i} > 1$ for all i.*

Proof. For $N \geq 0$, we rewrite μ_C^N as

$$\mu_C^N = \mu^{u}\lambda^{v}\mu^{k}(\mu^{-k}(t^{-v}\mu^{u}\lambda^{v})^{N}\mu^{k})\mu^{-u-k}\lambda^{-v}.$$

Fix a positive integer s that is large enough so that $(sq-u-k)/(sp-v) > r$, this is possible because $q/p > r$. Next, the product $\mu_C^{N+pqs}\lambda_C^{s} = \mu_C^{N}t^{ps}$ becomes $\mu_C^{N}\mu^{sq}\lambda^{sp}$, which is equal to

$$\mu^{u+k}\lambda^{v}(\mu^{-k}(t^{-v}\mu^{u}\lambda^{v})^{N}\mu^{k})(\mu^{qs-u-k}\lambda^{ps-v}).$$

This is a product of positive elements, because:

(a) $\mu^{u+k}\lambda^v > 1$ because $(u+k)/v > u/v > q/p > r$ (recalling that $pu - qv = 1$), and
(b) $\mu^{qs-u-k}\lambda^{ps-v} > 1$ because $(sq-u-k)/(sp-v) > r$,

while the quantity $\mu^{-k}(t^{-v}\mu^u\lambda^v)^N\mu^k$ is positive by assumption. Therefore, in this case we choose our sequence of rationals to be

$$r_i = \frac{pqs+i}{s}$$

for $i \geq 0$, this guarantees that the associated slopes α_{r_i} are positive in the given left-ordering. $\square$

Lemma 3.3. *If $\mu^{-k}(t^{-v}\mu^u\lambda^v)\mu^k \notin \mathcal{P}$ for all $k \geq 0$, and $(\mu^{-k}t^{-v}\mu^k)^{p-1}(\mu^u\lambda^v)^{p-1} \in \mathcal{P}$ for some $k \geq 0$, then there exists a sequence of rationals $\{r_i\}$ with $r_0 = pq$ such that $\alpha_{r_i} > 1$ for all i.*

Proof. Fix $k \geq 0$ such that $\mu^{-k}(t^{-v}\mu^u\lambda^v)\mu^k < 1$ and $(\mu^{-k}t^{-v}\mu^k)^{p-1}(\mu^u\lambda^v)^{p-1} > 1$, and let n be the smallest positive integer such that

$$(\mu^{-k}t^{-v}\mu^k)^n(\mu^u\lambda^v)^n > 1,$$

and

$$(\mu^{-k}t^{-v}\mu^k)^{n-1}(\mu^u\lambda^v)^{n-1} < 1,$$

note that $1 < n \leq p-1$ (If we take $n = p$, then the equality $(t^{-v})^p(\mu^u\lambda^v)^p = \mu$ reduces the first expression to the identity, so $n = p$ is not possible). Note that we may rearrange these two expressions, so that

$$\mu^{-k}t^{-vn}(\mu^u\lambda^v)^n\mu^k > 1,$$

and

$$\mu^{-k}(\mu^u\lambda^v)^{1-n}t^{-v(1-n)}\mu^k > 1.$$

Then, for $N \geq 1$, we can rewrite the expression for μ_C^N as follows:

$$\mu^{u+k}\lambda^v(\mu^{-k}t^{-v(1-n)}\mu^k)[(\mu^{-k}t^{-vn}(\mu^u\lambda^v)^n\mu^k)\mu^{-k}(\mu^u\lambda^v)^{1-n}t^{-v(1-n)}\mu^k]^{N-1}\mu^{-k}t^{-vn}.$$

In the above expression, the quantity inside the square brackets is a product of positive elements. Denote this quantity by P. Choose an integer s such that $(qs-k)/ps > r$. Then considering the slope $\mu_C^{N+pq(v+s)}\lambda_C^{v+s} = \mu_C^N t^{p(v+s)}$, we find

$$\mu_C^N t^{p(v+s)} = \mu^{u+k}\lambda^v(\mu^{-k}t^{-v(1-n)}\mu^k)P^{N-1}\mu^{qs-k}\lambda^{ps}t^{pv-vn}.$$

This is a product of positive elements, because:

(a) $\mu^{u+k}\lambda^v > 1$, since $(u+k)/v > q/p > r$ (as before).
(b) $\mu^{-k}t^{-v(1-n)}\mu^k > 1$, because if we consider its pth power, we can use the fact that t^p commutes with all peripheral elements so that

$$(\mu^{-k}t^{-v(1-n)}\mu^k)^p = t^{-pv(1-n)} > 1.$$

The final inequality follows from $-pv(1-n) > 0$.
(c) $\mu^{qs-k}\lambda^{ps} > 1$, because s is chosen so that $(qs-k)/ps > r$.
(d) $t^{pv-vn} > 1$, because $pv - vn > 0$.

Therefore, in this case we may choose our sequence of rationals to be

$$r_i = \frac{i + pq(v+s)}{v+s}$$

for $i \geq 0$, as the corresponding elements $\mu_C^{i+pq(v+s)}\lambda_C^{v+s}$ are positive in the left-ordering for $i \geq 0$. □

4. Surgery on satellites

Let T denote the solid torus containing a knot K^P, we require that K^P is not contained in any three-ball inside T. The knot K^P will be called the pattern knot. Let K^C denote a knot in S^3, K^C will be called the companion knot. We construct the satellite knot K with pattern K^P and companion K^C as follows.

Let $h : \partial T \to \partial(S^3 \setminus \nu(K^C))$ denote a diffeomorphism from the boundary of T to the boundary of the complement of $\nu(K^C)$, which carries the longitude of ∂T onto the longitude of the knot K^C. The knot K is then realized as the image of the knot K^P in the manifold

$$S^3 \setminus \nu(K^C) \sqcup_h T = S^3.$$

Lemma 4.1. [19, Proposition 3.4] *There exists a homomorphism $\phi : \pi_1(K) \to \pi_1(K^P)$ that perserves peripheral structure.*

Proof. We can compute the fundamental group $\pi_1(K)$ by using the Seifert-Van Kampen theorem. Since

$$S^3 \setminus \nu(K) = S^3 \setminus \nu(K^C) \sqcup_h T \setminus \nu(K^P),$$

the group $\pi_1(K)$ is the free product $\pi_1(K^C) * \pi_1(T \setminus \nu(K^P))$, with amalgamation as follows: the meridian of K^C is identified with the meridian of T, and the longitude of K^C is identified with the longitude of T.

Let N denote the normal closure in $\pi_1(K)$ of the commutator subgroup of $\pi_1(K^C)$. The quotient $\pi_1(K)/N$ can be considered as the result of killing the longitude of T. Topologically we can think of this quotient as gluing a second solid torus T' to the torus T containing K^P, in such a way that the meridian of T' is glued to the longitude of T. The result is that $\pi_1(K^C)$ collapses to a single infinite cyclic subgroup, and the group $\pi_1(K)/N$ is isomorphic to $\pi_1(K^P)$. The desired homomorphism ϕ is the quotient map $\pi_1(K) \to \pi_1(K)/N$. □

Proposition 4.1. *Suppose that K is a satellite knot with pattern knot K^P, and $r \in \mathbb{Q}$ is any rational number. If $\pi_1(S^3_r(K^P))$ is left-orderable and $S^3_r(K)$ is irreducible, then $\pi_1(S^3_r(K))$ is left-orderable.*

Proof. By Lemma 4.1, there exists a homomorphism ϕ: $\pi_1(K) \to \pi_1(K^P)$ that preserves peripheral structure, so there exists an induced map

$$\phi_r : \pi_1(S^3_r(K)) \to \pi_1(S^3_r(K^P)),$$

for every $r \in \mathbb{Q}$. Whenever $\pi_1(S^3_r(K^P))$ is left-orderable (and hence non-trivial, since our definition of left-orderable does not allow $\mathcal{P} = \emptyset$) the image of ϕ_r is non-trivial and $\pi_1(S^3_r(K))$ is left-orderable [2, Theorem 1.1] . □

Proof of Theorem 1.4. By [7], pq-surgery on a (p,q)-cable knot yields a reducible manifold. Since the minimal geometric intersection number between reducible slopes is ± 1 [8], r-surgery on a (p,q)-cable yields an irreducible manifold whenever $r < pq-p-q$. Moreover, a (p,q)-cable knot can be described as a satellite knot with pattern knot $T_{p,q}$, the (p,q)-torus knot. Therefore, for $r < pq-p-q$ we can apply Proposition 4.1 to conclude that $\pi_1(S^3_r(K))$ will be left-orderable whenever $\pi_1(S^3_r(T_{p,q}))$ is left-orderable.

We may now combine known results for surgery on torus knots in this setting. On the one hand, $\pi_1(S^3_r(T_{p,q}))$ is an L-space whenever $r \geq 2g-1$ [14, Proposition 9.5] (see in particular [9, Lemma 2.13]), where $g = g(T_{p,q})$ is the Seifert genus given by $g(T_{p,q}) = \frac{1}{2}(p-1)(q-1)$. On the other, since $S^3_r(T_{p,q})$ is Seifert fibred or a connect sum of lens spaces for every r [13], $S^3_r(T_{p,q})$ is an L-space if and only if $\pi_1(S^3_r(T_{p,q}))$ is not left-orderable [1] (see also [16, 21]). In particular, $\pi_1(S^3_r(T_{p,q}))$ is left-orderable whenever r is less than $2g(T_{p,q})-1$ and the result follows. □

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