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DEGENERATE FLAG VARIETIES AND THE MEDIAN GENOCCHI NUMBERS

Evgeny Feigin

Abstract. We study the $\mathbb{G}_a^M$ degenerations $\mathcal{F}^a_\lambda$ of the type A flag varieties $\mathcal{F}_\lambda$. We describe these degenerations explicitly as subvarieties in the products of Grassmannians. We construct cell decompositions of $\mathcal{F}^a_\lambda$ and show that for complete flags the number of cells is equal to the normalized median Genocchi numbers h_n. This leads to a new combinatorial definition of the numbers h_n. We also compute the Poincaré polynomials of the complete degenerate flag varieties via a natural statistics on the set of Dellac's configurations, similar to the length statistics on the set of permutations. We thus obtain a natural q-version of the normalized median Genocchi numbers.

0. Introduction

Let $\mathfrak{g} = \mathfrak{sl}_n$, $G = SL_n$. Fix the Cartan decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$, where $\mathfrak{b}$ is a Borel subalgebra, $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. In [Fe3] we considered the degenerate algebra $\mathfrak{g}^a = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is an abelian Lie algebra isomorphic to $\mathfrak{n}^-$ as a vector space. The corresponding Lie group is a semi-direct product $G^a = B \ltimes \mathbb{G}_a^M$, where $\mathbb{G}_a$ is the additive group of the field and $M = \dim \mathfrak{n}$. For a dominant integral weight λ let V_λ be the highest weight λ irreducible $\mathfrak{g}$-module with a highest weight vector v_λ. The increasing PBW filtration $F_\bullet$ on V_λ is defined as follows:

$$F_0 = \mathbb{C}v_\lambda,\ F_{s+1} = \mathrm{span}\{xv :\ x \in \mathfrak{g}, v \in F_s\}, s \geq 0$$

(see [Fe1, Fe2, FFoL1, FFoL2, K2]). The associated graded space $V^a_\lambda = F_0 \oplus F_1/F_0 \oplus F_2/F_1 \oplus \cdots$ can be naturally endowed with the structure of a $\mathfrak{g}^a$- and G^a-module. A degenerate flag variety $\mathcal{F}^a_\lambda$ is a subvariety in $\mathbb{P}(V^a_\lambda)$ defined by $\mathcal{F}^a_\lambda = \overline{\mathbb{G}_a^M \cdot \mathbb{C}v_\lambda}$. These are the $\mathbb{G}_a^M$-degenerations of the classical (generalized) flag varieties $\mathcal{F}_\lambda$ (see [A, AS, Fe3, HT]). For example, $\mathcal{F}^a_{\omega_d} \simeq Gr(d,n)$ for all fundamental weights. Recall also that in the classical case (for $\mathfrak{g} = \mathfrak{sl}_n$) the varieties $\mathcal{F}_\lambda = G \cdot \mathbb{C}v_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$ are the usual flag varieties (maybe partial). In particular, if λ is regular, i.e. $(\lambda, \omega_d) > 0$ for all d, then $\mathcal{F}_\lambda$ is isomorphic to the variety $\mathcal{F}_n$ of complete flags in n-dimensional space V. Fix a basis $v_1, \dots, v_n$ of V.

For all weights λ, μ there exists an embedding of G^a-modules $V^a_{\lambda+\mu} \hookrightarrow V^a_\lambda \otimes V^a_\mu$ sending $v_{\lambda+\mu}$ to $v_\lambda \otimes v_\mu$ (see [FFoL1, FFoL2]). This induces the embedding of varieties $\mathcal{F}^a_{\lambda+\mu} \hookrightarrow \mathcal{F}^a_\lambda \times \mathcal{F}^a_\mu$. Thus for any λ we obtain an embedding of $\mathcal{F}^a_\lambda$ into the product of Grassmannians. Our first result is an explicit description of this embedding. We state the theorem here for complete flag varieties $\mathcal{F}^a_n$. For this we need one more piece of notations. Let $pr_d : V \to V$ be the projection along the space $\mathbb{C}v_d$ to the linear span of the vectors v_i, $i \neq d$.

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Theorem 0.1. *The image of the embedding of the variety $\mathcal{F}_n^a$ in the product of Grassmann varieties $\prod_{d=1}^{n-1} Gr(d,n)$ is equal to the set of chains of subspaces $(V_1,\dots,V_{n-1})$, $V_d \in Gr(d,n)$ such that*

$$pr_{d+1}(V_d) \hookrightarrow V_{d+1}, \quad 1 \le d \le n-2.$$

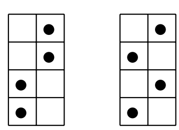
Our next goal is to compute the Poincaré polynomial of $\mathcal{F}_n^a$. Recall that in the classical case the flag variety $\mathcal{F}_n$ can be written as a disjoint union of $n!$ cells, each cell being associated with a torus fixed point. The fixed points are labeled by permutations from S_n. The length statistics $\sigma \to l(\sigma)$ gives the complex dimension of the cells. Therefore, the Poincaré polynomial $P_{\mathcal{F}_n}(t)$ of $\mathcal{F}_n$ is equal to $P_{\mathcal{F}_n}(t) = \sum_{\sigma \in S_n} t^{2l(\sigma)}$.

As an immediate corollary of Theorem 0.1 we obtain that the fixed points of the torus $T \subset G^a$ action on $\mathcal{F}_n^a$ are labeled by the sequences $I^1,\dots,I^{n-1}$, $I^d \subset \{1,\dots,n\}$, $\#I^d = d$, satisfying

$$I^d \setminus \{d+1\} \hookrightarrow I^{d+1}, \qquad d = 1,\dots,n-2. \tag{0.1}$$

(Note that this set of sequences has a subset with $I^d \hookrightarrow I^{d+1}$, which can be naturally identified with the permutations S_n). Our first task is to compute the number of such fixed points. To this end, recall the normalized median Genocchi numbers h_n, $n = 1,2,\dots$ (also referred to as the normalized Genocchi numbers of second kind). These numbers have several definitions [De, Du, DR, DZ, G, Kr, Vien] (see Section 3 for a review). Here we give the Dellac definition, which is the earliest one and which fits our construction in the best way.

Consider a rectangle with n columns and $2n$ rows. It contains $n \times 2n$ boxes labeled by pairs (l,j), with $l = 1,\dots,n$ and $j = 1,\dots,2n$. A Dellac configuration D is a subset of boxes, subject to the following conditions: first, each column contains exactly two boxes from D and each row contains exactly one box from D, and, second, if the (l,j)-th box is in D, then $l \le j \le n+l$. Let DC_n be the set of such configurations. Then h_n is the number of elements in DC_n. The first several median Genocchi numbers (starting from h_1) are as follows: $1, 2, 7, 38, 295$. For instance, the two Dellac configurations for $n = 2$ are as follows: (we specify boxes in a configuration by putting fat dots inside)



We prove the following theorem:

Theorem 0.2. *The number of sequences $I^1,\dots,I^{n-1}$ as above, satisfying* (0.1) *is equal to h_n.*

We also prove that the Dellac definition [De] is equivalent to the Dumont–Kreweras definition [Du, Kr] (this fact is known to experts [G],[S] but we were unable to find the proof in the literature).

Recall that the length of a permutation $\sigma \in S_n$ can be defined as the number of pairs $1 \le l_1 < l_2 \le n$ satisfying $\sigma(l_1) > \sigma(l_2)$. We define a length $l(D)$ of a Dellac configuration D as the number of squares $(l_1,j_1),(l_2,j_2) \in D$ such that $l_1 < l_2$ and $j_1 > j_2$. We prove the following theorem:

Theorem 0.3. *The Poincaré polynomial $P_{\mathcal{F}_n^a}(t)$ is given by $\sum_{D \in DC_n} t^{2l(D)}$.*

Our paper is organized in the following way:
In Section 1 we recall main definitions and theorems from [Fe3],
In Section 2 we describe explicitly the image of the embedding of the varieties $\mathcal{F}^a_\lambda$ into the product of Grassmannians and construct the cell decomposition of $\mathcal{F}^a_\lambda$.
In Section 3 we study the combinatorics of the median Genocchi numbers and compute the Poincaré polynomials of the complete degenerate flag varieties.

1. PBW deformation

1.1. Definitions. We first recall basic definitions and constructions from [FFoL1] and [Fe3]. Let $\mathfrak{g}$ be a simple Lie algebra with the Cartan decomposition $\mathfrak{g} = \mathfrak{n}\oplus\mathfrak{h}\oplus\mathfrak{n}^-$. We denote by M the number of positive roots of $\mathfrak{g}$, i.e. $M = \dim \mathfrak{n}$. Let $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$ be a Borel subalgebra. Then the deformed algebra $\mathfrak{g}^a$ is defined as a sum of two subalgebras $\mathfrak{g}^a = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is an abelian Lie algebra isomorphic to $\mathfrak{n}^-$ as a vector space. The subalgebra $(\mathfrak{n}^-)^a \hookrightarrow \mathfrak{g}^a$ is an abelian ideal and the action of $\mathfrak{b}$ on $(\mathfrak{n}^-)^a$ is induced from the identification $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$.

Let G be the Lie group of the Lie algebra $\mathfrak{g}$. Let N, T, N^-, B be the Lie groups of the Lie algebras $\mathfrak{n}$, $\mathfrak{h}$, $\mathfrak{n}^-$, $\mathfrak{b}$. The deformed Lie group G^a is defined as a semi-direct product of B and the normal subgroup $\mathbb{G}_a^M$, where $\mathbb{G}_a$ is the additive group of the field (thus $\mathbb{G}_a^M$ is the Lie group of the Lie algebra $(\mathfrak{n}^-)^a$). The Borel group B acts on the vector space $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$ via the restriction of the adjoint action and therefore there exists a natural homomorphism from B to $Aut(\mathbb{G}_a^M)$, defining the semi-direct product $G^a = B \ltimes \mathbb{G}_a^M$.

For a dominant integral weight λ we denote by V_λ the corresponding irreducible highest weight $\mathfrak{g}$-module with a highest weight vector v_λ. The Lie algebra $\mathfrak{g}^a$ and the Lie group G^a act on the deformed representations V_λ^a, where λ are dominant integral weights of $\mathfrak{g}$. The representations V_λ^a are defined as associated graded $gr_\bullet V_\lambda$ of the representation V_λ with respect to the PBW filtration F_s:

$$F_s = \mathrm{span}\{x_1 \cdots x_l v_\lambda : \ x_i \in \mathfrak{g}, l \le s\}.$$

So $V_\lambda^a = \bigoplus_{s\ge 0} V_\lambda^a(s)$, where $V_\lambda^a(0) = \mathbb{C}v_\lambda$ and $V_\lambda^a(s) = F_s/F_{s-1}$ for $s > 0$. It is easy to see that the action of $\mathfrak{n}^-$ on V_λ becomes abelian on V_λ^a (i.e. it induces the action of $(\mathfrak{n}^-)^a$) and the action of the Borel subalgebra induces the action of (the same algebra) $\mathfrak{b}$. The actions of $(\mathfrak{n}^-)^a$ and $\mathfrak{b}$ glue together to the action of $\mathfrak{g}^a$.

Remark 1.1. Let $\tilde{\mathfrak{g}}^a = \mathfrak{g}^a \oplus \mathbb{C}p$ be the central, extension of $\mathfrak{g}^a$ with a single element p subject to the relations $[p, \mathfrak{b}] = 0$, $[p, f_\alpha] = f_\alpha$ for any positive root α and the corresponding weight element $f_\alpha \in (\mathfrak{n}^-)^a$. Thus the Cartan subalgebra of $\tilde{\mathfrak{g}}^a$ has one extra dimension. We note that the $\mathfrak{g}^a$-module structure of V_λ^a naturally lifts to the structure of representation of $\tilde{\mathfrak{g}}^a$ by setting $pv_\lambda = 0$ (in general, $p|_{V_\lambda^a(s)} = s$). An eigenvalue of the operator p is sometimes referred to as a PBW degree. The character of V_λ^a with respect to $\mathfrak{h} \oplus \mathbb{C}p$ was computed in [FFoL1] for $\mathfrak{sl}_n$ and in [FFoL2] for symplectic Lie algebras. We denote the Lie group of $\tilde{\mathfrak{g}}^a$ by $\tilde{G}^a$, which differs from G^a by an additional $\mathbb{C}^*$.

Consider the action of G^a on the projective space $\mathbb{P}(V_\lambda^a)$. Recall that in the classical situation the (generalized) flag varieties are defined as $\mathcal{F}_\lambda = G \cdot \mathbb{C}v_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$ (see [K1]). The degenerate flag varieties $\mathcal{F}^a_\lambda \hookrightarrow \mathbb{P}(V_\lambda^a)$ are defined as the closures of the G^a

orbit (or, equivalently, of the $\mathbb{G}_a^M$ orbit) of the line $\mathbb{C}v_\lambda$. We note that in the classical case the orbit $G \cdot \mathbb{C}v_\lambda$ already covers the whole flag variety. This is not true in the degenerate case: the orbit $G^a \cdot \mathbb{C}v_\lambda$ is an affine cell, whose closure gives a projective singular variety $\mathcal{F}_\lambda^a$.

1.2. The type A case. From now on we assume that $\mathfrak{g} = \mathfrak{sl}_n$ and $G = SL_n$. Then all positive roots are of the form

$$\alpha_{i,j} = \alpha_i + \cdots + \alpha_j,\ 1 \le i \le j \le n-1$$

(for instance, $\alpha_{i,i} = \alpha_i$ are the simple roots). We denote by $f_{i,j} = f_{\alpha_{i,j}} \in \mathfrak{n}^-$ and $e_{i,j} = e_{\alpha_{i,j}} \in \mathfrak{n}$ the corresponding root elements. We have $\mathcal{F}^a_{\omega_d} \simeq \mathcal{F}_{\omega_d} \simeq Gr(d,n)$. The reason why the degenerate flag varieties are isomorphic to the non-degenerate ones for fundamental weights is that the radicals in $\mathfrak{sl}_n$, corresponding to ω_d, are abelian. In other words, define the set of positive roots

$$R_d = \{\alpha_{i,j} :\ 1 \le i \le d \le j \le n-1\}.$$

Define the subalgebra $\mathfrak{u}_d^- = \mathrm{span}\{f_\alpha :\ \alpha \in R_d\}$. Then $\mathfrak{u}_d^-$ is abelian and $V_{\omega_d} = U(\mathfrak{u}_d^-) \cdot v_\lambda$.

Remark 1.2. Let us explain the difference between the structure of $\mathfrak{g}$-module on V_{ω_d} and the structure of $\mathfrak{g}^a$-module on $V^a_{\omega_d}$. The operators f_α act trivially on $V^a_{\omega_d}$ unless $\alpha \in R_d$. Also, e_α act trivially on $V^a_{\omega_d}$ if $\alpha \in R_d$. Therefore, $\mathfrak{g}^a$ acts on $V^a_{\omega_d}$ via the projection to the subalgebra

$$\mathfrak{g}_d^a = \mathfrak{u}_d^- \oplus \mathfrak{h} \oplus \mathrm{span}\{e_\alpha :\ \alpha \notin R_d\}. \tag{1.1}$$

Similarly, the group G^a acts on $Gr(d,n)$ via the surjection to the Lie group of $\mathfrak{g}^a_d$. In particular, the group G^a does not act transitively on the deformed flag varieties even in the case of Grassmannians.

Remark 1.3. We note that though $\mathcal{F}^a_{\omega_d} \simeq \mathcal{F}_{\omega_d} \simeq Gr(d,n)$, the actions of the Borel groups $B \subset G$ and $B \subset G^a$ are very different. Let us consider the case $G = SL_2$. Then $\mathfrak{g}^a$ is spanned by three elements e^a, h^a and f^a subject to the relations

$$[h^a, e^a] = 2e^a,\ [h^a, f^a] = -2f^a,\ [e^a, f^a] = 0.$$

Let λ be a dominant weight of $\mathfrak{sl}_2$, $\lambda \in \mathbb{Z}_{\ge 0}$. Then V^a_λ is the direct sum of one-dimensional subspaces spanned by vectors v_l, $l = \lambda, \lambda-2, \ldots, -\lambda$ such that

$$h^a v_l = l v_l,\ f^a v_l = v_{l-2},\ e^a v_l = 0.$$

Therefore, the Borel subgroup B acts trivially on $\mathcal{F}^a_\lambda \simeq \mathbb{P}^1$. For instance, there exists one point of $\mathbb{P}^1$, which is fixed by the action of the whole group G^a.

Let us now recall the Plücker relations for $\mathcal{F}_\lambda$ [Fu] and the deformed Plücker relations for $\mathcal{F}^a_\lambda$ [Fe3].

Let $1 \le d_1 < \cdots < d_s \le n-1$ be a sequence of increasing numbers. Then for any positive integers $a_1, \ldots, a_s$ the variety $\mathcal{F}_{a_1\omega_{d_1} + \cdots + a_s\omega_{d_s}}$ is isomorphic to the partial flag variety

$$\mathcal{F}(d_1, \ldots, d_s) = \{V_1 \hookrightarrow V_2 \hookrightarrow \cdots \hookrightarrow V_s \hookrightarrow \mathbb{C}^n :\ \dim V_i = d_i\}.$$

In particular, if $s = 1$, then $\mathcal{F}(d)$ is the Grassmannian $Gr(d, n)$ and for $s = n - 1$ $\mathcal{F}(1, \dots, n-1)$ is the variety of the complete flags. We recall that

$$V_{\omega_d} = \Lambda^d(V_{\omega_1}) = \Lambda^d(\mathbb{C}^n)$$

and the embedding $Gr(d, n) \hookrightarrow \mathbb{P}(\Lambda^d V_{\omega_1})$ is defined as follows: a subspace with a basis $w_1, \dots, w_d$ maps to $\mathbb{C}w_1 \wedge \cdots \wedge w_d$. For general sequence $d_1, \dots, d_s$ one has embeddings:

$$\mathcal{F}(d_1, \dots, d_s) \hookrightarrow Gr(d_1, n) \times \cdots \times Gr(d_s, n) \hookrightarrow \mathbb{P}(V_{\omega_{d_1}}) \times \cdots \times \mathbb{P}(V_{\omega_{d_s}}).$$

The composition of these embeddings is called the Plücker embedding. The image is described explicitly in terms of Plücker relations. Namely, let $v_1, \dots, v_n$ be a basis of $\mathbb{C}^n = V_{\omega_1}$. Then one gets a basis v_J of V_{ω_d} $v_J = v_{j_1} \wedge \cdots \wedge v_{j_d}$ labeled by sequences $J = (1 \le j_1 < j_2 < \cdots < j_d \le n)$. Let $X_J \in V^*_{\omega_d}$ be the dual basis. We denote by the same symbols the coordinates of a vector $v \in V_{\omega_d}$: $X_J = X_J(v)$. The image of the embedding

$$\mathcal{F}(d_1, \dots, d_s) \hookrightarrow \times_{i=1}^s \mathbb{P}(V_{\omega_{d_i}})$$

is defined by the Plücker relations. These relations are labeled by a pair of numbers $p \ge q$, $p, q \in \{d_1, \dots, d_s\}$, by a number k, $1 \le k \le q$ and by a pair of sequences $L = (l_1, \dots, l_p)$, $J = (j_1, \dots, j_q)$, $1 \le l_\alpha, j_\beta \le n$. The corresponding relation is denoted by $R^k_{L,J}$ and is given by

$$R^k_{L,J} = X_L X_J - \sum_{1 \le r_1 < \cdots < r_k \le p} X_{L'} X_{J'}, \tag{1.2}$$

where L',J' are obtained from L, J by interchanging k-tuples $(l_{r_1}, \dots, l_{r_k})$ and $(j_1, \dots, j_k)$ in L and J respectively, i.e.

$$J' = (l_{r_1}, \dots, l_{r_k}, j_{k+1}, \dots, j_q),$$
$$L' = (l_1, \dots, l_{r_1 - 1}, j_1, l_{r_1+1}, \dots, l_{r_2-1}, j_2, \dots, l_p).$$

We note that for any $\sigma \in S_d$ the equality

$$X_{j_{\sigma(1)}, \dots, j_{\sigma(d)}} = (-1)^\sigma X_{j_1, \dots, j_d}$$

is assumed in (1.2). We denote the ideal generated by all $R^k_{L,J}$ by $I(d_1, \dots, d_s)$.

We introduce the notation

$$\mathcal{F}^a(d_1, \dots, d_s) = \mathcal{F}^a_{\omega_{d_1} + \cdots + \omega_{d_s}}, \ 1 \le d_1 < \cdots < d_s < n.$$

Definition 1.1. Let $I^a(d_1, \dots, d_s)$ be an ideal in the polynomial ring in variables $X^a_{j_1, \dots, j_d}$, $d = d_1, \dots, d_s$, $1 \le j_1 < \cdots < j_d < n$, generated by the elements $R^{k;a}_{L,J}$ given below. These elements are labeled by a pair of numbers $p \ge q$, $p, q \in \{d_1, \dots, d_s\}$, by an integer k, $1 \le k \le q$ and by sequences $L = (l_1, \dots, l_p)$, $J = (j_1, \dots, j_q)$, which are arbitrary subsets of the set $\{1, \dots, n\}$. The generating elements are given by the formulas

$$R^{k;a}_{L,J} = X^a_{l_1, \dots, l_p} X^a_{j_1, \dots, j_q} - \sum_{1 \le r_1 < \cdots < r_k \le p} X^a_{l'_1, \dots, l'_p} X^a_{j'_1, \dots, j'_q}, \tag{1.3}$$

where the terms of $R^{k;a}_{L,J}$ are the terms of $R^k_{L,J}$ (1.2) (with a superscript a, to be precise) such that

$$\{l_{r_1}, \dots, l_{r_k}\} \cap \{q+1, \dots, p\} = \emptyset. \tag{1.4}$$

Remark 1.4. The initial term $X^a_{l_1,\dots,l_p}X^a_{j_1,\dots,j_q}$ is also subject to the condition (1.4), i.e. it is not present in $R^{k;a}_{L,J}$ if $\{j_1,\dots,j_k\}\cap\{q+1,\dots,p\}\ne\emptyset$.

Example 1.1. Let $s=1$. Then $I^a(d)=I(d)$, since there are no numbers l such that $d+1\le l\le d$ and thus $R^{k;a}_{L,J}=R^k_{L,J}$ (up to a superscript a in the notations of variables X_J). Hence $\mathcal{F}^a_{\omega_d}\simeq\mathcal{F}_{\omega_d}$.

The following theorem is proved in [Fe3].

Theorem 1.1. *The variety* $\mathcal{F}^a(d_1,\dots,d_s)\hookrightarrow\times_{i=1}^s\mathbb{P}(\Lambda^{d_i}\mathbb{C}^n)$ *is defined by the ideal* $I^a(d_1,\dots,d_s)$.

Example 1.2. Let $s=2$, $d_1=1$, $d_2=n-1$. Then the classical flag variety $\mathcal{F}(1,n-1)$ is a subvariety in $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$ defined by a single relation

$$\sum_{i=1}^n(-1)^{i-1}X_iX_{1,\dots,i-1,i+1,\dots,n}=0.$$

The degenerate variety $\mathcal{F}(1,n-1)$ is also a subvariety in $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$, defined by a "degenerate" relation

$$X^a_1X^a_{2,\dots,n}+(-1)^{n-1}X^a_nX^a_{1,\dots,n-1}=0.$$

2. Cell decomposition

In this section, we describe explicitly the image of $\mathcal{F}^a_\lambda$ inside the product of Grassmannians and construct the cell decomposition of the degenerate flag varieties. We start with the case of $\lambda=\omega_d$.

2.1. Cell decomposition for Grassmannians. Recall that $\mathcal{F}^a_{\omega_d}\simeq\mathcal{F}_{\omega_d}\simeq Gr(d,n)$. Given an increasing tuple $L=(l_1<\dots<l_d)$ we set

$$p_L=\operatorname{span}(v_{l_1},\dots,v_{l_d})\in Gr(d,n).$$

The subspace p_L is T-invariant. Let k be a number such that $l_k\le d<l_{k+1}$.

Proposition 2.1. *The orbit* $G^a\cdot p_L$ *is an affine cell and* $Gr(d,n)$ *is the disjoint union of all such cells.*

Proof. Recall that G^a acts on $Gr(d,n)$ via the projection to the Lie group of $\mathfrak{g}_d$ (see (1.1)). Therefore the elements of $G^a\cdot p_L$ are exactly the subspaces of V having a basis $e_1,\dots,e_d$ of the form

$$e_j=v_{l_j}+\sum_{i=1}^{l_j-1}a_{i,j}v_i+\sum_{i=d+1}^{n}a_{i,j}v_i,\ j=1,\dots,k \tag{2.1}$$

$$e_j=v_{l_j}+\sum_{i=d+1}^{l_j-1}a_{i,j}v_i,\ j=k+1,\dots,d. \tag{2.2}$$

Such elements in $Gr(d,n)$ obviously form an affine cell and one has a decomposition $Gr(d,n)=\sqcup_L G^a\cdot p_L$. □

Remark 2.1. Formulas (2.1) and (2.2) can be combined together as follows. Let $[k]_+ = k$ if $k > 0$ and $[k]_+ = k + n$ if $k \le 0$. Then each element of $G^a \cdot p_L$ has a basis $e_1, \dots, e_d$ of the form

$$e_j = v_{l_j} + \sum_{i=1}^{[l_j-d]_+-1} a_{i,j} v_{[l_j-i]_+}. \tag{2.3}$$

Remark 2.2. The orbit $G^a \cdot p_L$ can be identified with a certain cell $B \cdot p_J$ in the usual cell decomposition of $Gr(d,n)$. Namely, define J as follows:

$$J = (l_{k+1} - d, l_{k+2} - d, \dots, l_d - d, l_1 - d + n, l_2 - d + n, \dots, l_k - d + n).$$

Then the map

$$\psi : V \to V,\ \psi(v_i) = v_{[i-d]_+}, i = 1, \dots, n$$

sends $G^a \cdot p_L$ to $B \cdot p_j$ (this is clear from the explicit description (2.1), (2.2)).

Example 2.1. Let $n = 9$, $d = 4$ and $L = (2, 3, 6, 7)$ (thus $k = 2$). Then the elements of $G^a \cdot p_L$ can be identified with the following matrices (the columns of a matrix form a basis of the corresponding subspace):

$$\begin{pmatrix} * & * & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ * & * & 0 & 0 \\ * & * & * & * \end{pmatrix}.$$

Here $*$ denotes arbitrary entries and hence the number of stars coincides with the dimension of the cell.

2.2. Chains of subspaces. In this section we fix the numbers $d_1, \dots, d_s$ and write $\mathcal{F}^a$ for $\mathcal{F}^a(d_1, \dots, d_s)$. Let $v_1, \dots, v_n$ be some basis of $V \simeq \mathbb{C}^n$. For $1 \le i < j \le n$ we define the projections $pr_{i+1,j} : V \to V$ by the formula

$$pr_{i+1,j}\left(\sum_{l=1}^{n} c_l v_l\right) = \sum_{l=1}^{i} c_l v_l + \sum_{l=j+1}^{n} c_l v_l.$$

The goal of this subsection is to prove the following theorem.

Theorem 2.1. *The variety $\mathcal{F}^a \hookrightarrow Gr(d_1, n) \times \cdots \times Gr(d_s, n)$ is formed by all sequences $V_1, \dots, V_s$, $V_l \in Gr(d_l, n)$ such that for all $1 \le l < m \le s$*

$$pr_{d_l+1,d_m} V_l \hookrightarrow V_m. \tag{2.4}$$

Remark 2.3. It is easy to see that the set of conditions (2.4) is equivalent to the subset with $m = l + 1$, i.e. to the set of conditions

$$pr_{d_l+1,d_{l+1}} V_l \hookrightarrow V_{l+1}, \quad l = 1, \dots, s - 1. \tag{2.5}$$

Lemma 2.1. *Let $(V_1, \dots, V_s) \in \mathcal{F}^a$. Then conditions (2.4) are satisfied.*

Proof. Let us first look at the big cell $\mathbb{G}_a^M \cdot \mathbb{C}v_\lambda \subset \mathcal{F}^a$. Note that the line $\mathbb{C}v_\lambda$ is represented by the point

$$\times_{i=1}^s \mathrm{span}(v_1, \dots, v_{d_i}) \in \times_{i=1}^s Gr(d_i, n).$$

Take an element $g = \exp(\sum s_{i,j} f_{i,j}) \in \mathbb{G}_a^M \subset G^a$. Then one has

$$g \cdot \mathrm{span}(v_1, \dots, v_d) = \mathrm{span}\left(v_1 + \sum_{j=d}^{n-1} s_{1,j} v_{j+1}, \dots, v_d + \sum_{j=d}^{n-1} s_{d,j} v_{j+1}\right).$$

Therefore conditions (2.4) hold for all points from the big cell of the degenerate flag varieties. Since $\mathcal{F}_\lambda^a$ is the closure of the big cell, the lemma is proved. ☐

Proposition 2.2. *Let $V_1, \dots, V_s$ be a set of subspaces of V satisfying* (2.4) *with* $\dim V_l = d_l$. *Then* $(V_1, \dots, V_s) \in \mathcal{F}^a$.

Proof. We know that the image of the embedding

$$\mathcal{F}^a \hookrightarrow \times_{i=1}^s Gr(d_i, n) \hookrightarrow \times_{i=1}^s \mathbb{P}(\Lambda^{d_i} V)$$

is defined by the set of relations $R_{J,I}^{k;a} = 0$. Our goal is to prove that (2.4) implies that all the relations $R_{J,I}^{k;a}$ vanish. Fix a pair $1 \le l \le m \le s$. In what follows we denote the projection pr_{d_l+1,d_m} simply by pr.

Let $(V_1, \dots, V_s)$ be a collection of subspaces satisfying (2.4). Fix tuples $I = (i_1, \dots, i_l)$ and $J = (j_1, \dots, j_m)$ and a number k. We prove that the relation $R_{J,I}^{k;a}$ vanishes on $(V_1, \dots, V_s)$. Without loss of generality we assume that $i_1, \dots, i_k \notin [d_l + 1, d_m]$. We also rearrange the entries of I in such a way that the elements from $I \cap [d_l + 1, d_m]$ are concentrated at the end of I, i.e. there exists a number b such that

$$i_1, \dots, i_b \notin [d_l + 1, d_m], \quad i_{b+1}, \dots, i_l \in [d_l + 1, d_m].$$

Obviously, $b \ge k$. Let $l - c = \dim(\ker pr \cap V_l)$. We fix a basis $e_1, \dots, e_l$ of V_l such that $pre_1, \dots, pre_c$ is a basis of prV_l and $e_{c+1}, \dots, e_l$ form a basis of $\ker pr \cap V_l$. We denote by $a_{s,t}$ the coefficients of the expansion of e_s in terms of v_t:

$$e_q = \sum_{r=1}^{l} a_{r,q} v_r.$$

The idea of the proof is to use the following decomposition of a Plücker coordinate X_I:

$$X_I = \sum_{1 \le \alpha_1 < \dots < \alpha_{l-b} \le l} \pm a_{i_{b+1},\alpha_1} \dots a_{i_l,\alpha_{l-b}} X_{i_1,\dots,i_b}. \tag{2.6}$$

Here $X_{i_1,\dots,i_b}$ is the $(i_1, \dots, i_b)$-th Plücker coordinate of the vector space $\mathrm{span}(e_{\beta_1}, \dots, e_{\beta_b})$, where the set of β's is complementary to the set of α's, i.e.

$$\{\beta_1, \dots, \beta_b\} \cup \{\alpha_1, \dots, \alpha_{l-b}\} = \{i_1, \dots, i_l\}.$$

The decomposition (2.6) induces the decomposition of the relation $R_{J,I}^{k;a}$, such that each term can be shown to vanish. Note that if $b > c$ then X_I vanishes on V_l. We thus assume that $b \le c$.

Define the subspace

$$E_\beta = pr(\mathrm{span}(e_{\beta_1}, \dots, e_{\beta_b})).$$

We know that $E_\beta \hookrightarrow V_m$. In addition, the coordinates $X_{(i_1,\dots,i_b)}$ of the space $\mathrm{span}(e_{\beta_1},\dots,e_{\beta_b})$ coincide with the Plücker coordinates $Y_{(i_1,\dots,i_b)}$ of E_β, because $i_1,\dots,i_b \notin [d_l+1,d_m]$ (we are using the notations Y_I to distinguish between Plücker coordinated of different spaces). Since $E_\beta \hookrightarrow V_m$, the classical relations $R^k_{J,(i_1,\dots,i_b)}$ vanish on the pair (E_β,V_m). Since

$$E_\beta \hookrightarrow \mathrm{span}(v_1,\dots,v_{d_l},v_{d_m+1},\dots,v_n),$$

a Plücker coordinate $Y_{q_1,\dots,q_b}$ of E_β vanishes unless non of the indices $q_\bullet$ are between d_l+1 and d_m. Hence the degenerate Plücker relation $R^{k;a}_{J,(i_1,\dots,i_b)}$ also vanishes on (E_β,V_m). Note also that the decomposition (2.6) induces the decomposition

$$R^{k;a}_{J,I} = \sum_{1\le\alpha_1<\dots<\alpha_{l-b}\le l} \pm a_{i_{b+1},\alpha_1}\dots a_{i_l,\alpha_{l-b}} R^{k;a}_{J,(i_{\beta_1},\dots,i_{\beta_b})}.$$

But as we have shown above, each of the relations $R^{k;a}_{J,(i_{\beta_1},\dots,i_{\beta_b})}$ vanishes on (V_l,V_m). Hence so does $R^{k;a}_{J,I}$. □

Example 2.2. Let $\lambda=\omega_1+\omega_{n-1}$, i.e. $s=2$, $d_1=1$, $d_2=n-1$. Then the image of $\mathcal{F}^a(1,n-1)$ inside $Gr(1,n)\times Gr(n-1,n)$ is formed by all pairs V_1,V_2 such that $pr_{2,n-1}V_1\hookrightarrow V_2$. Since $pr_{2,n-1}V_1\hookrightarrow \mathrm{span}(v_1,v_n)$, the image of the embedding $\mathcal{F}^a(1,n-1)\hookrightarrow \mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$ is defined by a single relation

$$X^a_1X^a_{2,\dots,n} + (-1)^{n-1}X^a_nX^a_{1,\dots,n-1}=0,$$

which agrees with Example 1.2.

Corollary 2.1. *Theorem 2.1 is true.*

Corollary 2.2. *Let $I^1,\dots,I^s$, $I^l\subset\{1,\dots,n\}$ be a collection of tuples such that the cardinality of I^l is d_l. Then a point $p_{I^1}\times\dots\times p_{I^s}$ belongs to $\mathcal{F}^a$ if and only if*

$$I^l\setminus\{d_l+1,\dots,d_{l+1}\}\subset I^{l+1}. \tag{2.7}$$

Example 2.3. Consider the case of the complete flags: $s=n-1$, $d_l=l$. Set $pr_l=pr_{l,l}$. Then the embedding of $\mathcal{F}^a$ into the product of Grassmannians is defined by the conditions

$$pr_{l+1}V_l\hookrightarrow V_{l+1},\ l=1,\dots,n-2 \tag{2.8}$$

and the conditions (2.7) read as $I^l\setminus\{l+1\}\subset I^{l+1}$ for $l=1,\dots,n-2$.

2.3. Cells for $\mathcal{F}^a$. Recall that the cell decomposition for a Grassmannian is given by the G^a-orbits of the torus fixed points. However this is not true for the case of general $\mathcal{F}^a_\lambda$. Moreover, the number of G^a-orbits can be infinite. The simplest example is as follows.

Example 2.4. Let $n=4$, $\lambda=\omega_1+\omega_3$. Then $\mathcal{F}^a_\lambda$ is embedded into $\mathbb{P}^3\times\mathbb{P}^3$ (two Grassmannians for $\mathfrak{sl}_4$) with the coordinates $(x_1:x_2:x_3:x_4)$ and $(x_{123}:x_{124}:x_{133}:x_{234})$. The variety $\mathcal{F}^a_{\omega_1+\omega_3}$ is defined by a single relation $x_1x_{234}-x_4x_{123}=0$. Therefore, $\mathcal{F}^a_{\omega_1+\omega_3}$ contains the product $\mathbb{P}^2\times\mathbb{P}^2$ defined by $x_1=x_{123}=0$. We note that the subgroup $\mathbb{G}_a^6$ of G^a acts trivially on this $\mathbb{P}^2\times\mathbb{P}^2$ (the PBW-degree in both V_{ω_1} and V_{ω_3} is at most one). Therefore, we are left with an action of the Borel subgroup. Let w_1,w_2,w_3,w_4 and $w_{123},w_{124},w_{134},w_{234}$ be the standard bases for V_{ω_1} and V_{ω_3}.

The group B acts on the span of w_2, w_3, w_4 (resp. on the span of $w_{124}, w_{134}, w_{234}$) as on the quotient of the vector representation (resp. the dual vector representation) by $\mathbb{C}w_1$ (resp. $\mathbb{C}w_{123}$). It is easy to see that the corresponding B-action on $\mathbb{P}^2 \times \mathbb{P}^2$ has infinitely many orbits.

In the following proposition we describe the cell decomposition for $\mathcal{F}^a = \mathcal{F}^a(d_1, \dots, d_s)$.

Proposition 2.3. *Let* $\mathbf{I} = (I^1, \dots, I^s)$ *be a set of sequences satisfying the condition* (2.7)*. Then there exists a cell decomposition* $\mathcal{F}^a = \sqcup_{\mathbf{I}} C_{\mathbf{I}}$*, where*

$$C_{\mathbf{I}} = (G^a \cdot p_{I^1} \times \dots \times G^a \cdot p_{I^s}) \cap \mathcal{F}^a.$$

In other words, a cell is given by the intersection of the degenerate flag variety, embedded into the product of Grassmannians, with the product of the corresponding cells in $Gr(d_i, n)$*.*

Proof. In Theorem 3.1 we compute the dimensions of $C_{\mathbf{I}}$. In the proof we construct explicitly the coordinates on $C_{\mathbf{I}}$ thus showing that $C_{\mathbf{I}}$ is a cell. $\square$

3. The median Genocchi numbers

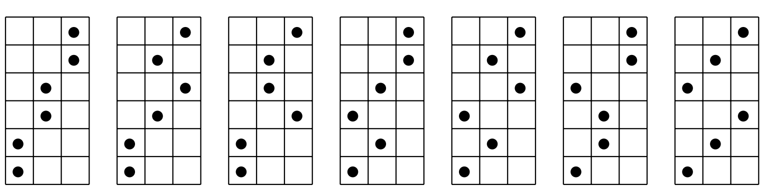
3.1. Combinatorics. Let h_n be the normalized Genocchi numbers of the second kind. They are also referred to as the normalized median Genocchi numbers. These numbers have several definitions (see [De, Du, Kr, S]). The first several h_n's are as follows: $1, 2, 7, 38, 295, 3098$. We first briefly recall definitions of these numbers.

We start with the Dellac definition (see [De]). Consider a rectangle with n columns and $2n$ rows. It contains $n \times 2n$ boxes labeled by pairs (l, j), where $l = 1, \dots, n$ is the number of a column and $j = 1, \dots, 2n$ is the number of a row. A Dellac configuration D is a subset of boxes, subject to the following conditions:

- each column contains exactly two boxes from D,
- each row contains exactly one box from D,
- if the (l, j)-th box is in D, then $l \le j \le n + l$.

Let DC_n be the set of such configurations. Then the number of elements in DC_n is equal to h_n.

We list all Dellac's configurations for $n = 3$. We specify boxes in a configuration by putting fat dots inside.

(3.1)  .

The Dellac definition is the earliest one, but the most well-known definition is via the Seidel triangle. The Seidel triangle is of the form

$$\begin{array}{ccccc}
1 & & & & \\
1 & & & & \\
1 & 1 & & & \\
2 & 1 & & & \\
2 & 3 & 3 & & \\
8 & 6 & 3 & & \\
8 & 14 & 17 & 17 & \\
56 & 48 & 34 & 17 & \\
56 & 104 & 138 & 155 & 155
\end{array}$$

By definition, the triangle is formed by the numbers $G_{k,n}$ (n is the number of a row and k is the number of a column) with $n = 1, 2, \dots$ and $1 \le k \le \frac{n+1}{2}$, subject to the relations $G_{1,1} = 1$ and

$$G_{k,2n} = \sum_{i \ge k} G_{i,2n-1},\ G_{k,2n+1} = \sum_{i \le k} G_{i,2n}.$$

The numbers $G_{n,2n-1}$ are called the Genocchi numbers of the first kind and the numbers $G_{1,2n}$ are called the Genocchi numbers of the second kind (or the median Genocchi numbers). Barsky [Ba] and then Dumont [Du] proved that the number $G_{1,2n+2}$ is divisible by 2^n. The normalized median Genocchi numbers h_n are defined as the corresponding ratios: $h_n = G_{1,2n+2}/2^n$.

In [Kr] Kreweras suggested another description of the numbers h_n. Namely, a permutation $\sigma \in S_{2n+2}$ is called a normalized Dumont permutation of the second kind if the following conditions are satisfied:

- $\sigma(k) < k$ if k is even,
- $\sigma(k) > k$ if k is odd,
- $\sigma^{-1}(2k) < \sigma^{-1}(2k+1)$ for $k = 1, \dots, n$.

The set of such permutations is denoted by $PD2N_n$ (P for permutations, D for Dumont, 2 for the second kind and N for normalized). According to Kreweras, the number of elements of $PD2N_n$ is equal to h_n. In Proposition 3.2 we show that the definitions of Dellac and Kreweras are equivalent (this seems to be known to expert — see [G, S], but we were not able to find a proof in the literature).

In the following proposition we show that the conditions from Example 2.3 give rise to a new definition of the numbers h_n.

Proposition 3.1. *The number of tuples $I^1, \dots, I^{n-1}$, with $I^l \subset \{1, \dots, n\}$, $\#I^l = l$ subject to the condition*

$$I^{l-1} \setminus \{l\} \subset I^l,\ l = 2, \dots, n-1 \tag{3.2}$$

is equal to h_n.

Proof. Let $\bar{h}_n$ be the number of tuples as above. We compare $\bar{h}_n$ with the Dellac definition of h_n. Given a set $I^1, \dots, I^{n-1}$ subject to the condition (3.2), we construct the corresponding Dellac's configuration D and then prove that this map is one-to-one. The rule is as follows. Let us explain what are the boxes of D in the lth column.

First, suppose $l \notin I^{l-1}$. Then because of the condition (3.2) the difference $I^l \setminus I^{l-1}$ contains exactly one number j. There are two cases:

- If $j > l$, then D contains boxes (l, l) and (l, j).
- If $j \le l$, then D contains boxes (l, l) and $(l, j + n)$.

Now, suppose $l \in I^{l-1}$. Then either $l \in I^l$, or $L \notin I^l$. If $l \in I^l$, then $I^l \setminus I^{l-1}$ contains exactly one number j. There are two cases:

- If $j > l$, then D contains boxes $(l, l + n)$ and (l, j).
- If $j \le l$, then D contains boxes $(l, l + n)$ and $(l, j + n)$.

Finally, let $l \in I^{l-1}$ and $l \notin I^l$. Then $I^l \setminus I^{l-1}$ contains exactly two numbers j_1 and j_2. There are four variants:

- If $j_1 > l$ and $j_2 > l$, then D contains boxes (l, j_1) and (l, j_2).
- If $j_1 > l$ and $j_2 \le l$, then D contains boxes (l, j_1) and $(l, n + j_2)$.
- If $j_1 \le l$ and $j_2 > l$, then D contains boxes $(l, j_1 + n)$ and (l, j_2).
- If $j_1 \le l$ and $j_2 \le l$, then D contains boxes $(l, j_1 + n)$ and $(l, j_2 + n)$.

This rule explains how to pick boxes in columns from 1 to $n - 1$. To complete the configuration we simply pick two boxes in the last column in the unique way to make D a configuration.

In order to prove that this map is a bijection, we construct the inverse map. Let D be a Dellac configuration. We define I^l inductively. First, let $l = 1$. Then the box $(1, 1)$ necessarily belongs to D. Let $j > 1$ and D contains $(1, j)$. Then if $j = n + 1$, then $I^1 = (1)$. Otherwise $I^1 = (j)$.

Now assume that I^{l-1} is already defined. First, suppose that the (l, l)-th box belongs to D. Then there exists one more box (l, j) in D with $n + l \ge j > l$. If $j \le n$ we set $I^l = I^{l-1} \cup \{j\}$. Otherwise, we set $I^l = I^{l-1} \cup \{j - n\}$. Second, suppose that the (l, l)-th box does not belong to D. Since the lth row of D contains exactly one box, there exists $l_1 < l$ such that the (l_1, l)-th box belongs to D. Therefore, $l \subset I^{l-1}$. There exist exactly two boxes (l, j_1) and (l, j_2) in D in the lth column. Then we set $I^l = I^{l-1} \setminus \{l\} \cup \{\bar{j}_1, \bar{j}_2\}$, where $\bar{j} = j$, if $j \le n$ and $\bar{j} = j - n$ otherwise. $\square$

Example 3.1. Let $n = 3$. The pairs I^1, I^2, corresponding to the Dellac configurations (3.1) are as follows (the order is the same as on picture (3.1)):

$$\{(2), (13)\}, \quad \{(2), (23)\}, \quad \{(2), (12)\}, \quad \{(3), (13)\},$$
$$\{(3), (23)\}, \quad \{(1), (13)\}, \quad \{(1), (12)\}.$$

We now compare the definitions by Dellac and by Kreweras.

Proposition 3.2. *The number of elements in $PD2N_n$ is equal to the number of elements in DC_n.*

Proof. We construct a bijection $A : PD2N_n \to DC_n$. Let $\sigma \in PD2N_n$. We determine the boxes in the kth column of $A(\sigma)$ using the values of $\sigma^{-1}(2k)$ and $\sigma^{-1}(2k + 1)$.

Let us start with $k = 1$. We note that $\sigma(2) = 1$, $\sigma(4)$ is equal to 2 or to 3. In addition, $\sigma^{-1}(2) = 1$ or 4 and the possible values of $\sigma^{-1}(3)$ are $4, 6, \ldots, 2n + 2$. Therefore, all possible values of the pair $(\sigma^{-1}(2), \sigma^{-1}(3))$ are as follows:

$$(1, 4),\ (4, 6),\ (4, 8), \ldots, (4, 2n + 2).$$

If the first possibility occurs, then by definition the first column of $A(\sigma)$ contains boxes $(1,1)$ (as any Dellac's configuration) and $(1,n+1)$. If $\sigma^{-1}(2)=4$ and $\sigma^{-1}(3)=2l+2$, then the first column of $A(\sigma)$ contains boxes $(1,1)$ and $(1,l)$.

Now let us consider the case $k=n$. We note that $\sigma(2n+1)=2n$, $\sigma(2n-1)$ is equal to $2n$ or to $2n+1$. In addition, $\sigma^{-1}(2n+1)=2n+2$ or $2n-1$ and the possible values of $\sigma^{-1}(2n)$ are $1,3,\dots,2n-1$. Therefore, all possible values of the pair $(\sigma^{-1}(2n),\sigma^{-1}(2n+1))$ are as follows:

$$(2n-1,2n+2),\ (1,2n-1),\ (3,2n-1),\dots,(2n-3,2n-1).$$

If the first possibility occurs, then by definition the nth column of $A(\sigma)$ contains boxes $(n,2n)$ (as any Dellac's configuration) and (n,n). If

$$(\sigma^{-1}(2n),\sigma^{-1}(2n+1))=(2l-1,2n-1),$$

then the first column of $A(\sigma)$ contains boxes $(n,2n)$ and $(n,n+l)$.

Finally, take $k=2,\dots,n-1$. We note that the possible values of $\sigma^{-1}(2k)$ are $1,3,\dots,2k-1,2k+2,\dots,2n$. Also, the possible values of $\sigma^{-1}(2k+1)$ are $3,5,\dots,2k-1,2k+2,\dots,2n,2n+2$. We now define the kth column of $A(\sigma)$ as follows:

(i) If the pair $(\sigma^{-1}(2k),\sigma^{-1}(2k+1))$ contains $2l-1$, $l=1,\dots,k$, then the kth column of $A(\sigma)$ contains a box $(k,n+l)$.
(ii) If the pair $(\sigma^{-1}(2k),\sigma^{-1}(2k+1))$ contains $2l+2$, $l=k,\dots,n$, then the kth column of $A(\sigma)$ contains a box (k,l).

We note that $A(\sigma)\in DC_n$. In fact, by definition any column of $A(\sigma)$ contains exactly two boxes and every row contains exactly one box (this follows from the definition above and because σ is one-to-one). In order to prove that A is a bijection it suffices to note that formulas (i) and (ii) allow to construct explicitly the map A^{-1}. $\square$

Example 3.2. Let $n=3$. The elements of $PD2N_3$ corresponding to the Dellac configurations on picture (3.1) are as follows (the order is the same as on picture (3.1)):

$$(41627385),\quad (61427385),\quad (41526387),\quad (41627583),$$
$$(61427583),\quad (21637485),\quad (21436587).$$

We recall that the main ingredient for the Kreweras construction of $PD2N_n$ is the following triangle:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 3 & & 2 & & & \\
 & & 7 & & 12 & & 12 & & 7 & & \\
 & 38 & & 69 & & 81 & & 69 & & 38 & \\
295 & & 552 & & 702 & & 702 & & 552 & & 295
\end{array}$$

The rule is as follows: denote the numbers in the nth line by $h_{n,1},\dots,h_{n,n}$. For example, $h_{4,2}=12$. Then the Kreweras triangle is defined by

$$h_{n,1}=h_{n-1,1}+\dots+h_{n-1,n-1},\ h_{n,2}=2h_{n,1}-h_{n-1,1},$$
$$h_{n,k}=2h_{n,k-1}-h_{n,k-2}-h_{n-1,k-2}-h_{n-1,k-1},\ k\geq 3.$$

Kreweras proved that $h_{n+1,1}$ is the nth Genocchi number h_n and in general $h_{n+1,k}$ is the number of the normalized Dumont permutations $\sigma \in S_{2n+2}$ of the second kind such that $\sigma(1) = 2k$. The following is an immediate corollary from the explicit bijections above.

Corollary 3.1. *The number of the Dellac configurations $D \in DC_n$ such that* $\min\{i : (i, n+1) \in D\} = k$ *is equal to $h_{n,k}$. The number of tuples $I^1, \dots, I^{n-1}$ subject to the condition $I^{l-1} \setminus \{l\} \subset I^l$ with an extra condition* $\min\{j :\ 1 \in I^j\} = k$ *is equal to $h_{n,k}$.*

3.2. The Poincaré polynomials. For a tuple $\mathbf{I} = (I^1, \dots, I^{n-1})$ subject to the relation $I^{l-1} \setminus \{l\} \subset I^l$ we denote by $D_{\mathbf{I}}$ the corresponding Dellac configuration. For a Dellac configuration $D \in DC_n$ we define the length $l(D)$ of D as the number of pairs (l_1, j_1), (l_2, j_2) such that the boxes (l_1, j_1) and (l_2, j_2) are both in D and $l_1 < l_2$, $j_1 > j_2$. We call such a pair of boxes (l_1, j_1), (l_2, j_2) a disorder. This definition resembles the definition of the length of a permutation. We note that in the classical case the dimension of a cell attached to a permutation σ in a flag variety is equal to the number of pairs $j_1 < j_2$ such that $\sigma(j_1) > \sigma(j_2)$ (which equals to the length of σ).

Theorem 3.1. *The dimension of a cell $C_{\mathbf{I}}$ is equal to $l(D_{\mathbf{I}})$.*

Proof. We prove the dimension formula by constructing explicitly the coordinates on the cell $C_{\mathbf{I}}$. Let

$$\mathbf{I} = (I^1, \dots, I^{n-1}),\ I^d = (i_1^d < \dots < i_d^d).$$

Recall the description of the cells $C_{I^d} \subset Gr(d, n)$ from Proposition 2.1. Using this description we construct the coordinates on $C_{\mathbf{I}}$ inductively on d. Let $(V_1, \dots, V_{n-1}) \in C_{\mathbf{I}}$. For a number k we set $[k]_+ = k$ if $k > 0$ and $[k]_+ = k + n$ if $k \le 0$.

We start with $d = 1$. An element $V_1 \in C_{I^1}$ is of the form $\mathbb{C}e_1^1$ with

$$e_1^1 = v_{i_1^1} + a_1^1 v_{[i_1^1-1]_+} + \dots + a^1_{[i_1^1-1]_+-1} v_2$$

(see Remark 2.1). We state that $[i_1^1 - 1]_+ - 1$ (which is exactly the number of the degrees of freedom we have so far) is exactly the number of boxes $(l, j) \in D_{\mathbf{I}}$ such that $l > 1$ and $j < i_1^1$ (note that the box $(1, 1)$ is necessarily in $D_{\mathbf{I}}$, but it does not add anything to the length of $D_{\mathbf{I}}$, since for any $(l.j) \in D_{\mathbf{I}}$ with $l > 1$ we have $j > 1$). In fact, the first column of $D_{\mathbf{I}}$ contains boxes in the first row and in the $([i_1^1 - 1]_+ + 1)$-st row (see the proof of Proposition 3.1). Since any row of $D_{\mathbf{I}}$ contains exactly one box, the rows number $2, \dots, [i_1^1 - 1]_+$ are occupied by boxes in the columns from 2 to n. Therefore, the box $(1, [i_1^1 - 1]_+ + 1)$ produces exactly $[i_1^1 - 1]_+ - 1$ disorders.

The second step is to construct the coordinates on those subspaces from C_{I^2} which contain $pr_2 V_1$. There are two possibilities: either $i_1^1 = 2$ or $i_1^1 \ne 2$. In the first case the condition $pr_2 V_1 \hookrightarrow V_2$ is empty. Therefore, we have to choose two basis vectors e_1^2, e_2^2 of $V_2 \in C_{I^2}$, with the coordinates

$$e_1^2 = v_{i_1^2} + a_1^1 v_{[i_1^2-1]_+} + \dots + a^1_{[i_1^2-2]_+-1} v_3,$$
$$e_2^2 = v_{i_2^2} + a_1^2 v_{[i_2^2-1]_+} + \dots + a^2_{[i_2^2-2]_+-2} v_3.$$

We note that the number of coefficients of e_2^2 is $[i_2^2 - 2]_+ - 2$, because $i_1^2 < i_2^2$ and hence adding appropriately normalized vector e_1^2 one can vanish the coefficient of e_2^2 in front of $v_{i_1^2}$. We note that since $i_1^1 = 2$, the second column of $D_{\mathbf{I}}$ contains boxes in the rows $([i_1^2 - 2]_+ + 2)$ and $([i_1^2 - 2]_+ + 2)$ (see the proof of Proposition 3.1). We state that

$[i_1^2-2]_+-1+[i_2^2-2]_+-2$ (the number of degrees of freedom we have fixing the vectors e_1^2 and e_2^2) is exactly the number of boxes in the columns $3,4,\dots,n$, having disorders with boxes in the second column. In fact, each row from 3 to $[i_1^2-2]_+-1$ contains one box in the columns 3 and greater (recall $i_1^1=2$). This produces $[i_1^2-2]_+-1$ disorders with the box $(2,[i_1^2-2]_+-1)$. Similarly, we obtain $[i_2^2-2]_+-2$ disorders with the second box in the second column.

Now assume $i_1^1\neq 2$. Then the space pr_2V_1 is nontrivial and spanned by a single vector $e_1^2=pr_2e_1^1$. Therefore in order to specify V_2 we need to fix one more vector e_2^2 such that $\mathrm{span}(e_1^2,e_2^2)\in C_{I^2}$. Recall that since $i_1^1\neq 2$ we have $I^2\setminus I^1=\{j\}$. Also, the second column of $D_{\mathbf{I}}$ contains boxes in the second row and in the row number $[j-2]_++2$ (see the proof of Proposition 3.1). The box $(2,2)$ does not produce any disorder with boxes in the columns greater than 2. As for the box $(2,[j-2]_++2)$, the number of disorders it produces is equal to the number of degrees of freedom of choosing the vector e_2^2 (the argument is very similar to the ones above in the case $i_1^1=2$).

Now let us consider the general induction step. Assume that we have already computed the number of degrees of freedom while fixing the subspaces $V_1,\dots,V_{d-1}$. Our goal is to show that the number of degrees of freedom of V_d is equal to the number of disorders produced by the boxes in the d'th column with the boxes in columns l with $l>d$. As in the previous case, one has to consider two cases: $d\in I^{d-1}$ and $d\notin I^{d-1}$. The proof is very similar to the one in the case $d=2$ and we omit it. □

Corollary 3.2. *The Poincaré polynomial $P_n(t)=P_{\mathcal{F}^a}(t)$ is given by*

$$P_n(t)=\sum_{D\in DC_n} t^{2l(D)}.$$

Let $q=t^2$. Then P_n are polynomials in q with $P_n(1)=h_n$. Thus the Poincaré polynomials of the degenerate flag varieties provide a natural q-version of the normalized median Genocchi numbers (it would be interesting to compare our q-version with the one in [HZ]).

Example 3.3. The first four polynomials $P_n(q)$ are as follows:

$$P_1(q)=1,\qquad P_2(q)=1+q,$$
$$P_3(q)=1+2q+3q^2+q^3,$$
$$P_4(q)=1+3q+7q^2+10q^3+10q^4+6q^5+q^6.$$

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Department of Mathematics, National Research University Higher School of Economics, Vavilova street 7, Moscow 117312, Russia

Tamm Theory Division, Lebedev Physics Institute, Moscow 117924, Russia
E-mail address: `evgfeig@gmail.com`