Math. Res. Lett. **19** (2012), *no.* 01, 13–29

THETA FUNCTIONS AND ARITHMETIC QUOTIENTS OF LOOP GROUPS

Dongwen Liu

Abstract. In this paper, we observe that isomorphism classes of certain meterized vector bundles over $\mathbb{P}^1_{\mathbb{Z}} - \{0, \infty\}$ can be parameterized by arithmetic quotients of loop groups. We construct an asymptotic version of theta functions, which are defined on these quotients. Then we prove the convergence and extend the theta functions to loop symplectic groups. We interpret them as sections of line bundles over an infinite-dimensional torus, discuss the relations with loop Heisenberg groups, and give an asymptotic multiplication formula.

1. Introduction

Let E be a vector bundle of rank n on the affine scheme $X := \mathbb{P}^1_{\mathbb{Z}} - \{0, \infty\} = \mathrm{Spec} R$, where $R = \mathbb{Z}[t, t^{-1}]$. Then the space of global global sections $H^0(E)$ of E is isomorphic to R^n. Fix $q \in \mathbb{R}$, $q > 1$. If we choose a trivialization $H^0(E) \simeq R^n$, then we may define an inner product on $H^0(E)$ via the following inner product $(,)_1$ on R^n:

$$(w(t), v(t))_1 = \int_{q^{-1}S^1} w(t)^T \overline{v(t)}\, dt, \tag{1.1}$$

where $q^{-1}S^1 = \{t \in \mathbb{C} : |t| = q^{-1}\}$, $\overline{v(t)}$ denotes the complex conjugate, and the superscript ^{T} is the transpose (here we write $w(t), v(t)$ as column vectors). The Lebesgue measure on $q^{-1}S^1$ is normalized such that $\mathrm{meas}(q^{-1}S^1) = 1$. For any $g \in G := GL_n(\mathbb{R}[t, t^{-1}])$, we define

$$(w, v)_g := (g^{-1}w, g^{-1}v)_1. \tag{1.2}$$

We extend these inner products to $\mathbb{R}[t, t^{-1}]^n = R^n \otimes_{\mathbb{Z}} \mathbb{R}$ by the same formulas.

Our starting point is to introduce the arithmetic quotient (see Section 2 for precise details)

$$\mathcal{Q}_{n,q} = \widetilde{\Gamma} \backslash \widetilde{G} / \widetilde{K},$$

where Γ, K are certain "discrete" and "maximal compact" subgroup of G respectively, $\widetilde{G}$ is a central extension of G by $\mathbb{R}^\times$, and $\widetilde{\Gamma}$, $\widetilde{K}$ are double covers of Γ, K. The key observation (Theorem 2.1) is that $\mathcal{Q}_{n,q}$ classifies above metrized vector bundles on X with an additional datum called *covolume theory*. Roughly speaking, a covolume theory, denoted by c, will assign a positive number $c(L)$ to each lattice L of $H^0(E)$ in such a way that the information of relative covolumes between different lattices will

Received by the editors April 2, 2011.
2000 *Mathematics Subject Classification.* Primary 22E66, 22E67.
Key words and phrases. theta functions, arithmetic quotients.

be captured. Via this moduli interpretation of $\mathcal{Q}_{n,q}$, for any $\widetilde{g} \in \widetilde{G}$ we may associate a triple $(E, (\,,\,)_g, c)$. Then we construct an asymptotic version of theta functions on $\widetilde{G}$

$$\vartheta(\widetilde{g}) := \lim_{L} c(L) \sum_{v \in L} e^{-\pi(v,v)_g},$$

where L runs over all lattices L of $H^0(E)$, and the limit is taken with respect to inclusions of lattices. The interesting point of our construction is the (tautological!) fact that ϑ is defined on $\mathcal{Q}_{n,q}$. In other words, we obtain an automorphic function on $\widetilde{G}$.

The most technical problem is the convergence of the limit defining $\vartheta(\widetilde{g})$. Using a variant of the lemmas in [4] and [12], together with some elementary Fourier analysis, in particular the Poisson summation formula, we establish in Section 3 the uniform convergence of $\vartheta(\widetilde{g})$ for $\widetilde{g}$ varying in certain Siegel subset of $\widetilde{G}$. In the course of the proof, the use of Iwasawa decomposition for loop groups simplifies our considerations. Motivated by the work of Zhu [12] on Weil representations, we also extend the theta functions to loop symplectic groups in Section 4. In all the above, the reparametrization $t \mapsto qt$ for the variable t, plays an important role for the convergence result. Note that it is also a crucial ingredient in the proof of the convergence of the Eisenstein series (see [1, 3]) on loop groups.

We try to give some interpretations for our theta functions from representation-theoretic and geometric point of view. Mumford's idea [5] is to bring together three ways of viewing theta functions:

(a) as holomorphic functions in the vector/period matrix,
(b) as matrix coefficients of a representation of the Heisenberg/metaplectic groups,
(c) as sections of line bundles on abelian varieties/moduli space of abelian varieties.

In Section 5, we interpret the theta functions as global sections of line bundles over an infinite-dimensional torus, discuss the action of loop Heisenberg groups, and give an asymptotic multiplication formula, which is analogous to the theta relations given by Mumford in [5].

Inspirations from Kapranov's work [2] are indispensable for the writing of this paper, especially for introducing the notion of covolume theory. We also use the notions of *semi-infinite Grassmannian* and *dimension theory* in [2] to write our multiplication formula in Section 5 in a more conceptual way.

It should not be too surprising to get some generalizations of part of this paper to more general framework, say, to the situation of higher local fields and adelic spaces, or the so-called C_n-categories (see [6–8, 10]). For instance, Parshin [9] considered Heisenberg groups associated with objects from $C_0^{f.g.}$, the category of finitely generated abelian groups. The loop Heisenberg groups introduced in Section 5 should be viewed as Heisenberg groups associated with objects from $C_1^{f.g.}$, the category of filtered abelian groups with finitely generated quotients.

Notations. *For an abelian group A, we write $A_{\mathbb{R}} = A \otimes_{\mathbb{Z}} \mathbb{R}$. For a finite-dimensional real vector space V, write $V^* = Hom_{\mathbb{R}}(V, \mathbb{R})$. Let $(\,,\,) : V \times V^* \to \mathbb{R}$ be the canonical pairing. If L is a lattice of V, denote by $L^* \subset V^*$ the dual lattice with respect to the bi-character $e^{-2\pi i(,)}$. Let $\mathcal{S}(V)$ be the space of Schwartz functions on V. If a Haar* *measure on V has been chosen, then define the Fourier transform $\mathscr{F} : \mathcal{S}(V) \to \mathcal{S}(V^*)$ by*

$$\mathscr{F} f(y) = \int_V f(x)\, e^{-2\pi i (x,y)} dx.$$

The Haar measure on V^* is determined by the condition that $\mathscr{F}$ is an isometry.

For a positive integer l, $\mathbb{Z}_l$ stands for $\mathbb{Z}/l\mathbb{Z}$ (in contrast to the usual meaning of l-adic integers for prime l), U_l is the group of lth roots of unity. The pairing $\mathbb{Z}_l \times U_l \to U_l \subset S^1$ gives the Pontryagin duality. Denote $\zeta_l = e^{2\pi i/l}$.

For $d \in \mathbb{Z}$ we write $L^d_{\mathbb{Z}} = t^d \mathbb{Z}[t^{-1}]^n$, $L^d_{\mathbb{R}} = t^d \mathbb{R}[t^{-1}]^n = L^d_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.

For an invertible matrix g write $g^* = (g^{-1})^T$.

2. Moduli interpretations of arithmetic quotients

Our purpose in this section is to give a moduli interpretation of arithmetic quotients of loop groups, which in turn classify the vector bundles with metrics described in the introduction. We start from the loop group $GL_n(\mathbb{R}[t,t^{-1}])$. It is well known that this group has a central extension $\widetilde{GL}_n(\mathbb{R}[t,t^{-1}])$ which we shall now define.

A *lattice* L of a free $\mathbb{R}[t,t^{-1}]$-module V of rank n is a free $\mathbb{R}[t^{-1}]$-submodule of rank n. In other words L is an $\mathbb{R}[t^{-1}]$-span of a basis of V. Any two lattices L_1, L_2 in V are *commensurable*, in the sense that the quotients $L_1/(L_1 \cap L_2)$ and $L_2/(L_1 \cap L_2)$ are finite-dimensional over $\mathbb{R}$. For example any lattice in $\mathbb{R}[t,t^{-1}]^n$ is commensurable with $\mathbb{R}[t^{-1}]^n$.

Let $L^0_{\mathbb{R}}$ be the lattice $\mathbb{R}[t^{-1}]^n$, and $g \in GL_n(\mathbb{R}[t,t^{-1}])$. Since $gL^0_{\mathbb{R}}/(L^0_{\mathbb{R}} \cap gL^0_{\mathbb{R}})$ and $L^0_{\mathbb{R}}/(L^0_{\mathbb{R}} \cap gL^0_{\mathbb{R}})$ are finite-dimensional vector spaces over $\mathbb{R}$, we can talk about their top wedge powers. Let $\det(L^0_{\mathbb{R}}, gL^0_{\mathbb{R}})$ be the tensor product

$$\wedge^{\mathrm{top}}(gL^0_{\mathbb{R}}/L^0_{\mathbb{R}} \cap gL^0_{\mathbb{R}}) \otimes_{\mathbb{R}} \wedge^{\mathrm{top}}(L^0_{\mathbb{R}}/L^0_{\mathbb{R}} \cap gL^0_{\mathbb{R}})^{-1},$$

where $(\cdot)^{-1} = \mathrm{Hom}_{\mathbb{R}}(\cdot, \mathbb{R})$ denotes the dual vector space. Then $\det(L^0_{\mathbb{R}}, gL^0_{\mathbb{R}})$ is a one-dimensional vector space over $\mathbb{R}$. Let $\det(L^0_{\mathbb{R}}, gL^0_{\mathbb{R}})^{\times}$ be the set of nonzero vectors in $\det(L^0_{\mathbb{R}}, gL^0_{\mathbb{R}})$, which form an $\mathbb{R}^{\times}$-torsor. Now define the group

$$\widetilde{GL}_n(\mathbb{R}[t,t^{-1}]) = \{(g,\omega_g) | g \in GL_n(\mathbb{R}[t,t^{-1}]), \omega_g \in \det(L^0_{\mathbb{R}}, gL^0_{\mathbb{R}})^{\times}\}. \tag{2.1}$$

The multiplication in the group is given by

$$(g,\omega_g)(h,\omega_h) = (gh, \omega_g \wedge g\omega_h),$$

where $g\omega_h$ is image of ω_h under the natural map

$$\det(L^0_{\mathbb{R}}, hL^0_{\mathbb{R}}) \xrightarrow{g} \det(gL^0_{\mathbb{R}}, ghL^0_{\mathbb{R}}),$$

and $\omega_g \wedge g\omega_h$ is defined via the isomorphism

$$\det(L^0_{\mathbb{R}}, gL^0_{\mathbb{R}}) \wedge \det(gL^0_{\mathbb{R}}, ghL^0_{\mathbb{R}}) \longrightarrow \det(L^0_{\mathbb{R}}, ghL^0_{\mathbb{R}}).$$

It turns out that $\widetilde{GL}_n(\mathbb{R}[t,t^{-1}])$ is a central extension of $GL_n(\mathbb{R}[t,t^{-1}])$ by $\mathbb{R}^{\times}$, then we have the short exact sequence

$$1 \longrightarrow \mathbb{R}^{\times} \longrightarrow \widetilde{GL}_n(\mathbb{R}[t,t^{-1}]) \longrightarrow GL_n(\mathbb{R}[t,t^{-1}]) \longrightarrow 1. \tag{2.2}$$

We are concerned with two subgroups of $GL_n(\mathbb{R}[t,t^{-1}])$ together with their extensions. The first one is $GL_n(\mathbb{Z}[t,t^{-1}])$, which plays the role of a discrete subgroup, or more precisely the so-called arithmetic subgroup. This group acts on $R^n = \mathbb{Z}[t,t^{-1}]^n$. Similarly as above, let us define a lattice in R^n to be a free $\mathbb{Z}[t^{-1}]$-submodule of rank n. Let $L^0_{\mathbb{Z}} = \mathbb{Z}[t^{-1}]^n$. For any $g \in GL_n(\mathbb{Z}[t,t^{-1}])$, $gL^0_{\mathbb{Z}}/(L^0_{\mathbb{Z}} \cap gL^0_{\mathbb{Z}})$ and $L^0_{\mathbb{Z}}/(L^0_{\mathbb{Z}} \cap gL^0_{\mathbb{Z}})$ are free $\mathbb{Z}$-modules of finite rank, and we let $\det(L^0_{\mathbb{Z}}, gL^0_{\mathbb{Z}})$ be the rank one free $\mathbb{Z}$-module

$$\wedge^{\mathrm{top}}(gL^0_{\mathbb{Z}}/L^0_{\mathbb{Z}} \cap gL^0_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \wedge^{\mathrm{top}}(L^0_{\mathbb{Z}}/L^0_{\mathbb{Z}} \cap gL^0_{\mathbb{Z}})^{-1}.$$

Note that there is a natural isomorphism

$$\varphi\colon \det(L^0_{\mathbb{Z}}, gL^0_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \det(L^0_{\mathbb{R}}, gL^0_{\mathbb{R}}). \tag{2.3}$$

Define the group

$$\widetilde{GL}_n(\mathbb{Z}[t,t^{-1}]) = \{(g,\omega_g) | g \in GL_n(\mathbb{Z}[t,t^{-1}]), \omega_g \text{ a basis of } \det(L^0_{\mathbb{Z}}, gL^0_{\mathbb{Z}})\} \tag{2.4}$$

which is a double cover of $GL_n(\mathbb{Z}[t,t^{-1}])$, i.e., there is a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{GL}_n(\mathbb{Z}[t,t^{-1}]) \longrightarrow GL_n(\mathbb{Z}[t,t^{-1}]) \longrightarrow 1.$$

Via (2.3) we have an embedding $\widetilde{GL}_n(\mathbb{Z}[t,t^{-1}]) \hookrightarrow \widetilde{GL}_n(\mathbb{R}[t,t^{-1}])$, which is compatible with the canonical embedding $\mathbb{Z}_2 \hookrightarrow \mathbb{R}^\times$. In other words, we have the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] & \widetilde{GL}_n(\mathbb{Z}[t,t^{-1}]) \arrow[r] \arrow[d] & GL_n(\mathbb{Z}[t,t^{-1}]) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{R}^\times \arrow[r] & \widetilde{GL}_n(\mathbb{R}[t,t^{-1}]) \arrow[r] & GL_n(\mathbb{R}[t,t^{-1}]) \arrow[r] & 1
\end{tikzcd}$$

This construction of double cover is quite natural (see [2] Section 2.0).

Recall (1.2) that for any $g \in GL_n(\mathbb{R}[t,t^{-1}])$, we have an inner product $(,)_g$ on $\mathbb{R}[t,t^{-1}]^n$. Write $\|\cdot\|_g$ for the induced norm. The following lemma is an easy exercise.

Lemma 2.1. *Fix g and for any subspace H of $\mathbb{R}[t,t^{-1}]^n$ write $\overline{H}$ for the completion with respect to $\|\cdot\|_g$. Then for any two lattices L_1, L_2 of $\mathbb{R}[t,t^{-1}]^n$,*

$$\overline{L}_1 \cap \overline{L}_2 = \overline{L_1 \cap L_2}, \quad \frac{L_i}{L_1 \cap L_2} \simeq \frac{\overline{L}_i}{\overline{L}_1 \cap \overline{L}_2}, \quad i = 1,2.$$

Using this lemma, we may induce a metric on $\det(L^0_{\mathbb{R}}, gL^0_{\mathbb{R}})$, which we still denote by $\|\cdot\|_g$. The subgroup

$$K = \{g \in GL_n(\mathbb{R}[t,t^{-1}]) | g(q^{-1}t)g(q^{-1}t^{-1})^T = 1\} \tag{2.5}$$

plays the role of a maximal compact subgroup. In fact K is the isometry group of $(,)_1$. Hence $\|\cdot\|_{gk} = \|\cdot\|_g$ for any $g \in GL_n(\mathbb{R}[t,t^{-1}])$, $k \in K$. We remark that by definition there is a family of group homomorphisms

$$ev(t)\colon K \longrightarrow U(n) \tag{2.6}$$

for $t \in q^{-1}S^1$, via evaluation maps. For fixed $k \in K$, $ev(\cdot)(k) : q^{-1}S^1 \to U(n)$ is smooth. The double cover $\widetilde{K}$ of K is defined by

$$\widetilde{K} = \{(g, \omega_g) | g \in K, \omega_g \in \det(L^0_{\mathbb{R}}, gL^0_{\mathbb{R}})^\times, \|\omega_g\|_1 = 1\}. \tag{2.7}$$

Similarly, we also have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] & \widetilde{K} \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{R}^\times \arrow[r] & \widetilde{GL}_n(\mathbb{R}[t, t^{-1}]) \arrow[r] & GL_n(\mathbb{R}[t, t^{-1}]) \arrow[r] & 1
\end{tikzcd}$$

We introduce the arithmetic quotient

$$\mathcal{Q}_{n,q} := \widetilde{GL}_n(\mathbb{Z}[t, t^{-1}]) \backslash \widetilde{GL}_n(\mathbb{R}[t, t^{-1}]) / \widetilde{K}. \tag{2.8}$$

The main result of this section is the following moduli interpretation of $\mathcal{Q}_{n,q}$. Let E be a vector bundle of rank n on X, and $\| \cdot \|$ be a metric on E such that after a trivialization of E it is induced by (1.2) for some g. A *covolume theory* of $(E, \| \cdot \|)$ is a rule c which assigns a positive real number $c(L)$ to each lattice L of $H^0(E)$ such that

$$c(L') = c(L) cov_{\|\cdot\|} \left(\frac{L'}{L}, \frac{L'_{\mathbb{R}}}{L_{\mathbb{R}}} \right)$$

whenever $L \subset L'$ are two lattices of $H^0(E)$. Here $cov_{\|\cdot\|}$ denotes the covolume induced by the given metric $\| \cdot \|$, and $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$. Note that once we know $c(L)$ for one lattice L, then we know $c(L')$ for any lattice L'. In fact, take a lattice $L'' \subset L, L'$ and define

$$c(\| \cdot \|, L, L') = \frac{cov_{\|\cdot\|} \left(\frac{L'}{L''}, \frac{L'_{\mathbb{R}}}{L''_{\mathbb{R}}} \right)}{cov_{\|\cdot\|} \left(\frac{L}{L''}, \frac{L_{\mathbb{R}}}{L''_{\mathbb{R}}} \right)}. \tag{2.9}$$

Then (2.9) does not depend on the choice of L'' and satisfies

$$c(L') = c(L) c(\| \cdot \|, L, L'). \tag{2.10}$$

We remark that all covolume theories attached to $(E, \| \cdot \|)$ form an $\mathbb{R}^\times_+$-torsor.

Consider all the triples $(E, \| \cdot \|, c)$ as above. We say that two such triples $(E_i, \| \cdot \|_i, c_i)$, $i = 1, 2$ are isomorphic if there is an isomorphism $\eta : E_1 \to E_2$ of vector bundles such that η induces an isometry $H^0(E_1) \to H^0(E_2)$ and $c_2(\eta(L_1)) = c_1(L_1)$ for any lattice L_1 of $H^0(E_1)$.

Theorem 2.1. *$\mathcal{Q}_{n,q}$ classifies the isomorphism classes of all such triples $(E, \| \cdot \|, c)$.*

Proof. Given $(g, \omega_g) \in \widetilde{GL}_n(\mathbb{R}[t, t^{-1}])$, we associate the triple $(R^n, \| \cdot \|_g, c)$ such that the covolume theory c is determined by $c(L^0_{\mathbb{Z}}) = \|\omega_g\|_g^{-1}$.

(1) Let $(\gamma, \omega_\gamma) \in \widetilde{GL}_n(\mathbb{Z}[t, t^{-1}])^n$. Then associated with $(\gamma g, \omega_\gamma \wedge \gamma\omega_g)$, we have the triple $(R^n, \| \cdot \|_{\gamma g}, c')$ such that c' is determined by

$$c'(L^0_{\mathbb{Z}}) = \|\omega_\gamma \wedge \gamma\omega_g\|^{-1}_{\gamma g}.$$

Consider the automorphism of trivial vector bundles $\eta_\gamma : R^n \to R^n, v \mapsto \gamma v$. This induces an isometry because we have

$$\|\gamma v\|_{\gamma g} = \|g^{-1}v\|_1 = \|v\|_g.$$

Moreover, we have

$$\begin{aligned} c'(\gamma L^0_{\mathbb{Z}}) &= c'(L^0_{\mathbb{Z}})c(\|\cdot\|_{\gamma g}, L^0_{\mathbb{Z}}, \gamma L^0_{\mathbb{Z}}) \\ &= \|\omega_\gamma \wedge \gamma\omega_g\|^{-1}_{\gamma g} c(\|\cdot\|_{\gamma g}, L^0_{\mathbb{Z}}, \gamma L^0_{\mathbb{Z}}) \\ &= \|\omega_g\|^{-1}_g \|\omega_\gamma\|^{-1}_{\gamma g} c(\|\cdot\|_{\gamma g}, L^0_{\mathbb{Z}}, \gamma L^0_{\mathbb{Z}}) \\ &= \|\omega_g\|^{-1}_g = c(L^0_{\mathbb{Z}}). \end{aligned}$$

This proves that η_γ induces an isomorphism between the triples $(R^n, \|\cdot\|_g, c)$ and $(R^n, \|\cdot\|_{\gamma g}, c')$.

(2) Let $(k, \omega_k) \in \widetilde{K}$. Then associated with $(gk, \omega_g \wedge g\omega_k)$, we have the triple $(R^n, \|\cdot\|_g, c'')$ such that c'' is determined by

$$c''(L^0_{\mathbb{Z}}) = \|\omega_g \wedge g\omega_k\|^{-1}_g.$$

In this case, the identity map induces an isometry, and

$$c''(L^0_{\mathbb{Z}}) = \|\omega_g\|^{-1}_g \|\omega_k\|^{-1}_1 = \|\omega_g\|^{-1}_g = c(L^0_{\mathbb{Z}}).$$

Hence, the triples $(R^n, \|\cdot\|_g, c)$ and $(R^n, \|\cdot\|_g, c'')$ are isomorphic via the identity map.

We have constructed a map from $\mathcal{Q}_{n,q}$ to the set of isomorphism classes of such triples. The inverse map is also clear and it is easy to prove this is a bijection. $\square$

3. Theta functions on the arithmetic quotients

Suppose we have a triple $(E, \|\cdot\|, c)$ as in the previous section. Via isomorphisms we may always assume that $E = R^n$ and $\|\cdot\| = \|\cdot\|_g$ for some $g \in GL(\mathbb{R}[t, t^{-1}])$. Let $\widetilde{g} = (g, \omega_g) \in \widetilde{GL}_n(\mathbb{R}[t, t^{-1}])$ such that $\widetilde{g}$ gives the triple $(R^n, \|\cdot\|_g, c)$ (see the proof of Theorem 2.1). We define

$$\vartheta(\widetilde{g}) = \lim_L \vartheta_L(\widetilde{g}) := \lim_L c(L) \sum_{v \in L} e^{-\pi(v,v)_g}, \tag{3.1}$$

where L runs over all lattices of $R^n \simeq H^0(E)$, and the limit is taken with respect to the inclusions of lattices. To make the definition more transparent, consider the filtration of lattices

$$\cdots \subset L^d_{\mathbb{Z}} \subset L^{d+1}_{\mathbb{Z}} \subset \cdots,$$

where $L^d_{\mathbb{Z}} := t^d L^0_{\mathbb{Z}}$. Then $\bigcup_{d\in\mathbb{Z}} L^d_{\mathbb{Z}} = R^n$ and $\vartheta(\widetilde{g})$ is the value (assume the existence), which satisfies the condition that for any $\varepsilon > 0$, there exists $d \in \mathbb{Z}$ such that

$$|\vartheta_L(\widetilde{g}) - \vartheta(\widetilde{g})| < \varepsilon, \tag{3.2}$$

whenever L is a lattice containing $L^d_{\mathbb{Z}}$. Of course, in this description, we may replace $\{L^d_{\mathbb{Z}}, d \in \mathbb{Z}\}$ by any exhausting filtration of lattices. Clearly $\vartheta(\widetilde{g})$ only depends on the isomorphism class of the triple, and from Theorem 2.1 it is a tautological fact that ϑ descends to a function on $\mathcal{Q}_{n,q}$, i.e., ϑ is left invariant under $\widetilde{GL}_n(\mathbb{Z}[t, t^{-1}])$ and right invariant under $\widetilde{K}$.

In this section, we shall prove uniform convergence properties of $\vartheta(\widetilde{g})$, for $\widetilde{g}$ varying in certain "compact" subset of $\widetilde{GL}_n(\mathbb{R}[t,t^{-1}])$. Equivalently, we are considering behaviors of $\vartheta([\widetilde{g}])$ when $[\widetilde{g}]$ lies in certain subset of the arithmetic quotient $\mathcal{Q}_{n,q}$. To this end, we need the following Iwasawa decomposition:

Lemma 3.1. *Let $B=\{g(t)\in GL_n(\mathbb{R}[t^{-1}])|g(\infty)$ is upper triangular$\}$. The sequence (2.2) splits over B. If $\widetilde{B}$ is the preimage of B under the projection*

$$\widetilde{GL}_n(\mathbb{R}[t,t^{-1}])\to GL_n(\mathbb{R}[t,t^{-1}]),$$

then

$$\widetilde{GL}_n(\mathbb{R}[t,t^{-1}])=\widetilde{B}\widetilde{K}=\mathbb{R}^\times B\widetilde{K}.$$

Proof of this lemma is by using standard theory of Tits systems. We also need a lemma that is similar to [12] Lemma 4.8 and [4] Lemma 3.3.

Lemma 3.2. *Let $V_1,\ldots,V_m$ be finite dimensional real vector spaces, $V=V_1\oplus\cdots\oplus V_m$. Suppose $f(x_1,\ldots,x_m)\in\mathcal{S}(V)$ is a Schwartz function, which takes values in $\mathbb{R}$ and its partial Fourier transforms $F_if\geq 0$ $(i=1,\ldots,m)$, i.e.,*

$$\int_{V_i}f(x_1,\ldots,x_m)e^{-2\pi i(x_i,y_i)}dx_i\geq 0,$$

where $y_i\in V_i^*$. Let $L_i\subset V_i$ be a lattice (i.e. a free $\mathbb{Z}$-module which spans V_i over $\mathbb{R}$) and $L=L_1\oplus\cdots\oplus L_m$. If $U:V\to V$ is a blockwise unipotent linear operator, i.e.,

$$UV_i\subset V_1\oplus\cdots\oplus V_i,\quad U|_{V_i}\equiv 1 \ mod\ V_1\oplus\cdots\oplus V_{i-1},$$

then

$$\sum_{k\in L}f(Uk)\leq\sum_{k\in L}f(k).$$

We give some notations. Let $N=\{g(t)\in B|g(\infty)$ is unipotent$\}$, and A be the group of diagonal matrices in $GL_n(\mathbb{R})$, then $B=AN$. For each $d\in\mathbb{Z}$, let $L^d_{\mathbb{R}}=t^dL^0_{\mathbb{R}}$, $V_d=t^d\mathbb{R}^n$ and $L_d=t^d\mathbb{Z}^n$, then

$$L^d_{\mathbb{R}}=\bigoplus_{i\leq d}V_i,\quad L^d_{\mathbb{Z}}=\bigoplus_{i\leq d}L_i. \tag{3.3}$$

For $d_1<d_2\in\mathbb{Z}$, let

$$V_{d_1,d_2}=\bigoplus_{d_1\leq i\leq d_2}V_i,\quad L_{d_1,d_2}=\bigoplus_{d_1\leq i\leq d_2}L_i. \tag{3.4}$$

Note that elements of N act on $L^d_{\mathbb{R}}$ as blockwise unipotent linear operators.

Let us define the notion of Siegel subsets of $\widetilde{GL}_n(\mathbb{R}[t,t^{-1}])$ which is suitable for our purpose. $GL_n(\mathbb{R}[t,t^{-1}])$ has an ind-scheme structure. Indeed for each $i\in\mathbb{N}$ let X_i be the subset of $GL_n(\mathbb{R}[t,t^{-1}])$ consisting of elements whose entries are contained in $V_{-i,i}$. Then $GL_n(\mathbb{R}[t,t^{-1}])=\bigcup_{i\in\mathbb{N}}X_i$, and one may choose an embedding

$$\varphi_i:X_i\hookrightarrow\mathbb{A}^{N(i)}(\mathbb{R}) \tag{3.5}$$

of affine subscheme, where $N(i) \in \mathbb{N}$ depends on i. For example, we may take $N(i) = n^2(2i+1) + 2ni + 1$. We call $C \subset GL_n(\mathbb{R}[t,t^{-1}])$ a compact subset, if C is contained in X_i for some i, and $\varphi_i(C) \subset \mathbb{R}^{N(i)}$ is compact in the real topology. It is easy to check that this notion is well defined, i.e., does not depend on the choice of (3.5). Similarly, we can define compact subsets of B. Recall from Lemma 3.1 the Iwasawa decomposition. We define a *Siegel subset* of $\widetilde{GL}_n(\mathbb{R}[t,t^{-1}])$ to be a subset of the form $C_Z C_B \widetilde{K}$ where C_Z, C_B are compact subsets of $\mathbb{R}^\times$ and B respectively.

Now, we are ready to state and prove the main result of this section.

Theorem 3.1. *The limit (3.1) defining $\vartheta(\widetilde{g})$ exists and converges uniformly for $\widetilde{g}$ varying in any Siegel subset of $\widetilde{GL}_n(\mathbb{R}[t,t^{-1}])$.*

Proof. Let $u \in N$, $a \in A$, and suppose that $\widetilde{g}$ gives the triple $(R^n, \|\cdot\|_{au}, c)$. Our strategy is to first show that the sequence

$$\vartheta_d(\widetilde{g}) := c(L_{\mathbb{Z}}^d) \sum_{v \in L_{\mathbb{Z}}^d} e^{-\pi(v,v)_{au}} \tag{3.6}$$

is bounded, and then compare $\vartheta_L(\widetilde{g})$ and $\vartheta_d(\widetilde{g})$ when L is a lattice containing $L_{\mathbb{Z}}^d$. More precisely, we shall prove that

$$\vartheta_L(\widetilde{g}) = (1 + o_d(1))\vartheta_d(\widetilde{g}), \tag{3.7}$$

where $o_d(1) \to 0$ as $d \to \infty$, and the rate of convergence depends on d but not on L. We proceed in 3 steps.

Step 1: Boundedness of subsequence. The trick is reducing to the diagonals. Let us first prove that for each $d \in \mathbb{Z}$,

$$\sum_{v \in L_{\mathbb{Z}}^d} e^{-\pi(v,v)_{au}} \leq \sum_{v \in L_{\mathbb{Z}}^d} e^{-\pi(v,v)_a}. \tag{3.8}$$

Lemma 3.2 does not apply directly in this infinite-dimensional case. However, consider

$$\sum_{v \in L_{d_1,d}} e^{-\pi(v,v)_{au}} \tag{3.9}$$

for $d_1 < d$. Let π_1, π_2 be the obvious projections

$$\pi_1 : L_{\mathbb{R}}^d \to V_{d_1,d}, \quad \pi_2 : L_{\mathbb{R}}^d \to L_{\mathbb{R}}^{d_1-1}.$$

Since $V_{d_1,d}$ and $L_{\mathbb{R}}^{d_1-1}$ are orthogonal with respect to $(,)_1$, we obtain

$$\begin{aligned}(v,v)_{au} &= (u^{-1}a^{-1}v, u^{-1}a^{-1}v)_1 \\ &\geq (\pi_1 u^{-1}a^{-1}v, \pi_1 u^{-1}a^{-1}v)_1.\end{aligned}$$

Apply Lemma 3.2 for V_i, $a^{-1}L_i$, $i = d_1, \dots, d$, $f(v) = e^{-\pi(v,v)_1}$ and the operator $U = \pi_1 u^{-1} : V_{d_1,d} \to V_{d_1,d}$, it follows that

$$(3.9) \leq \sum_{v \in a^{-1}L_{d_1,d}} e^{-\pi(Uv,Uv)_1} \leq \sum_{v \in a^{-1}L_{d_1,d}} e^{-\pi(v,v)_1} = \sum_{v \in L_{d_1,d}} e^{-\pi(v,v)_a}.$$

Let $d_1 \to -\infty$ we get (3.8).

On the other hand, we have the equality of covolumes

$$cov_{\|\cdot\|_{au}}\left(\frac{L^d_{\mathbb{Z}}}{L^{d_1}_{\mathbb{Z}}}, \frac{L^d_{\mathbb{R}}}{L^{d_1}_{\mathbb{R}}}\right) = cov_{\|\cdot\|_{a}}\left(\frac{L^d_{\mathbb{Z}}}{L^{d_1}_{\mathbb{Z}}}, \frac{L^d_{\mathbb{R}}}{L^{d_1}_{\mathbb{R}}}\right). \tag{3.10}$$

Indeed, take a basis $\{v_i\}$ of $L_{d_1,d}$. For $v \in L^d_{\mathbb{R}}$ denote by $v^{\perp}$ the orthogonal projection of v to $\left(L^{d_1}_{\mathbb{R}}\right)^{\perp}$ with respect to the metric $\|\cdot\|_{au}$. It is easy to check that

$$v^{\perp} = au\pi_1(u^{-1}a^{-1}v) = auUa^{-1}v.$$

Then

$$\begin{aligned}
&\text{LHS of (3.10)}\\
&\quad = \sqrt{\det\big((v_i^{\perp}, v_j^{\perp})_{au}\big)}\\
&\quad = \sqrt{\det((Ua^{-1}v_i, Ua^{-1}v_j)_1)}\\
&\quad = \sqrt{\det((a^{-1}v_i, a^{-1}v_j)_1)}\\
&\quad = \text{RHS of (3.10)}.
\end{aligned}$$

Write $a = \operatorname{diag}\{a_1, \dots, a_n\}$. Apply (3.8) and (3.10), we obtain that for $d > 0$,

$$\begin{aligned}
\vartheta_d(\widetilde{g}) &\le c(L^0_{\mathbb{Z}}) cov_{\|\cdot\|_a}\left(\frac{L^d_{\mathbb{Z}}}{L^0_{\mathbb{Z}}}, \frac{L^d_{\mathbb{R}}}{L^0_{\mathbb{R}}}\right) \sum_{v\in L^d_{\mathbb{Z}}} e^{-\pi(v,v)_a}\\
&= c(L^0_{\mathbb{Z}}) q^{-n\frac{d(d+1)}{2}} |\det a|^{-d} \prod_{j=1}^{n}\prod_{i=-\infty}^{d}\sum_{m\in\mathbb{Z}} \exp\big(-\pi a_j^{-2} q^{-2i} m^2\big).
\end{aligned}$$

By the Poisson summation formula,

$$\sum_{m\in\mathbb{Z}} \exp\big(-\pi a_j^{-2} q^{-2i} m^2\big) = |a_j| q^i \sum_{m\in\mathbb{Z}} \exp\big(-\pi a_j^{2} q^{2i} m^2\big).$$

Hence

$$\vartheta_d(\widetilde{g}) \le c(L^0_{\mathbb{Z}}) \prod_{j=1}^{n}\prod_{i=0}^{\infty}\sum_{m\in\mathbb{Z}} \exp\big(-\pi a_j^{-2} q^{2i} m^2\big) \prod_{i=1}^{d}\sum_{m\in\mathbb{Z}} \exp\big(-\pi a_j^{2} q^{2i} m^2\big), \tag{3.11}$$

and the right-hand-side (RHS) is convergent as $d \to \infty$.

Step 2: Comparison with the subsequence. Let L be a lattice containing $L^d_{\mathbb{Z}}$, and let L' be a complement of $L^d_{\mathbb{Z}}$ in L such that $L' \subset t^{d+1}\mathbb{Z}[t]$. Similarly as above, for $v' \in L'_{\mathbb{R}}$ write $v'^{\perp}$ for the orthogonal projection of v' to $(L^d_{\mathbb{R}})^{\perp}$. Write $v'_0 = v' - v'^{\perp} \in L^d_{\mathbb{R}}$. Then

$$\begin{aligned}
\vartheta_L(\widetilde{g}) &= c(L) \sum_{v\in L} e^{-\pi(v,v)_{au}}\\
&= c(L) \sum_{v\in L^d_{\mathbb{Z}}}\sum_{v'\in L'} e^{-\pi(v+v', v+v')_{au}}\\
&= c(L) \sum_{v'\in L'} e^{-\pi(v'^{\perp}, v'^{\perp})_{au}} \sum_{v\in L^d_{\mathbb{Z}}} e^{-\pi(v+v'_0, v+v'_0)_{au}}.
\end{aligned} \tag{3.12}$$

It is clear that $L'_{\mathbb{R}} \perp L_{\mathbb{R}}^{d-2\delta(u)-1}$, where $\delta(u)$ is the highest power of t^{-1} appearing in the entries of u. As a consequence, we have $v'_0 \in V_{d-2\delta(u),d}$. For $v \in L^d_{\mathbb{R}}$ let $v_\perp$ be the orthogonal projection of v to $V^\perp_{d-2\delta(u),d}$, and let $v_0 = v - v_\perp \in V_{d-2\delta(u),d}$. Then

$$\begin{aligned}(3.13)\qquad &\sum_{v\in L^d_{\mathbb{Z}}} e^{-\pi(v+v'_0,v+v'_0)_{au}}\\ &= \sum_{v\in t^{d-2\delta(u)-1}L^0_{\mathbb{R}}} e^{-\pi(v_\perp,v_\perp)_{au}} \sum_{v_1\in L_{d-2\delta(u),d}} e^{-\pi(v_1+v_0+v'_0,v_1+v_0+v'_0)_{au}}.\end{aligned}$$

Let $(,)^*_{au}$ be the induced inner product on $V^*_{d-2\delta(u),d}$. Choose the Haar measure on $V_{d-2\delta(u),d}$ such that the covolume of $L_{d-2\delta(u),d}$ equals one (i.e. choose the ordinary Lebesgue measure on each $V_i = t^i\mathbb{R} \simeq \mathbb{R}$ via $rt^i \mapsto r$). Then for fixed $x_0 \in V_{d-2\delta(u),d}$, we have the Fourier transform

$$\mathscr{F}\left(e^{-\pi(x+x_0,x+x_0)_{au}}\right)(y) = \frac{e^{2\pi i(x_0,y)-\pi(y,y)^*_{au}}}{cov_{\|\cdot\|_{au}}(L_{d-2\delta(u),d}, V_{d-2\delta(u),d})}.$$

By the Poisson summation formula,

$$\sum_{v_1\in L_{d-2\delta(u),d}} e^{-\pi(v_1+x_0,v_1+x_0)_{au}} = \sum_{v_1^*\in L^*_{d-2\delta(u),d}} \frac{e^{2\pi i(x_0,v_1^*)-\pi(v_1^*,v_1^*)^*_{au}}}{cov_{\|\cdot\|_{au}}(L_{d-2\delta(u),d}, V_{d-2\delta(u),d})}.$$

Previous reasoning for (3.8) implies that

$$\begin{aligned}&\left|\sum_{v_1^*\in L^*_{d-2\delta(u),d}-\{0\}} e^{2\pi i(x_0,v_1^*)-\pi(v_1^*,v_1^*)^*_{au}}\right|\\ &\le \sum_{v_1^*\in L^*_{d-2\delta(u),d}-\{0\}} e^{-\pi(v_1^*,v_1^*)^*_{au}}\\ &\le \sum_{v_1^*\in L^*_{d-2\delta(u),d}-\{0\}} e^{-\pi(v_1^*,v_1^*)^*_{a}}\\ &= \prod_{j=1}^n \prod_{i=d-2\delta(u)}^d \sum_{m\in\mathbb{Z}} \exp\left(-\pi a_j^2 q^{2i} m^2\right) - 1\\ &\le \prod_{j=1}^n \prod_{i=d-2\delta(u)}^\infty \sum_{m\in\mathbb{Z}} \exp\left(-\pi a_j^2 q^{2i} m^2\right) - 1\\ &= o_d(1).\end{aligned}$$

Therefore, we have proved that

$$\sum_{v_1\in L_{d-2\delta(u),d}} e^{-\pi(v_1+x_0,v_1+x_0)_{au}} = \frac{1+o_d(1)}{cov_{\|\cdot\|_{au}}(L_{d-2\delta(u),d}, V_{d-2\delta(u),d})},$$

where the rate of convergence of $o_d(1)$ does not depend on x_0. Hence from (3.13), we obtain

$$(3.14)\qquad \sum_{v\in L^d_{\mathbb{Z}}} e^{-\pi(v+v'_0,v+v'_0)_{au}} = (1+o_d(1)) \sum_{v\in L^d_{\mathbb{Z}}} e^{-\pi(v,v)_{au}},$$

where again the magnitude of $o_d(1)$ does not depend on v'_0. Similar arguments apply for the summation over L' in (3.12), and yield

$$\sum_{v' \in L'} e^{-\pi (v'^{\perp}, v'^{\perp})_{au}} = \frac{1 + o_d(1)}{cov_{\|\cdot\|_{au}}\left(\frac{L}{L^d_{\mathbb{Z}}}, \frac{L_{\mathbb{R}}}{L^d_{\mathbb{R}}}\right)}, \tag{3.15}$$

where the magnitude of $o_d(1)$ does not depend on L', hence not on L. Combine (3.12), (3.14) and (3.15), together with the identity

$$c(L) = c(L^d_{\mathbb{Z}}) cov_{\|\cdot\|_{au}}\left(\frac{L}{L^d_{\mathbb{Z}}}, \frac{L_{\mathbb{R}}}{L^d_{\mathbb{R}}}\right),$$

we obtain (3.7).

Step 3: Conclusions. After the first two steps, we have proved the existence of $\vartheta(\widetilde{g})$. If we restrict to a Siegel subset, then by definition, we have in the previous settings that $c(L^0_{\mathbb{Z}})$ and au vary in some compact subsets of $\mathbb{R}^{\times}_{+}$ and B respectively, and moreover $\delta(u)$ is bounded. Consequently, the bound (3.11) and the quantities $o_d(1)$ appearing in Step 2 are all uniform. □

4. Generalization to loop symplectic groups

Recall that the Siegel upper half-space $\mathscr{H}_n$ for $Sp_{2n}(\mathbb{R})$ is the set of all $n \times n$ complex symmetric matrices with positive definite imaginary part. $Sp_{2n}(\mathbb{R})$ acts on $\mathscr{H}_n$ via linear fractional transformations. $L^2(\mathbb{R}^n)$ is a model for the Weil representation of $Sp_{2n}(\mathbb{R})$, and the dense subspace $\mathcal{S}(\mathbb{R}^n)$ is closed under this action. For $\Omega \in \mathscr{H}_n$ the corresponding Gaussian function $f_\Omega(x) = e^{\pi i x \Omega x^T}$ is in $\mathcal{S}(\mathbb{R}^n)$. The theta function $\theta : \mathscr{H}_n \to \mathbb{C}$ given by

$$\theta(\Omega) = \sum_{m \in \mathbb{Z}^n} f_\Omega(m) = \sum_{m \in \mathbb{Z}^n} e^{\pi i m \Omega m^T}$$

is automorphic for some arithmetic subgroup of $Sp_{2n}(\mathbb{R})$. In [12] Zhu has generalized this classical theory of Weil representations and theta functions (cf. [11]) to the loop group $Sp_{2n}(\mathbb{R}((t)))$. His method also works in our situation. We do not attempt to build the theory of Weil representations for our loop group in full generality. Instead we shall focus on theta functions. From now on, we use row vectors instead of column vectors, and matrix group acts from the right.

Define $W = V \oplus V^* = \mathbb{R}[t,t^{-1}]^n \oplus \mathbb{R}[t,t^{-1}]^n$, where the canonical pairing (v, v^*) equals the constant term of $v(t)v^*(t)^T$. The space W has a symplectic form given by $\langle v_1 + v_1^*, v_2 + v_2^* \rangle = (v_1, v_2^*) - (v_2, v_1^*)$. We define $Sp_{2n}(\mathbb{R}[t,t^{-1}])$ to be the isometry group of $(W, \langle , \rangle)$.

Denote by $V_{\mathbb{C}}$ and $V^*_{\mathbb{C}}$ the complexifications of V and V^*; let $\mathscr{H}$ be the set of $\Omega = X + iY \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, V^*_{\mathbb{C}})$ such that $X, Y \in \mathrm{Hom}_{\mathbb{R}}(V, V^*)$ are are self-dual and $Y > 0$. Note that $\mathrm{Im}(\Omega) = Y$ induces an inner product $(w,v)_{iY}$ on V via $(w,v)_{iY} := (w, vY)$. Let $iI \in \mathscr{H}$ such that I induces the inner product $(,)_1$ defined by (1.1). In concrete terms, the operator $I : V \to V^*$ maps $t^i e_j$ to $q^{-2i} t^{-i} e_j$, where $i \in \mathbb{Z}$ and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. For any

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in Sp_{2n}(\mathbb{R}[t,t^{-1}]),$$

g acts on $\mathscr{H}$ by $g \cdot \Omega = (\alpha\Omega + \beta)(\gamma\Omega + \delta)^{-1}$. Consider the diagonal embedding

$$d : GL_n(\mathbb{R}[t, t^{-1}]) \hookrightarrow Sp_{2n}(\mathbb{R}[t, t^{-1}]), \quad g \mapsto d(g) = \begin{bmatrix} g^* & 0 \\ 0 & g \end{bmatrix},$$

where $g^* = (g^{-1})^T$. Then $d(g) \cdot iI$ induces the inner product $(,)_g$. This suggests that we may generalize the theta function to the loop symplectic group.

Remark 4.1. Here our diagonal embedding differs from the usual one because in previous sections, we let $GL_n(V)$ act on V from the left. For the setting of Weil representations by convention the symplectic group acts from the right.

We first generalize the notion of covolume theory. Define a sesquilinear form $(,)_\Omega : V_\mathbb{C} \times V_\mathbb{C} \to \mathbb{C}$ by $(v, w)_\Omega = -i(\bar{v}, w\Omega)$. It is easy to check that $\mathrm{Re}(v, v)_\Omega > 0$ for any $v \neq 0$. If $L \subset L'$ are two lattices of $R^n = \mathbb{Z}[t, t^{-1}]^n$, the orthogonal complement $L_\mathbb{C}^\perp$ of $L_\mathbb{C}$ in $L'_\mathbb{C}$ with respect to the form $(,)_\Omega$ is well defined. If $\{v_i\}$ is a complementary basis (i.e. $\{v_i\}$ span a complement of L in L'), then we define the *covolume* of L in L' to be the complex number

$$c(\Omega, L, L') = \sqrt{\det(v_i^\perp, v_j^\perp)_\Omega}, \tag{4.1}$$

where $v^\perp$ stands for the projection of $v \in L'_\mathbb{C}$ to $L_\mathbb{C}^\perp$. Here for the square root, we take the positive branch. Similarly as in section 2, a *covolume theory* for Ω is a rule c assigning a nonzero complex number $c(L)$ to each lattice L of R^n such that $c(L') = c(L)c(\Omega, L, L')$ whenever $L \subset L'$ are two lattices.

If $\Omega = g \cdot iI$ for some $g \in Sp_{2n}(\mathbb{R}[t, t^{-1}])$ and c is a covolume theory for Ω, then we define

$$\vartheta(\Omega, c) = \lim_L c(L) \sum_{v \in L} e^{-\pi(v,v)_\Omega}. \tag{4.2}$$

Proposition 4.1. *The limit defining $\vartheta(\Omega, c)$ is convergent.*

The proof of Proposition 4.1 relies on the Iwasawa decomposition for loop symplectic groups, which we shall recall now. Let

$$K_{sp} = \{g \in Sp_{2n}(\mathbb{R}[t, t^{-1}]) | g(q^{-1}t)g(q^{-1}t^{-1})^T = 1\}. \tag{4.3}$$

It can be shown that K_{sp} is the stabilizer of iI in $Sp_{2n}(\mathbb{R}[t, t^{-1}])$. Let B_0 denote the Borel subgroup of $Sp_{2n}(\mathbb{R})$ given by

$$B_0 = \left\{ \begin{bmatrix} A^* & C \\ 0 & A \end{bmatrix} \in Sp_{2n}(\mathbb{R}) \middle| A \text{ is upper triangular} \right\}, \tag{4.4}$$

and define

$$B_{sp} = \{g \in Sp_{2n}(\mathbb{R}[t^{-1}]) | g(\infty) \in B_0\}. \tag{4.5}$$

Then we have the Iwasawa decomposition

$$Sp_{2n}(\mathbb{R}[t, t^{-1}]) = B_{sp}K_{sp}. \tag{4.6}$$

Proof of Proposition 4.1: It suffices to consider $\Omega = g \cdot iI$ with $g \in B_{sp}$. We use the fact that B_{sp} is generated by elements of the form

$$\begin{bmatrix} \alpha & 0 \\ \gamma & \delta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}. \tag{4.7}$$

Note that in this case $\delta = \alpha^* \in GL_n(\mathbb{R}[t^{-1}])$, $\beta, \gamma \in M_{n\times n}(\mathbb{R}[t^{-1}])$. Moreover $\delta \in B$. We may write $g = g_l \cdots g_1$ such that each g_j is of the form (4.7). Hence, we have a chain of transformations

$$iI = \Omega_0 \xrightarrow{g_1} \Omega_1 \xrightarrow{g_2} \Omega_2 \xrightarrow{g_3} \cdots \xrightarrow{g_l} \Omega_l = \Omega.$$

For each j one has either $\Omega_j = \alpha_j \Omega_{j-1}\alpha_j^T + \beta_j \alpha_j^T$ or $\Omega_j = (\gamma_j \delta_j^T + \delta_j \Omega_{j-1}^{-1}\delta_j^T)^{-1}$. One may keep track of these finitely many steps and apply the same method but with some variant of the arguments used in Section 3. $\square$

5. Interpretations of theta functions

We start from general theta functions with parameters. Keep the notations in the last section. Let $\Omega = g \cdot iI \in \mathscr{H}$, where $g \in Sp_{2n}(\mathbb{R}[t, t^{-1}])$. Let c be a covolume theory for Ω. For $a \in V$, $b \in V^*$, $z \in V^* + V\Omega = V_{\mathbb{C}}^* = \mathbb{C}[t, t^{-1}]^n$, we define the theta function

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega, c) = \lim_L c(L) \sum_{v \in L} \exp(-\pi(v+a, v+a)_\Omega + 2\pi i(v+a, z+b)). \tag{5.1}$$

We have the quasi-periodicity: for $v \in \mathbb{Z}[t, t^{-1}]^n$,

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z+v, \Omega, c) = e^{2\pi i(a,v)} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega, c), \tag{5.2}$$

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z+v\Omega, \Omega, c) = e^{\pi(v,v)_\Omega - 2\pi i(v, z+b)} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega, c). \tag{5.3}$$

Let $L_\Omega = \mathbb{Z}[t, t^{-1}]^n + \mathbb{Z}[t, t^{-1}]^n \Omega \subset V_{\mathbb{C}}^* = \mathbb{C}[t, t^{-1}]^n$. Let l be a positive integer and fix a covolume theory c_l for $l\Omega$. Each $a \in \mathbb{Z}[t, t^{-1}]^n$ gives a theta function

$$f_l^a(z) := \vartheta \begin{bmatrix} a/l \\ 0 \end{bmatrix} (lz, l\Omega, c_l). \tag{5.4}$$

For a lattice L of $\mathbb{Z}[t, t^{-1}]^n$, introduce the partial theta function

$$f_{l,L}^a(z) = \vartheta_L \begin{bmatrix} a/l \\ 0 \end{bmatrix} (lz, l\Omega, c_l). \tag{5.5}$$

From quasi-periodicity (5.2), (5.3) it follows that for $v \in \mathbb{Z}[t, t^{-1}]^n$,

$$f_l^a(z+v) = f_l^a(z), \tag{5.6}$$

$$f_l^a(z+v\Omega) = e^{\pi l(v,v)_\Omega - 2\pi i l(v,z)} f_l^a(z). \tag{5.7}$$

Following [12], let S_l be the set of $a \in \mathbb{Z}[t, t^{-1}]^n$ such that all the coefficients in a lie in $\{0, 1, \ldots, l-1\}$. Let

$$\mathscr{B}_l = \{z \in V_{\mathbb{C}}^* | f_l^a(z) = 0 \text{ for all } a \in S_l\}/L_\Omega,$$

which is called the set of "base points" in the complex torus $V_{\mathbb{C}}^*/L_\Omega$. Let $\mathbb{P}^{S_l} = \mathbb{P}(\mathbb{C}^{S_l})$ be the infinite projective space, then there is a canonical holomorphic map

$$\phi_l : V_{\mathbb{C}}^*/L_\Omega - \mathscr{B}_l \to \mathbb{P}^{S_l}$$

given by $\phi_l(z) = [f_l^a(z)]_{a \in S_l}$. The well definedness of ϕ_l follows from quasi-periodicity (5.6), (5.7). It is known that $\mathscr{B}_l = \emptyset$ for $l \geq 2$ and ϕ_l is an embedding for $l \geq 3$.

We introduce a line bundle $\mathscr{L}$ over $V_{\mathbb{C}}^*/L_\Omega$. Define an action of L_Ω on the trivial bundle $V_{\mathbb{C}}^* \times \mathbb{C} \to V_{\mathbb{C}}^*$ by

$$(v_1 + v_2\Omega) \cdot (z, \lambda) = (z + v_1 + v_2\Omega, e^{\pi(v_2,v_2)\Omega - 2\pi i(v_2,z)}\lambda).$$

Let $\mathscr{L}$ be the quotient line bundle. If f is a function on $V_{\mathbb{C}}^*$ satisfying (5.6) (5.7), and f is holomorphic, i.e. the restriction of f to any finite-dimensional complex subspace of $V_{\mathbb{C}}^*$ is holomorphic, then we say f is a global section of $\mathscr{L}^l$. Let $\Gamma(\mathscr{L}^l)$ be the set of global sections of $\mathscr{L}^l$.

We claim that f_l^a, $a \in S_l$ are global sections of $\mathscr{L}^l$. It suffices to prove the holomorphic property. This follows from certain uniform convergence (in the variable z) results, and one may argue similarly as before. We omit the details. If we fix a covolume theory c_l' for $l^{-1}\Omega$ and for $b \in S_l$ define

$$g_l^b(z) = \vartheta \begin{bmatrix} 0 \\ b/l \end{bmatrix} (z, l^{-1}\Omega, c_l'),$$

then g_l^b, $b \in S_l$ are also global sections of $\mathscr{L}^l$. In the special case $l = k^2$, there is another family of global sections

$$h_l^{a,b}(z) = \vartheta \begin{bmatrix} a/k \\ b/k \end{bmatrix} (kz, \Omega, c), \quad a, b \in S_k.$$

Let us introduce the loop Heisenberg group $Heis(n,l)$, which is the set $U_l \times \mathbb{Z}_l[t,t^{-1}]^n \times \mathbb{Z}_l[t,t^{-1}]^n$ subject to the following relations:

$$(1, x, 0)(1, x', 0) = (1, x + x', 0), \tag{5.8}$$

$$(1, 0, y)(1, 0, y') = (1, 0, y + y'), \tag{5.9}$$

$$(\zeta, 0, 0)(\zeta', 0, 0) = (\zeta\zeta', 0, 0), \quad \text{and} \quad U_l \times \{0\} \times \{0\} \text{ is the centre}, \tag{5.10}$$

$$(1, 0, y)(1, x, 0) = \zeta_l((x, y))(1, x, 0)(1, 0, y), \tag{5.11}$$

where $(x, y) \in \mathbb{Z}_l$ is the constant term of $x(t)y(t)^T$, and $\zeta_l(a) := (\zeta_l, a)$, $a \in \mathbb{Z}_l$, is the canonical pairing.

$\Gamma(\mathscr{L}^l)$ can be realized as a representation of $Heis(n,l)$ as follows. Note that there is a natural bijection between S_l and $\mathbb{Z}_l[t,t^{-1}]^n$. For $x \in \mathbb{Z}_l[t,t^{-1}]^n$ write $\widetilde{x}$ for the corresponding element in S_l. Conversely for $a \in S_l$ write $\bar{a} = a \bmod l \in \mathbb{Z}_l[t,t^{-1}]^n$. We define an action of $Heis(n,l)$ on $f(z) \in \Gamma(\mathscr{L}^l)$ by

$$(1, x, 0)f(z) = f(z + \widetilde{x}/l), \tag{5.12}$$

$$(1, 0, y)f(z) = \exp(-\pi(\widetilde{y}, \widetilde{y})_\Omega/l + 2\pi i(\widetilde{y}, z))f(z + \widetilde{y}\Omega/l), \tag{5.13}$$

$$(\zeta, 0, 0)f(z) = \zeta f(z). \tag{5.14}$$

One can check that this defines a group action of $Heis(n,l)$ on $\Gamma(\mathscr{L}^l)$. Acting on above three families of sections, we have the formulas

$$\begin{aligned} &(1, x, 0)f_l^a = \zeta_l((\bar{a}, x))f_l^a, && (1, 0, y)f_l^a = f_l^{a+\widetilde{y}}, \\ &(1, x, 0)g_l^b = g_l^{b+\widetilde{x}}, && (1, 0, y)g_l^b = \zeta_l(-(y, \bar{b}))g_l^b, \\ &(1, x, 0)h_l^{a,b} = h_l^{a,b+k\widetilde{x}}, && (1, 0, y)h_l^{a,b} = \zeta_k(-(y, \bar{b}))h_l^{a+k\widetilde{y},b}. \end{aligned}$$

Consider the graded algebra $\bigoplus_{l=0}^{\infty}\Gamma(\mathscr{L}^l)$, where $\Gamma(\mathscr{L}^0):=\mathbb{C}$. We shall give the explicit formula for $f_{l_1}^{a_1}f_{l_2}^{a_2}$, where $a_i\in S_{l_i}$, $i=1,2$, under the multiplication $\Gamma(\mathscr{L}^{l_1})\times\Gamma(\mathscr{L}^{l_2})\to\Gamma(\mathscr{L}^{l_3})$, where $l_3=l_1+l_2$. Intuitively, we may expand $f_{l_1}^{a_1}f_{l_2}^{a_2}$ as an infinite linear combination of $f_{l_3}^{a}$, $a\in S_{l_3}$. However, it turns out to be not the case. The formalism of the result is by first averaging partial theta functions and then taking the limit.

We need to introduce some notations for our formula. Let (l_1,l_2) be the greatest common divisor of l_1 and l_2, and let $l_i'=l_i/(l_1,l_2)$, $i=1,2,3$. Choose $j_1,j_2\in\mathbb{Z}$ such that $j_1l_1+j_2l_2=(l_1,l_2)$. For any $\eta\in\mathbb{Z}[t,t^{-1}]^n$, define

$$a_\eta=a_1+a_2+(l_1,l_2)\eta, \tag{5.15}$$

$$\widetilde{a}_\eta=l_2'a_1-l_1'a_2+l_1l_2'(j_1-j_2)\eta. \tag{5.16}$$

Note that $(\widetilde{a}_\eta \bmod l_1'l_2'l_3)$ does not depend on the choice of (j_1,j_2). Introduce a subset of $\mathbb{Z}[t,t^{-1}]^n$,

$$S_{l_1,l_2}^{a_1,a_2}=\{\eta\in\mathbb{Z}[t,t^{-1}]^n | a_\eta\in S_{l_3}\}. \tag{5.17}$$

It is clear that $S_{l_1,l_2}^{a_1,a_2}$ is a set of representatives for $\mathbb{Z}_{l_3'}[t,t^{-1}]^n$, i.e.,

$$\mathbb{Z}[t,t^{-1}]^n=\bigsqcup_{\eta\in S_{l_1,l_2}^{a_1,a_2}}\eta+l_3'\mathbb{Z}[t,t^{-1}]^n. \tag{5.18}$$

Similarly, for $d\in\mathbb{Z}$ such that $a_1,a_2\in L_{\mathbb{Z}}^d$ one has

$$L_{\mathbb{Z}}^d=\bigsqcup_{\eta\in S_{l_1,l_2}^{a_1,a_2}\cap L_{\mathbb{Z}}^d}\eta+l_3'L_{\mathbb{Z}}^d. \tag{5.19}$$

We also need some notions from [2]. Let us denote by $\mathscr{G}(V)$ the set of lattices of V and call it the *semi-infinite Grassmannian* of V. A *dimension theory* on V is a map $\mathscr{D}:\mathscr{G}(V)\to\mathbb{Z}$ such that whenever $L,L'\in\mathscr{G}(V)$ and $L\subset L'$, we have $\mathscr{D}(L')=\mathscr{D}(L)+\dim(L'/L)$. All dimension theories on V form a $\mathbb{Z}$-torsor.

Recall that we have fixed a covolume theory c_j for $j\Omega$, $\forall j>0$. We claim that there exist a dimension theory $\mathscr{D}$ on V and a constant $\lambda\in\mathbb{C}^\times$ such that

$$c_{l_1}c_{l_2}=\lambda\cdot(l_3')^{-\mathscr{D}}\cdot c_{l_1'l_2'l}c_{l_3}. \tag{5.20}$$

Note that we view both sides of (5.20) as functions $\mathscr{G}(V)\to\mathbb{C}^\times$. The claim follows from the simple identity

$$l_1l_2=l_1'l_2'l_3^2(l_3')^{-2}.$$

The choice of $(\mathscr{D},\lambda)$ is not unique, and we only need to fix such a pair. Now we are ready to give the asymptotic multiplication formula.

Proposition 5.1. *With above notations, and a choice of the pair $(\mathscr{D},\lambda)$ satisfying (5.20), we have*

$$f_{l_1}^{a_1}(z)f_{l_2}^{a_2}(z)=\lambda\cdot\lim_{d\to\infty}(l_3')^{-\mathscr{D}(L_{\mathbb{R}}^d)}\sum_{\eta\in S_{l_1,l_2}^{a_1,a_2}\cap L_{\mathbb{Z}}^d}f_{l_1'l_2'l_3,L_{\mathbb{Z}}^d}^{\widetilde{a}_\eta}(0)\cdot f_{l_3,L_{\mathbb{Z}}^d}^{a_\eta}(z).$$

Corollary 5.1. *If $l_1=l_2=l$, then*

$$f_l^{a_1}(z)f_l^{a_2}(z)=\lambda\cdot\lim_{d\to\infty}2^{-\mathscr{D}(L_{\mathbb{R}}^d)}\sum_{\eta\in S_2}f_{2l,L_{\mathbb{Z}}^d}^{a_1-a_2+l\eta}(0)f_{2l,L_{\mathbb{Z}}^d}^{a_1+a_2+l\eta}(z).$$

Remark 5.1. Corollary 5.1 is an infinite analog of Corollary 6.8 in [5], which plays a major role in Riemann's theta relation (see [5] Section 7).

Proof of Proposition 5.1: Assume that $a_1, a_2 \in L^d_{\mathbb{Z}}$ such that (5.19) is satisfied. We shall compute $f^{a_1}_{l_1, L^d_{\mathbb{Z}}}(z) f^{a_2}_{l_2, L^d_{\mathbb{Z}}}(z)$ explicitly. Consider the following expression for v_1, v_2 run over $L^d_{\mathbb{Z}} \times L^d_{\mathbb{Z}}$:

$$\begin{aligned} l_1(v_1 + \frac{a_1}{l_1}, v_1 + \frac{a_1}{l_1})_\Omega - 2i(v_1 + \frac{a_1}{l_1}, l_1 z) \\ + l_2(v_2 + \frac{a_2}{l_2}, v_2 + \frac{a_2}{l_2})_\Omega - 2i(v_2 + \frac{a_2}{l_2}, l_2 z). \end{aligned} \tag{5.21}$$

Make the change of variables

$$v_1 = j_1(\eta + l'_3 v) + l'_2 u, \quad v_2 = j_2(\eta + l'_3 v) - l'_1 u,$$

such that η runs over $S^{a_1, a_2}_{l_1, l_2} \cap L^d_{\mathbb{Z}}$, and u, v run over $L^d_{\mathbb{Z}}$. Then from (5.19) it follows that

$$L^d_{\mathbb{Z}} \times L^d_{\mathbb{Z}} \to (S^{a_1, a_2}_{l_1, l_2} \cap L^d_{\mathbb{Z}}) \times L^d_{\mathbb{Z}} \times L^d_{\mathbb{Z}}, \quad (v_1, v_2) \mapsto (\eta, u, v)$$

is a bijection. In terms of the new variables, after some heavy manipulations one can check that (5.21) equals

$$\begin{aligned} l'_1 l'_2 l_3 \left(u + \frac{\widetilde{a}_\eta}{l'_1 l'_2 l_3}, u + \frac{\widetilde{a}_\eta}{l'_1 l'_2 l_3} \right)_\Omega \\ + l_3 \left(v + \frac{a_\eta}{l_3}, v + \frac{a_\eta}{l_3} \right)_\Omega - 2i \left(v + \frac{a_\eta}{l_3}, l_3 z \right). \end{aligned}$$

Taking into account of (5.20), this completes the proof. □

Remark 5.2. When $q \in \mathbb{Q}$, the endomorphism ring of $V^*_{\mathbb{C}}/L_\Omega$ is much larger than $\mathbb{Z}$, namely, it contains $\mathbb{Z}[Nt, Nt^{-1}]$ for some sufficiently large integer N. This is an interesting analogue of abelian varieties with large endomorphism rings.

Acknowledgment

The author is grateful to Professor Y. Zhu for very stimulating and helpful discussions during this work.

References

[1] H. Garland, *Absolute convergence of Eisenstein series on loop groups*, Duke Math. J. **135**(2) (2006), 203–260.
[2] M. Kapranov, *Semiinfinite symmetric powers*, `arXiv:math/0107089`.
[3] D. Liu, *Eisenstein series on loop groups*, `arXiv:1103.4212`.
[4] D. Liu and Y. Zhu, *On the theta functional of Weil representations for symplectic loop groups*, J. Algebra **324** (2010), 3115–3130.
[5] D. Mumford, *Tata lectures on theta. III*, With the collaboration of Madhav Nori and Peter Norman. Progress in Mathematics, **97**, Birkhauser Boston, Inc., Boston, MA, 1991.
[6] D.V. Osipov, *Adeles on n-dimensional schemes and categories C_n*, Int. J. Math. **18**(3) (2007), 269–279.
[7] D.V. Osipov and A.N. Parshin, *Harmonic analysis on local fields and adelic spaces. I*, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **72**(5) (2008), 77–140; translation in Izv. Math. **72**(5) (2008), 915–976.
[8] D.V. Osipov and A.N. Parshin, *Harmonic analysis on local fields and adelic spaces. II*, `arXiv:0912.1577`.

[9] A.N. Parshin, *On holomorphic representations of discrete Heisenberg groups*, (Russian) Funktsional. Anal. i Prilozhen. **44(2)** (2010), 92–96; translation in Funct. Anal. Appl. **44**(2) (2010), 156–159.

[10] A.N. Parshin, *Representations of higher adelic groups and arithmetic*, Proceedings of the International Congress of Mathematicians. Volume I, 362–392, Hindustan Book Agency, New Delhi, 2010.

[11] A. Weil, *Sur certaines groups d'operators unitaires*, Acta Math. **11** (1964), 143–211.

[12] Y. Zhu, *Theta functions and Weil representations of loop symplectic groups*, Duke Math. J. **143**(1) (2008), 17–39.

School of Mathematical Science, Xiamen University, Xiamen 361005, Fujian, China
E-mail address: `math.dwliu@gmail.com`