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INVARIANCE OF THE JACOBIAN NEWTON DIAGRAM

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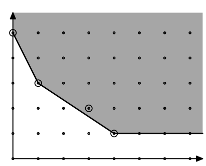
Abstract. We prove that the jacobian Newton diagram of the holomorphic mapping $(f,g) : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ depends only on the equisingularity class of the pair of curves $f=0$ and $g=0$.

1. Introduction

Write $\mathbb{R}_+ = \{\, x \in \mathbb{R} : x \geq 0 \,\}$. The Newton diagram Δ_h of a power series $h(x,y) = \sum_{i,j} c_{ij}x^iy^j$ is by definition the convex hull of the union

$$\bigcup_{\{(i,j):c_{ij}\neq 0\}} \{\, (i,j) + \mathbb{R}^2_+ \,\}.$$

Example 1.1. The Newton diagram of $h(x,y) = y^5 + 2xy^3 - x^3y^2 + 3x^4y$ is drawn in the figure. Black dots are the points of the first quadrant $\mathbb{R}^2_+$ corresponding to nonzero monomials of the series h.



Let $\phi : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$, $\phi^{-1}(0,0) = \{(0,0)\}$ be the germ of a holomorphic mapping given by $(x,y) = (f(u,v), g(u,v))$. Let $\operatorname{jac}\phi = \frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial u}$ be the usual jacobian determinant. The direct image of the curve germ $\operatorname{jac}\phi = 0$ by ϕ is called the *discriminant curve* of ϕ (see [2]). If $D(x,y) = 0$ is an analytic equation of the discriminant curve then the Newton diagram of D is called the *jacobian Newton diagram* of (f,g). We will write $\mathcal{N}_J(f,g)$ for the jacobian Newton diagram.

Let $h = h(u,v) \in \mathbb{C}\{u,v\}$, $h(0,0) = 0$ be a convergent power series and let $h = h_1^{m_1} \cdots h_s^{m_s}$ be a factorization of h in the ring $\mathbb{C}\{u,v\}$ with h_i irreducible and pairwise co-prime. Then every curve germ $h_i = 0$ is called a branch of the curve $h = 0$ and m_i is called the multiplicity of $h_i = 0$ in $h = 0$.

Definition 1.1. Let ξ, ξ', ν, ν' be germs of analytic curves in $(\mathbb{C}^2,0)$. We say that the pairs of curves ξ, ν and ξ', ν' are equisingular if there exists a homeomorphism $\Psi : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ preserving for each curve the multiplicity in it of each of its branches such that $\Psi(\xi) = \xi'$ and $\Psi(\nu) = \nu'$.

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2. Main result

Theorem 2.1. *Let* $(f,g):(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$, $(f,g)^{-1}(0,0)=\{(0,0)\}$ *be the germ of a holomorphic mapping. Then the jacobian Newton diagram* $\mathcal{N}_J(f,g)$ *depends only on the equisingularity class of the pair of curves* $f=0$ *and* $g=0$.

The proof is in the last section.

We give a survey of results related with Theorem 2.1. We need a few notions which will be used only in this section to explain connection between certain analytic factorizations of $\operatorname{jac}(f,g)$ and the jacobian Newton diagram $\mathcal{N}_J(f,g)$.

The Minkowski sum of Newton diagrams Δ_1 and Δ_2 is by definition $\Delta_1+\Delta_2=\{p+q : p\in\Delta_1, q\in\Delta_2\}$. The set of Newton diagrams is a semi-group with respect to Minkowski sum and the generators of this semi-group are elementary Newton diagrams illustrated in Figure 1.

The inclination of the elementary Newton diagram $\{\frac{a}{b}\}$ is the quotient $\frac{a}{b}$ (by convention $\frac{\infty}{b}=\infty$ and $\frac{a}{\infty}=0$). For an arbitrary Newton diagram Δ represented as a sum of elementary Newton diagrams let us denote $I(\Delta)$ the set of inclinations of elementary Newton diagrams of the sum. It is easy to check that $I(\Delta)$ does not depend on the choice of a representation. Coming back to Example 1.1 the Newton diagram Δ_h is the sum $\{\frac{1}{2}\}+\{\frac{3}{2}\}+\{\frac{\infty}{1}\}$ and $I(\Delta_h)=\{1/2,3/2,\infty\}$.

If h is any irreducible factor of $\operatorname{jac}(f,g)$ then $q(h)=\frac{i_0(g,h)}{i_0(f,h)}$, where $i_0(\cdot,\cdot)$ stands for the intersection multiplicity, is called the jacobian quotient of (f,g).

Definition 2.1. Let $\operatorname{jac}(f,g)=J_1\cdots J_n$ be an analytic factorization of the jacobian.

We will call $J_1\cdots J_n$ a Hironaka factorization if for every J_i $(1\le i\le n)$ the jacobian quotient $q(h)$ is constant for all irreducible factors h of J_i.

The Hironaka factorization $J_1\cdots J_n$ will be called minimal if jacobian quotients of irreducible factors of J_l and J_k are different for $1\le l<k\le n$.

Let $\operatorname{jac}(f,g)=h_1\cdots h_n$ be the factorization of the jacobian into irreducible factors. It is easy to check (see [13]) that

$$\mathcal{N}_J(f,g)=\sum_{i=1}^{n}\left\{\frac{i_0(g,h_i)}{i_0(f,h_i)}\right\}.$$

It follows directly from the above formula that

- the set of jacobian quotients of (f,g) is the set of inclinations of $\mathcal{N}_J(f,g)$,
- if $J_1\cdots J_r$ is a Hironaka factorization of $\operatorname{jac}(f,g)$ then

$$\mathcal{N}_J(f,g)=\sum_{i=1}^{r}\left\{\frac{i_0(g,J_i)}{i_0(f,J_i)}\right\},$$

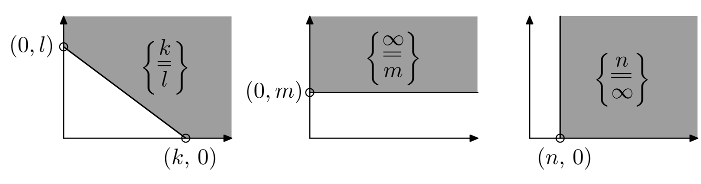

FIGURE 1. Elementary Newton diagrams

- if $\mathcal{N}_J(f,g) = \sum_{i=1}^{s}\{\frac{a_i}{b_i}\}$ with inclinations $\frac{a_i}{b_i}$ pairwise different then $\mathrm{jac}(f,g)$ has the minimal Hironaka factorization $J_1 \cdots J_s$ such that $i_0(g, J_i) = a_i$ and $i_0(f, J_i) = b_i$ for $i = 1, \dots, s$.

Let us consider a germ of a holomorphic mapping $(l, f) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $f = 0$ is a curve germ without multiple branches and $l = 0$ is a generic smooth curve (here generic means that the curve $l = 0$ is not tangent to any branch of the curve $f = 0$). Under these assumptions $\mathrm{jac}(l, f) = 0$ is called the polar curve of f and jacobian quotients of (l, f) are called polar quotients. A survey of recent results concerning polar curves is in [6].

Merle in [10] obtained the minimal Hironaka decomposition of the polar curve of the irreducible curve germ $f = 0$. Merle's result is rewritten in [13] as a formula for the jacobian Newton diagram of (l, f) (see also [6], Theorem 4.1).

In [7], Kuo and Lu described the contact orders of Newton–Puiseux roots of the partial derivative $f'_x(x, y) = 0$ with the Newton–Puiseux roots of $f(x, y) = 0$. They constructed the tree model $T(f)$ which encodes these contact orders. Using the Kuo–Lu tree $T(f)$ one can compute all polar quotients of (y, f). One can also give a formula for the jacobian Newton diagram of (y, f) in terms of $T(f)$ (see the last line before Example 5.2 in [5]).

In [4], the author studied the polar curve of a many-branched curve $f = 0$. He introduced a new type of tree $E(f)$ called now the Eggers tree of f. Eggers found the Hironaka factorization of the jacobian $\mathrm{jac}(l, f)$ such that the factors are indexed by vertices of $E(f)$. He also computed the intersection multiplicities of every factor with l and f. Since the Eggers tree $E(f)$ depends only on the equisingularity class of f, Theorem 2.1 in this particular case follows from [4].

The papers [7] and [4] provide methods of computing $\mathcal{N}_J(l, f)$ using invariants of equisingularity of f. Another way to obtain Theorem 2.1 in the polar case is to use a deformation argument. Teissier proved in [14] that for every μ^*-constant family of hypersurfaces with isolated singularities, the jacobian Newton diagram is constant. Since every two plane analytic curves of the same equisingularity type can be joined by a μ^*-constant family of plane curves (see [2], Proposition 5.2 for a direct construction of such a family) we get another proof of Theorem 2.1 in the polar case.

Consider now a general case of a holomorphic mapping germ $(f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, where $(f, g)^{-1}(0, 0) = \{(0, 0)\}$.

In [8], the authors additionally assumed that the curve $fg = 0$ has no multiple components. They defined the equivalence relation between vertices of the Kuo–Lu tree $T(fg)$. Then they obtained the Hironaka factorization of $\mathrm{jac}(f, g)$ such that the factors are indexed by equivalence classes of this relation. However, as Section 5 of [8] shows, the equisingularity class of the pair $f = 0$, $g = 0$ does not determine the intersection multiplicities of some factors with f and g. Hence, Theorem 4.1 does not follow from [8].

In [9] and [11], the authors resolved singularities of the curve $fg = 0$. Then they distinguished some subsets of the exceptional divisor called rupture zones and associated a factor of the jacobian $\mathrm{jac}(f, g)$ with every rupture zone. Maugendre [9] found the set of jacobian quotients using topological methods (see also [3] for an algebraic proof) and Michel [11] completed the work computing the intersection multiplicities of every factor with f and g. Since the decomposition of the jacobian obtained by Michel is a

Hironaka factorization, Theorem 2.1 follows from [11]. However, the proof presented in this article is much simpler as it uses only Theorem 3.1 and the following formula.

Theorem 2.2 ([2] Theorem 3.2). *Let $(f,g) : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ be a germ of a holomorphic mapping such that $(f,g)^{-1}(0,0) = \{(0,0)\}$. Let $D(x,y) = 0$ be the discriminant curve of (f,g). Take any curve germ $h(x,y) = 0$ and let $H(u,v) = h(f(u,v),g(u,v))$. Then*

$$\mu_0(H) - 1 = i_0(f,g)[\mu_0(h) - 1] + i_0(h,D),$$

where $\mu_0(h)$ denotes the Milnor number of the curve $h = 0$ at zero.

3. Invariance of a generic curve of the pencil

Theorem 3.1. *Let $(f,g) : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$, $(f,g)^{-1}(0,0) = \{(0,0)\}$ be the germ of a holomorphic mapping. Then for all $t \in \mathbb{C}$ but a finite number the equisingularity class of the curve $f(x,y) - tg(x,y) = 0$ depends only on the equisingularity class of the pair of curves $f = 0$ and $g = 0$.*

Proof. Our main reference is Chapter III of [12]. Let $R : M \to (\mathbb{C}^2,0)$ be the minimal good resolution of singularities of the curve $fg = 0$. The set $R^{-1}(\{fg = 0\})$ can be written as the union of irreducible components $E_1 \cup \cdots \cup E_n \cup E_{n+1} \cup \cdots \cup E_m$, where $E = E_1 \cup \cdots \cup E_n$ is the exceptional divisor $R^{-1}(0)$ and $E_{n+1}, \ldots, E_m$ are noncompact curves corresponding with branches of the curve $fg = 0$. Put $\tilde{f} = f \circ R$, $\tilde{g} = g \circ R$ and let a_i = order of $\tilde{f}$ along E_i, b_i = order of $\tilde{g}$ along E_i for $i = 1, \ldots, m$. Then, after renumbering $E_1, \ldots, E_m$ if necessary, the total dual resolution graph as well as the numbers a_i and b_i for $i = 1, \ldots, m$ depend only on the equisingularity class of the pair of curves $f = 0$ and $g = 0$.

Consider the meromorphic function $\tilde{f}/\tilde{g} : M \setminus E \to \mathbb{C} \cup \{\infty\}$. We will check that this function extends analytically to the whole M with the exception of a finite number of points. Let $-\!o\!-$ denotes any germ of a holomorphic function $u(x,y)$ such that $u(0,0) \neq 0$.

First take $P \in E_i$ $(1 \leq i \leq n)$ which is not an intersection point with another component E_j for $1 \leq j \leq m$. Choose a local analytical coordinate system (x,y) centered at P such that E_i has the equation $x = 0$. In these coordinates $\tilde{f} = -\!o\!- x^{a_i}$ and $\tilde{g} = -\!o\!- x^{b_i}$. We get $\tilde{f}/\tilde{g} = -\!o\!- x^{a_i - b_i}$.

Now take the intersection point P of E_i with another component E_j. Choose a local analytical coordinate system (x,y) centered at P such that E_i has the equation $x = 0$ and E_j has the equation $y = 0$. In these coordinates $\tilde{f} = -\!o\!- x^{a_i} y^{a_j}$ and $\tilde{g} = -\!o\!- x^{b_i} y^{b_j}$. We get $\tilde{f}/\tilde{g} = -\!o\!- x^{a_i - b_i} y^{a_j - b_j}$.

Let H be an analytic extension of $\tilde{f}/\tilde{g}$. Divide the set $\{E_1, \ldots, E_m\}$ into three subsets $A_+ = \{E_i : a_i - b_i > 0\}$, $A_0 = \{E_i : a_i - b_i = 0\}$ and $A_- = \{E_i : a_i - b_i < 0\}$. It follows from the above description of $\tilde{f}/\tilde{g}$ near E that H is not defined only at the intersection points of components from A_+ with components from A_-.

Let $E_i \in A_0$. Consider the restriction $H|_{E_i}$ of the meromorphic function H to E_i. Then $P \in E_i$ is a zero of $H|_{E_i}$ if and only if $\{P\} = E_i \cap E_j$ for some $E_j \in A_+$. Moreover, $\mathrm{ord}_P H|_{E_i} = a_j - b_j$. Hence the topological degree of $H|_{E_i}$ is the number $d_i = \sum (a_j - b_j)$ where the sum runs over all j such that $E_j \in A_+$ and the intersection $E_i \cap E_j$ is nonempty.

Choose a nonzero complex number t which is different from

- any critical value of meromorphic functions $H|_{E_i}$ where $E_i \in A_0$,
- any value $H(P)$ where P is the intersection point of some $E_i \in A_0$ with some E_j, $j \neq i$.

Let Γ be the proper preimage of the curve $f - tg = 0$. The curve Γ has an equation $H = t$ at every point where H is well defined. Hence Γ intersects transversally every $E_i \in A_0$ at d_i points and none of these points belong to $\bigcup_{j \neq i} E_j$.

Now we compute the equation of Γ near points where H is not defined. Take $E_i \in A_+$, $E_j \in A_-$ with nonempty intersection and denote $P_{i,j}$ their intersection point. There exists a local analytical coordinate system (x, y) centered at $P_{i,j}$ such that E_i has the equation $x = 0$ and E_j has the equation $y = 0$. In these coordinates $\tilde{f} - t\tilde{g} = \ominus\, x^{a_i} y^{a_j} - t\ominus\, x^{b_i} y^{b_j} = \ominus\, x^{b_i} y^{a_j} (\ominus\, x^{a_i - b_i} - t y^{b_j - a_j})$, hence Γ has the equation $\ominus\, x^{a_i - b_i} - t y^{b_j - a_j} = 0$.

We want to resolve singularities of the curve Γ to obtain a good (not necessarily minimal) resolution of singularities of $f - tg = 0$. Every function $h_{i,j}(x, y) = \ominus\, x^{a_i - b_i} - t y^{b_j - a_j}$ is nondegenerate with the Newton diagram $\left\{ \dfrac{a_i - b_i}{b_j - a_j} \right\}$. Hence by Theorem 4.3 of [12] there exists a canonical toric resolution of $h_{i,j}(x, y) = 0$ at the origin, that is the resolution of Γ at $P_{i,j}$, which depends only on the Newton diagram of $h_{i,j}$. Applying such a toric resolution at every point $P_{i,j}$ described above we obtain a good resolution of $f - tg = 0$. Moreover, the total dual resolution graph of this resolution depends only on the total dual resolution graph of R and on the numbers a_i and b_i for $i = 1, \dots, m$.

Since the total dual resolution graph of the plane curve singularity determines its equisingularity class (see [1], Chapter 8.4, Proposition 20) the proof is finished. $\square$

4. Proof of the main result

For every Newton diagram Δ and for every $\vec{v} = (v_1, v_2)$, where $v_1 > 0$, $v_2 > 0$ we define

$$l(\vec{v}, \Delta) = \min\{\, v_1 i + v_2 j : (i, j) \in \Delta\}.$$

Lemma 4.1. *Let $\vec{v} = (m, n)$, where n, m are co-prime positive integers. Then for generic $t \in \mathbb{C}$*

$$l\big(\vec{v}, \mathcal{N}_J(f, g)\big) = \mu_0(f^n - t g^m) - i_0(f, g)[(m-1)(n-1) - 1] - 1.$$

Proof. Let $D = 0$, where $D(x, y) = \sum c_{ij} x^i y^j$, be the equation of the discriminant curve of $\phi = (f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. Take a curve $x^n - t y^m = 0$.

Claim. For generic $t \in \mathbb{C}$ we have $i_0(x^n - t y^m, D) = l\big(\vec{v}, \mathcal{N}_J(f, g)\big)$.

Let $\tau = \sqrt[n]{t}$. Then $x = \tau s^m$, $y = s^n$ is a parameterization of the branch $x^n - t y^m = 0$. By the classical formula for the intersection multiplicity

$$i_0(x^n - t y^m, D) = \mathrm{ord}_s D(\tau s^m, s^n) = \mathrm{ord}_s \sum c_{ij} \tau^i s^{mi + nj} = l\big(\vec{v}, \mathcal{N}_J(f, g)\big)$$

provided τ is sufficiently general so that the sum $\sum_{mi + nj = l(\vec{v}, \mathcal{N}_J(f,g))} c_{ij} \tau^i$ is nonzero. The Claim is proved.

The pull-back of the curve $x^n - ty^m = 0$ by ϕ has an equation $f^n - tg^m = 0$. Thus by Theorem 2.2 we have

$$\mu_0(f^n - tg^m) - 1 = i_0(f, g)[\mu_0(x^n - ty^m) - 1] + i_0(x^n - ty^m, D),$$

which gives the lemma because $\mu_0(x^n - ty^m) = (m - 1)(n - 1)$. □

Proof of Theorem 2.1. It follows from Lemma 4.1 and Theorem 3.1 that for every vector $\vec{v} = (m, n)$, where m, n are co-prime positive integers, the number $l\big(\vec{v}, \mathcal{N}_J(f, g)\big)$ depends only on the equisingularity class of the pair $f = 0$ and $g = 0$. Since every Newton diagram Δ is equal to the intersection of half-planes determined by $l(\vec{v}, \Delta)$ the theorem is proved. □

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