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ON YAU RIGIDITY THEOREM FOR MINIMAL SUBMANIFOLDS IN SPHERES

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Abstract. In this paper, we investigate the well-known Yau rigidity theorem for minimal submanifolds in spheres. Using the parameter method of Yau and the DDVV inequality verified by Lu and Ge–Tang, we prove that if M is an n-dimensional oriented compact minimal submanifold in the unit sphere S^{n+p}, and if $K_M \geq \frac{p \cdot sgn(p-1)}{2(p+1)}$, then M is either a totally geodesic sphere, one of the Clifford minimal hypersurfaces in S^{n+1}, or the Veronese surface in S^4. Here $\mathrm{sgn}(\cdot)$ is the standard sign function. We also extend the rigidity theorem above to the case where M is a compact submanifold with parallel mean curvature in a space form.

1. Introduction

Rigidity of minimal submanifolds plays an important role in submanifold geometry. After the pioneering rigidity theorem proved by Simons [24], a series of striking rigidity results for minimal submanifolds were proved by several geometers [2, 14, 29]. Let M^n be an n-dimensional compact Riemannian manifold isometrically immersed into an $(n+p)$-dimensional complete and simply connected Riemannian manifold $F^{n+p}(c)$ with constant curvature c. Denote by K_M and H the sectional curvature and mean curvature of M, respectively. In 1975, Yau [29] proved the following celebrated rigidity theorem for minimal submanifolds in spheres under sectional curvature pinching condition.

Theorem A. *Let M be an n-dimensional oriented compact minimal submanifold in the unit sphere S^{n+p}. If $K_M \geq \frac{p-1}{2p-1}$, then M is either a totally geodesic sphere, one of the Clifford minimal hypersurfaces $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ in S^{n+1} for $k = 1, \ldots, n-1$, or the Veronese surface in S^4.*

The pinching constant above is the best possible in the case where $p = 1$, or $n = 2$ and $p = 2$. It improves the pinching constant of Simons [24] even though the latter is in the sense of the average of sectional curvatures. Later, Itoh [12, 13] proved that if M^n is an oriented compact minimal submanifold in S^{n+p} whose sectional curvature satisfies $K_M \geq \frac{n}{2(n+1)}$, then M is the totally geodesic sphere or the Veronese submanifold. Further discussions in this direction have been carried out by many authors [5, 15, 22, 26–28], etc. Even though, the following important problem remains unsolved.

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Open Problem B. *What is the best pinching constant for the rigidity theorem for oriented compact minimal submanifolds in a unit sphere under sectional (Ricci, scalar, resp.) curvature pinching condition?*

In particular, Lu's conjecture (see Conjecture 4 in [20]), a scalar curvature pinching problem for minimal submanifolds in a unit sphere, has not been verified yet. In this paper, using Yau's parameter method [29] and the DDVV inequality proved by Lu [20] Ge–Tang [7], we prove the following rigidity theorem for minimal submanifolds in spheres.

Theorem 1. *Let M be an n-dimensional oriented compact minimal submanifold in the unit sphere S^{n+p}. If*

$$K_M \geq \frac{p \cdot sgn(p-1)}{2(p+1)},$$

then M is either a totally geodesic sphere, one of the Clifford minimal hypersurfaces $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ in S^{n+1} for $k = 1, \ldots, n-1$, or the Veronese surface in S^4. Here $sgn(\cdot)$ is the standard sign function.

Remark 1. When $2 < p < n$, our pinching constant in Theorem 1 improves the ones given by Yau [29] and Itoh [13].

Generalizing Theorem 1, we obtain the following rigidity result for submanifolds with parallel mean curvature in space forms.

Theorem 2. *Let M be an n-dimensional oriented compact submanifold with parallel mean curvature ($H \neq 0$) in $F^{n+p}(c)$. If $c + H^2 > 0$ and*

$$K_M \geq \frac{(p-1) \cdot sgn(p-2)}{2p}(c + H^2),$$

then M is congruent to one of the following:

(i) $S^n(\frac{1}{\sqrt{c+H^2}})$;

(ii) *one of the Clifford hypersurfaces* $S^k(\frac{1}{\sqrt{c+\lambda^2(k,n,H,c)}}) \times S^{n-k}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$ *in* $F^{n+1}(c)$ *with* $c > 0$, *where* $\lambda(k,n,H,c) = \frac{1}{2k}[nH + \sqrt{n^2H^2 + 4k(n-k)c}]$, $k = 1, \ldots, n-1$;

(iii) *one of the Clifford minimal hypersurfaces* $S^k(\sqrt{\frac{k}{n(c+H^2)}}) \times S^{n-k}(\sqrt{\frac{n-k}{n(c+H^2)}})$ *in* $F^{n+1}(c + H^2)$, $k = 1, \ldots, n-1$;

(iv) *the Clifford torus* $S^1(r_1) \times S^1(r_2)$ *in* $F^3(c+H^2-H_0^2)$ *with constant mean curvature* H_0, *where* $r_1, r_2 = [2(c + H^2) \pm 2H_0(c + H^2)^{1/2}]^{-1/2}$, $0 \leq H_0 \leq H$, *and* $c + H^2 - H_0^2 > 0$;

(v) *the Veronese surface in* $F^4(c + H^2)$;

(vi) *the product of three spheres* $S^{k_1}(\sqrt{\frac{k_1}{k(c+\lambda^2(k,n,H,c))}}) \times S^{k-k_1}(\sqrt{\frac{k-k_1}{k(c+\lambda^2(k,n,H,c))}}) \times S^{n-k}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$ *in* $F^{n+2}(c)$ *with* $c > 0$, *where* $\lambda(k,n,H,c) = \frac{1}{2k}[nH + \sqrt{n^2H^2 + 4k(n-k)c}]$, $1 \leq k_1 < k \leq n-1$.

2. Notations and lemmas

Throughout this paper, let M^n be an n-dimensional compact Riemannian manifold isometrically immersed into an $(n+p)$-dimensional complete and simply connected space form $F^{n+p}(c)$. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+p; \ 1 \leq i, j, k, \ldots \leq n; \ n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

We let $\{e_A\}$ be local orthonormal frames in $F^{n+p}(c)$ such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection 1-forms of $F^{n+p}(c)$ respectively. Restricting these forms to M, we have

$$\begin{aligned} \omega_{\alpha i} &= \sum_j h^\alpha_{ij}\omega_j, h^\alpha_{ij} = h^\alpha_{ji}, \\ h &= \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \xi = \frac{1}{n}\sum_{\alpha,i} h^\alpha_{ii} e_\alpha, \\ R_{ijkl} &= c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \\ R_{\alpha\beta kl} &= \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}), \end{aligned} \tag{2.1}$$

where h, ξ, R_{ijkl}, and $R_{\alpha\beta kl}$ are the second fundamental form, the mean curvature vector, the curvature tensor and the normal curvature tensor of M, respectively. We define

$$S = |h|^2, \ H = |\xi|, \ H_\alpha = (h^\alpha_{ij})_{n\times n}.$$

The scalar curvature R of M is given by

$$R = n(n-1)c + n^2H^2 - S. \tag{2.2}$$

Denote $K_M(p,\pi)$ the sectional curvature of M for 2-plane $\pi \subset T_pM$ at point $p \in M$. Set $K_{\min}(p) = \min_{\pi\subset T_pM} K_M(p,\pi)$. From [29], we have the following lemma.

Lemma 1. *If M^n is a submanifold with parallel mean curvature and positive sectional curvature in $F^{n+p}(c)$, then M is pseudo-umbilical.*

Let M be a submanifold with parallel mean curvature vector ξ. Choose e_{n+1} such that it is parallel to ξ. Then we have

$$\mathrm{tr}H_{n+1} = nH, \quad \mathrm{tr}H_\alpha = 0, \quad \alpha \neq n+1. \tag{2.3}$$

Set

$$S_H = \mathrm{tr}H^2_{n+1}, \quad S_I = \sum_{\alpha\neq n+1} \mathrm{tr}H^2_\alpha. \tag{2.4}$$

If M is pseudo-umbilical and $H \neq 0$,

$$S_H = \mathrm{tr}H^2_{n+1} = nH^2. \tag{2.5}$$

Denoting the first and second covariant derivatives of h_{ij}^α by h_{ijk}^α and h_{ijkl}^α, respectively. Then by definition

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha - \sum_k h_{ik}^\alpha \omega_{kj} - \sum_k h_{kj}^\alpha \omega_{ki} - \sum_\beta h_{ij}^\beta \omega_{\beta\alpha},$$
$$\sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha - \sum_l h_{ijl}^\alpha \omega_{lk} - \sum_l h_{ilk}^\alpha \omega_{lj} - \sum_l h_{ljk}^\alpha \omega_{li} - \sum_\beta h_{ijk}^\beta \omega_{\beta\alpha}.$$

In particular, we have

$$h_{ijk}^\alpha = h_{ikj}^\alpha, \quad h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{mj}^\alpha R_{mikl} - \sum_\beta h_{ij}^\beta R_{\alpha\beta kl},$$

$$\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha$$
$$= \sum_k h_{kkij}^\alpha + \sum_k \left(\sum_m h_{km}^\alpha R_{mijk} + \sum_m h_{mi}^\alpha R_{mkjk} - \sum_\beta h_{ki}^\beta R_{\alpha\beta jk} \right). \tag{2.6}$$

The following lemma will be used in the proof of our main results.

Lemma 2 ([29])**.** *If M^n is a submanifold with parallel mean curvature in $F^{n+p}(c)$, then either $H \equiv 0$, or H is non-zero constant and $H_{n+1}H_\alpha = H_\alpha H_{n+1}$ for all α.*

For an $(n \times n)$-matrix $A = (a_{ij})$, we denote by $N(A)$ the square of the norm of A, i.e.,

$$N(A) = \operatorname{tr}(AA^T) = \sum_{i,j=1}^n a_{ij}^2.$$

Then the DDVV inequality proved by Lu [20] and Ge–Tang [7] is stated as follows.

DDVV inequality. *Let $B_1, \ldots, B_m$ be symmetric $(n \times n)$-matrices. Then*

$$\sum_{r,s=1}^m N(B_rB_s - B_sB_r) \le \left[\sum_{r=1}^m N(B_r) \right]^2, \tag{2.7}$$

where the equality holds if and only if under some rotation[1] *all B_r's are zero except two matrices, which can be written as*

$$\tilde{B}_1 = P \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t, \quad \tilde{B}_2 = P \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t.$$

Here P is an orthogonal $(n \times n)$-matrix.

For further discussions about the DDVV inequality, we refer to see [3,4,7,8,17–20].

[1] An orthogonal $m \times m$ matrix $R = (R_{rs})$ acts as a rotation on $(B_1, \ldots, B_m)$ by $(\tilde{B}_1, \ldots, \tilde{B}_m) = (B_1, \ldots, B_m)R$.

3. Proof of the theorems

When M^n is a minimal submanifold in S^{n+p}, we have $\mathrm{tr}H_\alpha = 0$ and $\sum_i h^\alpha_{iikl} = 0$ for all α. Consequently, from (2.6), we have

$$\Delta h^\alpha_{ij} = \sum_{k,m} h^\alpha_{km} R_{mijk} + \sum_{k,m} h^\alpha_{mi} R_{mkjk} - \sum_{k,\beta} h^\beta_{ki} R_{\alpha\beta jk}. \tag{3.1}$$

Thus

$$\sum_{i,j,\alpha} h^\alpha_{ij} \Delta h^\alpha_{ij} = \sum_{i,j,k,m,\alpha} (h^\alpha_{ij} h^\alpha_{km} R_{mijk} + h^\alpha_{ij} h^\alpha_{mi} R_{mkjk}) - \sum_{i,j,k,\alpha,\beta} h^\alpha_{ij} h^\beta_{ki} R_{\alpha\beta jk}. \tag{3.2}$$

Proof of Theorem 1. By using (2.1), we obtain

$$\begin{aligned}&\sum_{i,j,k,m,\alpha} h^\alpha_{ij} h^\alpha_{km} R_{mijk} + \sum_{i,j,k,m,\alpha} h^\alpha_{ij} h^\alpha_{mi} R_{mkjk} \\ &\quad = nS + \sum_{\alpha,\beta} \mathrm{tr}H_\beta \cdot \mathrm{tr}(H^2_\alpha H_\beta) - \sum_{\alpha,\beta} [\mathrm{tr}(H_\alpha H_\beta)]^2 - \sum_{\alpha,\beta} [\mathrm{tr}(H^2_\alpha H^2_\beta) - \mathrm{tr}(H_\alpha H_\beta)^2],\end{aligned}$$

and

$$\sum_{i,j,k,\alpha,\beta} h^\alpha_{ij} h^\beta_{ki} R_{\alpha\beta jk} = \sum_{\alpha,\beta} [\mathrm{tr}(H^2_\alpha H^2_\beta) - \mathrm{tr}(H_\alpha H_\beta)^2].$$

Since $(\mathrm{tr}(H_\alpha H_\beta))$ is a symmetric $(p \times p)$-matrix, we can choose the normal frame fields $\{e_\alpha\}$ such that

$$\mathrm{tr}(H_\alpha H_\beta) = \mathrm{tr}H^2_\alpha \cdot \delta_{\alpha\beta}.$$

Thus, we have

$$\sum_{\alpha,\beta} [\mathrm{tr}(H_\alpha H_\beta)]^2 = \sum_\alpha (\mathrm{tr}H^2_\alpha)^2. \tag{3.3}$$

From above equalities, we obtain

$$\begin{aligned}\sum_{i,j,\alpha} h^\alpha_{ij} \Delta h^\alpha_{ij} &= -anS + (1+a) \sum_{i,j,k,m,\alpha} (h^\alpha_{ij} h^\alpha_{km} R_{mijk} + h^\alpha_{ij} h^\alpha_{mi} R_{mkjk}) \\ &\quad + (a-1) \sum_{\alpha,\beta} [\mathrm{tr}(H^2_\alpha H^2_\beta) - \mathrm{tr}(H_\alpha H_\beta)^2] + a \sum_{\alpha,\beta} (\mathrm{tr}H^2_\alpha)^2\end{aligned} \tag{3.4}$$

for any real number a. For a fixed α, we choose the orthonormal frame fields $\{e_i\}$ such that $h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}$. Then, using the trick used by Yau [29], we obtain

$$\begin{aligned}\sum_{i,j,k,m} h^\alpha_{ij} h^\alpha_{km} R_{mijk} + \sum_{i,j,k,m} h^\alpha_{ij} h^\alpha_{mi} R_{mkjk} &= \sum_{i,k} \lambda^\alpha_i \lambda^\alpha_k R_{kiik} + \sum_{i,k} \lambda^\alpha_i \lambda^\alpha_i R_{ikik} \\ &= \frac{1}{2} \sum_{i,j} (\lambda^\alpha_i - \lambda^\alpha_j)^2 R_{ijij} \\ &\geq \frac{1}{2} K_{\min} \sum_{i,j} (\lambda^\alpha_i - \lambda^\alpha_j)^2 \\ &= nK_{\min}(\mathrm{tr}H^2_\alpha),\end{aligned} \tag{3.5}$$

which implies that

$$\sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mkjk} \geq n K_{\min} S. \tag{3.6}$$

On the other hand, by a direct computation and the DDVV inequality, we obtain

$$\begin{aligned} \sum_{\alpha,\beta} \mathrm{tr}(H_\alpha^2 H_\beta^2) - \mathrm{tr}(H_\alpha H_\beta)^2 &= \frac{1}{2} \sum_{\alpha,\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) \\ &\leq \frac{1}{2} \mathrm{sgn}(p-1) \Big(\sum_\alpha \mathrm{tr} H_\alpha^2 \Big)^2 \\ &= \frac{1}{2} \mathrm{sgn}(p-1) S^2, \end{aligned} \tag{3.7}$$

where $\mathrm{sgn}(\cdot)$ is the standard sign function. It follows from (3.4), (3.6) and (3.7) that

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2 + \sum_{i,j,\alpha} h_{ij}^{\alpha} \Delta h_{ij}^{\alpha} \\ &\geq \sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2 - anS + (1+a) n K_{\min} S + \left[\frac{a}{p} + \frac{\mathrm{sgn}(p-1)}{2}(a-1) \right] S^2 \end{aligned} \tag{3.8}$$

for $0 \leq a < 1$. Taking $a = \mathrm{sgn}(p-1)\frac{p}{p+2}$, we obtain

$$\frac{1}{2}\Delta S \geq nS \left[\left(1 + \mathrm{sgn}(p-1) \frac{p}{p+2} \right) K_{\min} - \mathrm{sgn}(p-1) \frac{p}{p+2} \right].$$

By the assumption and the maximum principle, S is a constant, and

$$S \left[\left(1 + \mathrm{sgn}(p-1) \frac{p}{p+2} \right) K_{\min} - \mathrm{sgn}(p-1) \frac{p}{p+2} \right] = 0.$$

If there is a point $q \in M$ such that $K_{\min}(q) > \frac{p \cdot \mathrm{sgn}(p-1)}{2(p+1)}$, then $S = 0$, i.e., M is totally geodesic. If $K_{\min} \equiv \frac{p \cdot \mathrm{sgn}(p-1)}{2(p+1)}$, then inequalities in (3.6), (3.7) and (3.8) become equalities. From the DDVV inequality, we obtain $p \leq 2$. This together with Theorem A implies that M is either one of the Clifford minimal hypersurfaces $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ in S^{n+1} for $k = 1, \ldots, n-1$, or the Veronese surface in S^4. This completes the proof of Theorem 1. □

Similar to (3.1) and (3.2), when M^n is a submanifold with parallel mean curvature in $F^{n+p}(c)$, we have $\xi = He_{n+1}$, and $\sum_i h_{iikl}^{\alpha} = 0$ for $\alpha \neq n+1$. It follows from (2.6) and Lemma 2 that

$$\Delta h_{ij}^{\alpha} = \sum_{k,m} h_{km}^{\alpha} R_{mijk} + \sum_{k,m} h_{mi}^{\alpha} R_{mkjk} - \sum_{k,\beta \neq n+1} h_{ki}^{\beta} R_{\alpha\beta jk}, \quad \alpha \neq n+1. \tag{3.9}$$

Thus

$$\begin{aligned} \sum_{i,j,\alpha \neq n+1} h_{ij}^{\alpha} \Delta h_{ij}^{\alpha} = &\sum_{i,j,k,m,\alpha \neq n+1} (h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk} + h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mkjk}) \\ &- \sum_{i,j,k,\alpha,\beta \neq n+1} h_{ij}^{\alpha} h_{ki}^{\beta} R_{\alpha\beta jk}. \end{aligned} \tag{3.10}$$

Before the proof of Theorem 2, we give the following lemma first.

Lemma 3. *Let M be an n-dimensional oriented compact submanifold with parallel mean curvature $(H \neq 0)$ in $F^{n+p}(c)$ with $p \leq 2$. If $c + H^2 > 0$ and $K_M \geq 0$, then M is congruent to one of the following:*

(i) $S^n(\frac{1}{\sqrt{c+H^2}})$;
(ii) *one of the Clifford hypersurfaces*
$S^k(\frac{1}{\sqrt{c+\lambda^2(k,n,H,c)}}) \times S^{n-k}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$ *in* $F^{n+1}(c)$ *with* $c > 0$,
where $\lambda(k,n,H,c) = \frac{1}{2k}[nH + \sqrt{n^2H^2 + 4k(n-k)c}]$, $k = 1, \dots, n-1$;
(iii) *one of the Clifford minimal hypersurfaces*
$S^k(\sqrt{\frac{k}{n(c+H^2)}}) \times S^{n-k}(\sqrt{\frac{n-k}{n(c+H^2)}})$ *in* $F^{n+1}(c + H^2)$, $k = 1, \dots, n-1$;
(iv) *the Clifford torus* $S^1(r_1) \times S^1(r_2)$ *in* $F^3(c+H^2-H_0^2)$ *with constant mean curvature* H_0, *where* $r_1, r_2 = [2(c + H^2) \pm 2H_0(c + H^2)^{1/2}]^{-1/2}$,
$0 \leq H_0 \leq H$, *and* $c + H^2 - H_0^2 > 0$;
(v) *the product of three spheres*
$S^{k_1}(\sqrt{\frac{k_1}{k(c+\lambda^2(k,n,H,c))}}) \times S^{k-k_1}(\sqrt{\frac{k-k_1}{k(c+\lambda^2(k,n,H,c))}}) \times S^{n-k}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$
in $F^{n+2}(c)$ *with* $c > 0$, *where*
$\lambda(k,n,H,c) = \frac{1}{2k}[nH + \sqrt{n^2H^2 + 4k(n-k)c}]$, $1 \leq k_1 < k \leq n-1$.

Proof. When $p = 1$, M is a compact hypersurface with nonzero constant mean curvature and non-negative sectional curvature in $F^{n+1}(c)$. In this case, Lemma 3 was proved by Nomizu and Smyth [21] for $c \geq 0$ and by Walter [25] for $c < 0$, i.e., M is either a totally umbilical sphere, or one of the Clifford hypersurfaces $S^k(\frac{1}{\sqrt{c+\lambda^2(k,n,H,c)}}) \times S^{n-k}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$ in $F^{n+1}(c)$ with $c > 0$, where

$$\lambda(k,n,H,c) = \frac{1}{2k}[nH + \sqrt{n^2H^2 + 4k(n-k)c}], \quad k = 1, \dots, n-1.$$

When $p = 2$, we have $K_M \geq 0$ and $H = \text{constant} \neq 0$.

If $n = 2$, we know from Theorem 4 in [29] that M is a surface in $F^3(c + H^2 - H_0^2)$ with constant mean curvature H_0. A direct computation shows that M is either a totally umbilical sphere, or the Clifford torus $S^1(r_1) \times S^1(r_2)$ in $F^3(c + H^2 - H_0^2)$ with constant mean curvature H_0. Here $0 \leq H_0 \leq H$, $c + H^2 - H_0^2 > 0$, and $r_1, r_2 = [2(c + H^2) \pm 2H_0(c + H^2)^{1/2}]^{-1/2}$.

If $n \geq 3$, it follows from Lemma 2 that the matrices H_{n+1} and H_{n+2} can be diagonalized simultaneously. Let $\{e_i\}$ be a frame such that

$$h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij},\ h_{ij}^{n+2} = \lambda_i^{n+2}\delta_{ij} \tag{3.11}$$

for all i, j. It is seen from Theorem 9 in [29] that M is either a minimal hypersurface in the totally umbilical hypersurface $F^{n+1}(c+H^2)$, or $M = M_1 \times M_2$, where M_i is a minimal hypersurface in a totally umbilical submanifold N_i of $F^{n+2}(c)$ for $i = 1, 2$.

For the first case, it follows from Theorem 1 that M is either the totally umbilical sphere $S^n(\frac{1}{\sqrt{c+H^2}})$, or one of the Clifford minimal hypersurfaces $S^k(\sqrt{\frac{k}{n(c+H^2)}}) \times S^{n-k}(\sqrt{\frac{n-k}{n(c+H^2)}})$ in $F^{n+1}(c + H^2)$ for $k = 1, \dots, n-1$.

For the second case, we see from (2.6), (3.11) and the assumption that

$$\begin{aligned}\frac{1}{2}\Delta S_H &= \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} \\ &\geq \sum_{i,j,k,m} h_{ij}^{n+1}h_{km}^{n+1}R_{mijk} + \sum_{i,j,k,m} h_{ij}^{n+1}h_{mi}^{n+1}R_{mkjk} \\ &= \frac{1}{2}\sum_{i,j}(\lambda_i^{n+1}-\lambda_j^{n+1})^2 R_{ijij} \\ &\geq 0.\end{aligned}$$

This together with the maximum principle implies that $R_{ijij}=0$ for $\lambda_i^{n+1}\neq\lambda_j^{n+1}$, $1\leq i,j\leq n$. Moreover, it follows from the proof of Theorem 9 in [29] that $\lambda_1^{n+1}=\cdots=\lambda_k^{n+1}=\lambda$, $\lambda_{k+1}^{n+1}=\cdots=\lambda_n^{n+1}=\mu$, where $\lambda\neq\mu$, $1\leq k\leq n-1$. Then we see from the Guass equation that

$$R_{ijij}=c+\lambda\mu+\lambda_i^{n+2}\lambda_j^{n+2}=0 \tag{3.12}$$

for $i=1,\ldots,k$ and $j=k+1,\ldots,n$.

If $\lambda_i^{n+2}=0$ for $i=1,\ldots,n$, then M lies in $F^{n+1}(c)$, and $\lambda=\lambda(k,n,H,c)$. It follows from a theorem due to Nomizu and Smyth [21] and Walter [25] that M is one of the Clifford hypersurfaces $S^k(\frac{1}{\sqrt{c+\lambda^2(k,n,H,c)}})\times S^{n-k}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$ in $F^{n+1}(c)$ with $c>0$ for $k=1,\ldots,n-1$.

If $\mathrm{tr}H_{n+2}^2\neq 0$, without loss of generality, we assume that $\lambda_1^{n+2}\neq 0$. This together with (3.12) implies that $\lambda_i^{n+2}=\lambda_j^{n+2}$ for $i,j=k+1,\ldots,n$. Since M_2 is a minimal hypersurface in N_2, we have $\sum_{i=k+1}^n\lambda_i^{n+2}=0$. Thus, we get $\lambda_i^{n+2}=0$ for $i=k+1,\ldots,n$, and M_2 is a totally geodesic hypersurface in N_2. Moreover, we have $c+\lambda\mu=0$. Note that $k\lambda+(n-k)\mu=nH$. A direct computation shows that

$$\lambda=\lambda(k,n,H,c)=\frac{1}{2k}[nH+\sqrt{n^2H^2+4k(n-k)c}], 2\leq k\leq n-1.$$

Hence, $N_1=S^{k+1}(\frac{1}{\sqrt{c+\lambda^2(k,n,H,c)}})$, $N_2=S^{n-k+1}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$ and $c>0$. Since M_1 is a minimal hypersurface in $S^{k+1}(\frac{1}{\sqrt{c+\lambda^2(k,n,H,c)}})$ with $K_{M_1}\geq 0$, it follows from Theorem 1 and the assumption that M_1 is congruent to $S^{k_1}(\sqrt{\frac{k_1}{k(c+\lambda^2(k,n,H,c))}})\times S^{k-k_1}(\sqrt{\frac{k-k_1}{k(c+\lambda^2(k,n,H,c))}})$ for $1\leq k_1<k$. Therefore, M is the product of three spheres $S^{k_1}(\sqrt{\frac{k_1}{k(c+\lambda^2(k,n,H,c))}})\times S^{k-k_1}(\sqrt{\frac{k-k_1}{k(c+\lambda^2(k,n,H,c))}})\times S^{n-k}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$ in $F^{n+2}(c)$ with $c>0$, where $1\leq k_1<k\leq n-1$.

This proves Lemma 3. □

Proof of Theorem 2. Applying (2.1), we obtain

$$\begin{aligned}&\sum_{i,j,k,m,\alpha\neq n+1} h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk}+\sum_{i,j,k,m,\alpha\neq n+1} h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk}\\ &\quad= ncS_I+\sum_{\alpha\neq n+1,\beta}\mathrm{tr}H_{\beta}\cdot\mathrm{tr}(H_{\alpha}^2H_{\beta})-\sum_{\alpha\neq n+1,\beta}[\mathrm{tr}(H_{\alpha}H_{\beta})]^2\\ &\qquad-\sum_{\alpha,\beta\neq n+1}[\mathrm{tr}(H_{\alpha}^2H_{\beta}^2)-\mathrm{tr}(H_{\alpha}H_{\beta})^2],\end{aligned}$$

and

$$\sum_{i,j,k,\alpha,\beta\neq n+1} h_{ij}^{\alpha}h_{ki}^{\beta}R_{\alpha\beta jk}=\sum_{\alpha,\beta\neq n+1}[\mathrm{tr}(H_{\alpha}^2H_{\beta}^2)-\mathrm{tr}(H_{\alpha}H_{\beta})^2].$$

Since $\alpha,\beta\neq n+1$, $(\mathrm{tr}(H_{\alpha}H_{\beta}))$ is a symmetric $(p-1)\times(p-1)$-matrix. We choose the normal vector fields $\{e_{\alpha}\}_{\alpha\neq n+1}$ such that

$$\mathrm{tr}(H_{\alpha}H_{\beta})=\mathrm{tr}H_{\alpha}^2\cdot\delta_{\alpha\beta},$$

which implies

$$\sum_{\alpha,\beta\neq n+1}[\mathrm{tr}(H_{\alpha}H_{\beta})]^2=\sum_{\alpha\neq n+1}(\mathrm{tr}H_{\alpha}^2)^2. \tag{3.13}$$

For any real number a, we have

(3.14)

$$\begin{aligned}\sum_{i,j,\alpha\neq n+1} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}&=(1+a)\sum_{i,j,k,m,\alpha\neq n+1}(h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk}+h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk})-ancS_I\\ &\quad+(a-1)\sum_{\alpha,\beta\neq n+1}[\mathrm{tr}(H_{\alpha}^2H_{\beta}^2)-\mathrm{tr}(H_{\alpha}H_{\beta})^2]+a\sum_{\alpha\neq n+1}(\mathrm{tr}H_{\alpha}^2)^2\\ &\quad+a\left\{-\sum_{\alpha\neq n+1}\mathrm{tr}(H_{\alpha}^2H_{n+1})\cdot\mathrm{tr}H_{n+1}+\sum_{\alpha\neq n+1}[\mathrm{tr}(H_{\alpha}H_{n+1})]^2\right\}.\end{aligned}$$

When $p\leq 2$, the assertion follows from Lemma 3.

When $p\geq 3$, it follows from Lemma 1 and the assumption that M is pseudo-umbilical, i.e., $h_{ij}^{n+1}=H\delta_{ij}$. Hence, we have

$$\begin{aligned}&\sum_{\alpha\neq n+1}\mathrm{tr}(H_{\alpha}^2H_{n+1})\cdot\mathrm{tr}H_{n+1}-\sum_{\alpha\neq n+1}[\mathrm{tr}(H_{\alpha}H_{n+1})]^2\\ &\quad=\sum_{i,j,k,m,\alpha\neq n+1} h_{ij}^{\alpha}h_{mi}^{\alpha}h_{mj}^{n+1}h_{kk}^{n+1}-\sum_{i,j,k,m,\alpha\neq n+1} h_{ij}^{\alpha}h_{km}^{\alpha}h_{mk}^{n+1}h_{ij}^{n+1}\\ &\quad=nH^2\sum_{i,j,\alpha\neq n+1}(h_{ij}^{\alpha})^2-H^2\sum_{\alpha\neq n+1}(\mathrm{tr}H_{\alpha})^2\\ &\quad=nH^2S_I.\end{aligned} \tag{3.15}$$

On the other hand, we get from (3.5)

$$\sum_{i,j,k,m,\alpha\neq n+1} h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk}+\sum_{i,j,k,m,\alpha\neq n+1} h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk}\geq nK_{\min}S_I. \tag{3.16}$$

By a direct computation and the DDVV inequality, we obtain

$$\begin{aligned}(3.17)\qquad \sum_{\alpha,\beta\neq n+1}\mathrm{tr}(H_\alpha^2H_\beta^2)-\mathrm{tr}(H_\alpha H_\beta)^2&=\frac{1}{2}\sum_{\alpha,\beta\neq n+1}N(H_\alpha H_\beta-H_\beta H_\alpha)\\&\le\frac{1}{2}\Big(\sum_{\alpha\neq n+1}\mathrm{tr}H_\alpha^2\Big)^2\\&=\frac{1}{2}S_I^2.\end{aligned}$$

It follows from (3.14)–(3.17) that

$$\begin{aligned}(3.18)\qquad \frac{1}{2}\Delta S_I&=\sum_{i,j,k,\alpha\neq n+1}(h_{ijk}^\alpha)^2+\sum_{i,j,\alpha\neq n+1}h_{ij}^\alpha\Delta h_{ij}^\alpha\\&\ge(1+a)nK_{\min}S_I+a\sum_{\alpha\neq n+1}(\mathrm{tr}H_\alpha^2)^2+\frac{1}{2}(a-1)S_I^2-an(c+H^2)S_I\\&\ge(1+a)nK_{\min}S_I+\Big(\frac{a}{p-1}+\frac{a-1}{2}\Big)S_I^2-an(c+H^2)S_I\\&=S_I\Big[(1+a)nK_{\min}+\Big(\frac{a}{p-1}+\frac{a-1}{2}\Big)S_I-an(c+H^2)\Big]\end{aligned}$$

for $0\le a<1$. Taking $a=\frac{p-1}{p+1}$, we get

$$\begin{aligned}\frac{1}{2}\Delta S_I&\ge nS_I[(1+a)K_{\min}-a(c+H^2)]\\&=nS_I\left[\left(1+\frac{p-1}{p+1}\right)K_{\min}-\frac{p-1}{p+1}(c+H^2)\right].\end{aligned}$$

It follows from the assumption and the maximum principle that S_I is a constant, and

$$S_I\left[\left(1+\frac{p-1}{p+1}\right)K_{\min}-\frac{p-1}{p+1}(c+H^2)\right]=0.$$

If there is a point $q\in M$ such that $K_{\min}(q)>\frac{(p-1)(c+H^2)}{2p}$, then $S_I=0$. It follows from the codimension reduction theorem due to Erbacher [6] that M is a compact hypersurface with non-zero constant mean curvature and positive sectional curvature in the totally geodesic submanifold $F^{n+1}(c)$. Therefore, M is the totally umbilical sphere $S^n(\frac{1}{\sqrt{c+H^2}})$.

If $K_{\min}\equiv\frac{(p-1)(c+H^2)}{2p}$, then inequalities in (3.16)–(3.18) become equalities. This, together with the DDVV inequality, implies that $p=3$ and $K_{\min}=\frac{c+H^2}{3}$. Taking $a=0$ in (3.18), we get $S_I=\frac{2n}{3}(c+H^2)$. By the same argument as in [2], we conclude that $n=2$. Hence, $K_M=\frac{c+H^2}{3}$ and M is the Veronese surface in $F^4(c+H^2)$. This completes the proof of Theorem 2. □

Combing Theorems 1, 2 and rigidity results in [13, 22, 28], we present a general version of the Yau rigidity theorem.

Generalized Yau rigidity theorem. *Let M be an n-dimensional oriented compact submanifold with parallel mean curvature in $F^{n+p}(c)$, where $c+H^2>0$. Set $\tau(m,n)=$*

$\min\{m \cdot sgn(m-1), n\}$. *Then we have*
(1) *if* $H = 0$ *and*

$$K_M \geq \frac{\tau(p,n)c}{2[\tau(p,n)+1]},$$

then M *is either a totally geodesic sphere, one of the Clifford minimal hypersurfaces* $S^k(\sqrt{\frac{k}{nc}}) \times S^{n-k}(\sqrt{\frac{n-k}{nc}})$ *in* $F^{n+1}(c)$ *for* $k = 1, \ldots, n-1$, *or the Veronese submanifold in* $F^{n+d}(c)$, *where* $d = \frac{1}{2}n(n+1) - 1$;
(2) *if* $H \neq 0$ *and*

$$K_M \geq \frac{\tau(p-1,n)(c+H^2)}{2[\tau(p-1,n)+1]},$$

then M *is congruent to one of the following:*

(i) $S^n(\frac{1}{\sqrt{c+H^2}})$;
(ii) *one of the Clifford hypersurfaces*
$S^k(\frac{1}{\sqrt{c+\lambda^2(k,n,H,c)}}) \times S^{n-k}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$ *in* $F^{n+1}(c)$ *with* $c > 0$,
where $\lambda(k,n,H,c) = \frac{1}{2k}[nH + \sqrt{n^2H^2 + 4k(n-k)c}]$, $k = 1, \ldots, n-1$;
(iii) *one of the Clifford minimal hypersurfaces*
$S^k(\sqrt{\frac{k}{n(c+H^2)}}) \times S^{n-k}(\sqrt{\frac{n-k}{n(c+H^2)}})$ *in* $F^{n+1}(c+H^2)$, $k = 1, \ldots, n-1$;
(iv) *the Clifford torus* $S^1(r_1) \times S^1(r_2)$ *in* $F^3(c+H^2-H_0^2)$ *with constant mean curvature* H_0, *where* $r_1, r_2 = [2(c+H^2) \pm 2H_0(c+H^2)^{1/2}]^{-1/2}$,
$0 \leq H_0 \leq H$, *and* $c + H^2 - H_0^2 > 0$;
(v) *the Veronese submanifold in* $F^{n+d}(c+H^2)$, *where* $d = \frac{1}{2}n(n+1) - 1$;
(vi) *the product of three spheres*
$S^{k_1}(\sqrt{\frac{k_1}{k(c+\lambda^2(k,n,H,c))}}) \times S^{k-k_1}(\sqrt{\frac{k-k_1}{k(c+\lambda^2(k,n,H,c))}}) \times S^{n-k}(\frac{\lambda(k,n,H,c)}{\sqrt{c^2+c\lambda^2(k,n,H,c)}})$
in $F^{n+2}(c)$ *with* $c > 0$, *where*
$\lambda(k,n,H,c) = \frac{1}{2k}[nH + \sqrt{n^2H^2 + 4k(n-k)c}], 1 \leq k_1 < k \leq n-1$.

Recently Andrews and Baker [1] generalized a weaker version of Huisken's convergence theorem [10] for mean curvature flow of convex hypersurfaces in $\mathbb{R}^{n+1}$ to higher codimensional cases. Motivated by the generalized Yau rigidity theorem, we would like to propose the following conjecture on mean curvature flow in higher codimensions, which can be considered as a generalization of the Huisken convergence theorem [10].

Conjecture A. *Let* $M_0 = F_0(M)$ *be an* n*-dimensional compact submanifold in an* $(n+p)$*-dimensional space form* $F^{n+p}(c)$ *with* $c + H^2 > 0$. *If the sectional curvature of* M_0 *satisfies*

$$K_{M_0} > \frac{\tau(p,n)(c+H^2)}{2[\tau(p,n)+1]},$$

then the mean curvature flow

$$\begin{cases} \frac{\partial}{\partial t}F(x,t) = n\xi(x,t), \quad x \in M,\ t \geq 0, \\ F(\cdot,0) = F_0(\cdot), \end{cases}$$

exists smooth solution $F_t(\cdot)$, *and* $F_t(\cdot)$ *converges to a round point in finite time, or* $c > 0$ *and* $F_t(\cdot)$ *converges to a totally geodesic sphere as* $t \to \infty$. *In particular,* M *is diffeomorphic to* S^n.

At this moment, only a very few cases of the above conjecture is known [1,10,11,16]. When $p = 1$ and $c = 0$, the conjecture was verified by Huisken [10]. When $p = 1$ and $c = 1$, a weaker version of the conjecture was proved by Huisken [11]. We hope our results will be helpful in generalizing the result of Andrews–Baker [1] and Liu *et al.* [16]. Motivated by the generalized Yau rigidity theorem and a convergence theorem for Ricci flow in [9], we propose the following conjecture on the normalized Ricci flow.

Conjecture B. *Let (M, g_0) be an n-dimensional compact submanifold in an $(n+p)$-dimensional space form $F^{n+p}(c)$ with $c + H^2 > 0$. If the sectional curvature of M satisfies*

$$K_M > \frac{\tau(p,n)(c+H^2)}{2[\tau(p,n)+1]},$$

then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)} + \frac{2}{n} r_{g(t)} g(t),$$

exists for all time and converges to a constant curvature metric as $t \to \infty$. Moreover, M is diffeomorphic to S^n.

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References

[1] B. Andrews and C. Baker, *Mean curvature flow of pinched submanifolds to spheres*, J. Differ. Geom., **85** (2010), 357–396.

[2] S. S. Chern, M. do Carmo, and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, in 'Functional Analysis and Related Fields,' Springer-Verlag, New York, 1970.

[3] T. Choi and Z. Lu, *On the DDVV conjecture and the comass in calibrated geometry (I)*, Math. Z., **260** (2008), 409–429.

[4] P. J. De Smet, F. Dillen, L. Verstraelen, and L. Vrancken, *A pointwise inequality in submanifold theory*, Arch. Math., **35** (1999), 115–128.

[5] N. Ejiri, *Compact minimal submanifolds of a sphere with positive Ricci curvature*, J. Math. Soc. Japan, **31** (1979), 251–256.

[6] J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differ. Geom., **5** (1971), 333–340.

[7] J. Q. Ge and Z. Z. Tang, *A proof of the DDVV conjecture and its equality case*, Pacific J. Math., **237** (2008), 87–95.

[8] J. Q. Ge and Z. Z. Tang, *A survey on the DDVV conjecture*, Harmonic maps and differential geometry, 247–254, Contemp. Math., **542**, Amer. Math. Soc., Providence, RI, 2011.

[9] J. R. Gu and H. W. Xu, *The sphere theorem for manifolds with positive scalar curvature*, `arXiv:math.DG/1102.2424`, to appear in J. Differ. Geom.

[10] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differ. Geom., **20** (1984), 237–266.

[11] G. Huisken, *Deforming hypersurfaces of the sphere by their mean curvature*, Math. Z., **195** (1987), 205–219.

[12] T. Itoh, *On veronese manifolds, J. Math. Soc. Japan*, **27** (1975), 497–506.

[13] T. Itoh, Addendum to my paper '*On Veronese manifolds*', J. Math. Soc. Japan., **30** (1978), 73–74.
[14] B. Lawson, *Local rigidity theorems for minimal hyperfaces*, Ann. Math., **89** (1969), 187–197.
[15] A. M. Li and J. M. Li, *An intrinsic rigidity theorem for minimal submanifolds in a sphere*, Arch. Math., **58** (1992), 582–594.
[16] K. F. Liu, H. W. Xu, F. Ye, and E. T. Zhao, *Mean curvature flow of higher codimension in hyperbolic spaces*, `arXiv:math.DG/1105.5686`, to appear in Comm. Anal. Geom.
[17] Z. Lu, *On the DDVV conjecture and the comass in calibrated geometry (II)*, `arXiv:math.DG/0708.2921v1`.
[18] Z. Lu, *Proof of the normal scalar curvature conjecture*, `arXiv:math.DG/0711.3510v1`.
[19] Z. Lu, *Recent developments of the DDVV conjecture*, Bull. Transil. Univ. Brasov Ser. B, **14** (2008), 133–144.
[20] Z. Lu, *Normal scalar curvature conjecture and its applications*, J. Funct. Anal., **261** (2011), 1284–1308.
[21] K. Nomizu and B. Smyth, *A formula of Simons' type and hypersurfaces with constant mean curvature*, J. Differential Geom., **3** (1969), 367–377.
[22] Y. B. Shen, *Submanifolds with nonnegative sectional curvature*, Chin. Ann. Math. Ser. B, **5** (1984), 625–632.
[23] K. Shiohama and H. W. Xu, *A general rigidity theorem for complete submanifolds*, Nagoya Math. J., **150** (1998), 105–134.
[24] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. Math., **88** (1968), 62–105.
[25] R. Walter, *Compact hypersurfaces with a constant higher mean curvature function*, Math. Ann., **270** (1985), 125–145.
[26] H. W. Xu, *A rigidity theorem for submanifolds with parallel mean curvature in a sphere*, Arch. Math., **61** (1993), 489–496.
[27] H. W. Xu, *On closed minimal submanifolds in pinched Riemannian manifolds*, Trans. Amer. Math. Soc., **347** (1995), 1743–1751.
[28] H. W. Xu and W. Han, *Geometric rigidity theorem for submanifolds with positive curvature*, Appl. Math. J. Chinese Univ. Ser. B, **20** (2005), 475–482.
[29] S. T. Yau, *Submanifolds with constant mean curvature I, II*, Amer. J. Math., **96, 97** (1974, 1975), 346–366, 76–100.

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