

THE CONJUGACY RELATION ON UNITARY REPRESENTATIONS

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ABSTRACT. In a 1965 paper, E.G. Effros asked the question if the conjugacy relation for unitary representations of a locally compact second countable group is a Borel equivalence relation. In this paper, we answer this question affirmatively. This also settles a recent question raised by A.S. Kechris regarding the hierarchy complexity of unitary equivalence of probability measure preserving actions of countable discrete groups.

1. Introduction

Let H be an infinite-dimensional separable complex Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. As usual, let $\mathcal{B}(H)$ denote the set of bounded operators on H . Let $U(H)$ denote the unitary group of H , which we equip with the strong (equivalently, weak) operator topology¹. Recall that $U(H)$ is a Polish group when given this topology.

Let Γ be a discrete countable group. A unitary representation of Γ is a homomorphism $\pi : \Gamma \rightarrow U(H)$. Define

$$\text{Rep}(\Gamma, H) = \{\pi : \Gamma \rightarrow U(H) : (\forall \gamma_1, \gamma_2 \in \Gamma) : \pi(\gamma_1 \gamma_2) = \pi(\gamma_1) \pi(\gamma_2)\},$$

the *space of unitary representations of Γ on H* , which we equip with the subspace topology inherited from $U(H)^\Gamma$, when the latter is given the product topology. Since $\text{Rep}(\Gamma, H)$ is a closed subset of $U(H)^\Gamma$, $\text{Rep}(\Gamma, H)$ is a Polish space. For notational ease, we usually write π_γ for $\pi(\gamma)$, when $\pi \in \text{Rep}(\Gamma, H)$.

We say that two representations $\sigma, \pi \in \text{Rep}(\Gamma, H)$ are *conjugate* if there is $U \in U(H)$ such that $U\sigma_\gamma U^{-1} = \pi_\gamma$ for all $\gamma \in \Gamma$, and we write $\sigma \simeq \pi$ if this is the case. It is clear that the conjugacy relation \simeq is induced by the natural conjugation action of $U(H)$ on $\text{Rep}(\Gamma, H)$, and so \simeq is an analytic equivalence relation, i.e., an analytic subset of $\text{Rep}(\Gamma, H)^2$.

The question was raised by Effros in [2, pp. 1163–1164] if \simeq is in fact Borel. The purpose of this paper is to prove that this is indeed the case:

Theorem 1.1. *The conjugacy relation in $\text{Rep}(\Gamma, H)$ is $F_{\sigma\delta}$.*

This also answers a question raised by Kechris in [9, p. 123, (IIIb) and (IVb)], who asked if unitary equivalence of probability measure preserving actions of Γ is Borel. We note that Theorem 1.1 stands in contrast to the recent result of Foreman *et al.* [4], who have shown that the conjugacy relation on ergodic transformations in the group

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¹We refer to [12] for basic definitions.

of measure preserving transformations $\text{Aut}(X, \mu)$ is complete analytic, and so cannot be Borel, when (X, μ) is a standard non-atomic probability space.

Theorem 1.1 applies more generally to representations of *separable involutive Banach algebras* in the sense of [11, Appendix B], and so in particular it applies to separable C^* -algebras and to unitary representations of second countable locally compact groups. If Γ is a separable involutive Banach algebra, then we let $\text{Rep}(\Gamma, H)$ denote the set of $*$ -homomorphisms from Γ to $\mathcal{B}(H)$. We give $\text{Rep}(\Gamma, H)$ the initial topology induced by the maps

$$\sigma \mapsto \sigma_\gamma(\xi),$$

ranging over all $\gamma \in \Gamma$ and $\xi \in H$. It can be seen that $\text{Rep}(\Gamma, H)$ is Polish in this topology. The (unitary) conjugacy relation \simeq in $\text{Rep}(\Gamma, H)$ is defined as in the case of countable discrete groups, and we have:

Theorem 1.2. *Let Γ be a separable involutive Banach algebra. Then the conjugacy relation in $\text{Rep}(\Gamma, H)$ is $F_{\sigma\delta}$.*

Theorems 1.1 and 1.2 have identical proofs, and below we will use Γ to stand for either a countable discrete group or a separable involutive Banach algebra.

The paper is organized as follows: First, in Section 2, we will recall various basic notions from the theory of von Neumann algebras and the direct integral theory of unitary representations, and establish a simple lemma that will facilitate hierarchy complexity calculations later. The proof of Theorems 1.1 and 1.2 is given in Section 3, where we also show that the result is optimal, and discuss some consequences related to Borel reducibility of equivalence relations.

2. Preliminaries

In what follows, H will always denote an infinite-dimensional separable complex Hilbert space equipped with the topology induced by its norm. We let $H^1 = \{\xi \in H : \|\xi\| \leq 1\}$ be the unit ball of H and $\mathcal{B}^1(H) = \{T \in \mathcal{B}(H) : \|T\| \leq 1\}$, where $\|T\|$ is the operator norm of T .

2.1. Elementary relations and functions. The following Lemma collects the basic complexity calculations that we need for our arguments.

Lemma 2.1. (a) *The function*

$$\mathcal{B}(H) \times H \times H \rightarrow \mathbb{C} : (T, \xi, \eta) \mapsto \langle T\xi, \eta \rangle$$

is continuous when $\mathcal{B}(H)$ has the weak topology.

(b) *The function $\mathcal{B}(H) \times H \rightarrow H : (T, \xi) \mapsto T\xi$ is continuous when $\mathcal{B}(H)$ is given the strong topology (but not the weak.)*

(c) *The relation $Q^< \subseteq H \times H \times \mathbb{C} \times \mathbb{R}_+ \times \mathcal{B}(H)$ defined by*

$$Q^<(\xi, \eta, z, \varepsilon, T) \iff |\langle T\xi, T\eta \rangle - z| < \varepsilon$$

is F_σ when $\mathcal{B}(H)$ is given the weak topology.

(d) *The map $H \times H \times \mathcal{B}(H) \times \mathcal{B}(H) \rightarrow \mathbb{C} : (\xi, \eta, T, S) \mapsto \langle T\xi, S\eta \rangle$ is continuous when the first $\mathcal{B}(H)$ has the weak topology and the second has the strong topology (or when the reverse is the case.)*

Proof. Verification of (a) and (b) are routine and left to the reader, while (d) follows directly from (a) and (b).

(c) Fix an orthonormal basis (e_i) for H . Then

$$|\langle T\xi, T\eta \rangle - z| < \varepsilon \iff (\exists N)(\exists q \in \mathbb{Q}_+)(\forall k \geq N) \left| \sum_{l=1}^k \langle T\xi, e_l \rangle \langle T\eta, e_l \rangle^* - z \right| \leq \varepsilon - q.$$

Since the coefficients in the sum are weakly continuous, this gives a F_σ definition of $Q^<$. \square

Definition 2.2. The relation $R \subseteq \text{Rep}(\Gamma, H) \times \text{Rep}(\Gamma, H) \times \mathcal{B}(H)$ is defined by

$$R(\sigma, \pi, T) \iff (\forall \gamma \in \Gamma) T\sigma_\gamma = \pi_\gamma T.$$

The set $R_{\sigma, \pi} = \{T \in \mathcal{B}(H) : R(\sigma, \pi, T)\}$ is the set of *intertwiners* of σ and π . We let $R_\sigma = R_{\sigma, \sigma}$, and

$$R^1 = R \cap \text{Rep}(\Gamma, H) \times \text{Rep}(\Gamma, H) \times \mathcal{B}^1(H).$$

Finally, let $R_{\sigma, \pi}^{\text{p.i.}}$ denote the set of partial isometries in $R_{\sigma, \pi}^1$, i.e. $u \in R_{\sigma, \pi}^1$ such that u^*u and uu^* are (orthogonal) projections.

Lemma 2.3. *The relations R and R^1 are closed when $\mathcal{B}(H)$ has the weak topology. Moreover, if $T \in R_{\sigma, \pi}$ then $T^* \in R_{\pi, \sigma}$. In particular, R_σ is a von Neumann algebra acting on H .*

Proof. We claim that

$$(2.1) \quad R(\sigma, \pi, T) \iff (\forall \gamma \in \Gamma)(\forall \varepsilon > 0)(\forall \xi, \eta \in H) |\langle \sigma_\gamma(\xi), T^* \eta \rangle - \langle T\xi, \pi_{\gamma^{-1}}(\eta) \rangle| \leq \varepsilon.$$

Note that this equivalence gives a weakly closed definition of R by (d) of Lemma 2.1 and the fact that $T \mapsto T^*$ is weakly continuous. To see the equivalence, just note that it follows from the right-hand side of (2.1) that

$$(\forall \varepsilon > 0)(\forall \xi, \eta \in H) |\langle T\sigma_g(\xi), \eta \rangle - \langle \pi_g T\xi, \eta \rangle| \leq \varepsilon,$$

so $\|T\sigma_g\xi - \pi_g T\xi\| = 0$ for all $\xi \in H$, and so $T\sigma_g = \pi_g T$. Finally, (2.1) gives $R(\sigma, \pi, T) \iff R(\pi, \sigma, T^*)$. \square

2.2. Some von Neumann algebra notions. In this section, we recall the basic notions from the theory of von Neumann algebras that are needed for the proof of Theorems 1.1 and 1.2.

The set of (orthogonal) projections in $\mathcal{B}(H)$ will be denoted $P(H)$, and for M a von Neumann algebra (e.g., $M = R_\sigma$), we let $P(M)$ be the set of projections in M . Also, M^+ denotes the set of positive operators in the von Neumann algebra M , and, as usual, for self-adjoint $x, y \in M$, we write $x \leq y$ when $y - x \in M^+$.

If $\sigma \in \text{Rep}(\Gamma, H)$ and $p \in P(R_\sigma)$, then the range of p is a σ -invariant closed subspace, and we denote by $\sigma|_p$ the restriction of σ to $\text{ran}(p)$. If $\sigma, \pi \in \text{Rep}(\Gamma, H)$ and $p \in P(R_\sigma)$, $q \in P(R_\pi)$, then we will say that p is *equivalent* to q , written $p \sim q$, if $\sigma|_p \simeq \pi|_q$. This is clearly the same as saying that there is a partial isometry $u \in R_{\sigma, \pi}$ such that $u^*u = p$ and $uu^* = q$. In particular, $p, q \in P(R_\sigma)$ are equivalent precisely when they are Murray–von Neumann equivalent in R_σ ; see [1]. Also, for $p \in R_\sigma$ and $q \in R_\pi$, write $p \precsim q$ if p is equivalent to a sub-projection of q . Further, we will write $\sigma \precsim \pi$ if $1 \in R_\sigma$ is equivalent to a projection in R_π , and so in particular, we

have $\sigma|p \precsim \pi|q$ if and only if $p \precsim q$. Throughout the paper we will use the basic (Murray–von Neumann) comparison theory of projections freely, and refer to [1] for background.

When Γ is group, we will say that the representations σ and π are *disjoint*, written $\sigma \perp \pi$, if $R_{\sigma, \pi} = \{0\}$. When Γ is an involutive Banach algebra, we define $\sigma \perp \pi$ to hold if either σ or π is a trivial representation (i.e., $\sigma_a = 0$ for all $a \in \Gamma$), or $R_{\sigma, \pi} = \{0\}$.² Effros has shown that \perp is a Borel relation; see [2].

Definition 2.4. A representation $\sigma \in \text{Rep}(\Gamma, H)$ is called a *factor representation* (or a *primary representation*), if R_σ is a factor, i.e., if the center of R_σ consists of multiples of 1.

Remark 2.5. It is useful to note that for primary representations, the relation $\not\prec$ is transitive: If $\sigma, \pi, \rho \in \text{Rep}(\Gamma, H)$ are primary and $\sigma \not\prec \pi$ and $\pi \not\prec \rho$, then $\sigma \not\prec \rho$. This follows since in a factor M , it holds for all $p, q \in P(M)$ that either $p \precsim q$ or $q \precsim p$. Since $\not\prec$ is also reflexive and symmetric, it is an equivalence relation on the set of factor representations.

The following is a well-known consequence of the polar decomposition theorem (see e.g. [13, 1.12.1]):

Lemma 2.6. *Let $\sigma, \pi \in \text{Rep}(\Gamma, H)$ and suppose $T \in R_{\sigma, \pi}$. Let $T = u|T|$ be the polar decomposition. Then $|T| \in R_\sigma$ and $u \in R_{\sigma, \pi}$. In particular, if $p_0 \in P(R_\sigma)$, $q_0 \in P(R_\pi)$ and $\sigma|p_0 \not\prec \pi|q_0$ then there is a subprojection of p_0 which is equivalent to a subprojection of q_0 .*

Proof. Let $T \in R_{\sigma, \pi}$. Then from Lemma 2.3 we have $T^*T \in R_\sigma$, and since R_σ is a von Neumann algebra, we have $|T| = (T^*T)^{\frac{1}{2}} \in R_\sigma$. Moreover, $p = u^*u \in P(R_\sigma)$, since it is the projection onto $\ker(|T|)^\perp$. Since we have

$$u\sigma_\gamma(|T|\xi) = T\sigma_\gamma(\xi) = \pi_\gamma T(\xi) = \pi_\gamma(u|T|\xi),$$

and since $\text{ran}(|T|)$ is dense in $\text{ran}(p)$ (see [13, 1.12.1]), it follows that

$$u\sigma_\gamma(\xi) = up\sigma_\gamma(\xi) = u\sigma_\gamma(p\xi) = \pi_\gamma u(p\xi) = \pi_\gamma u(\xi)$$

for all $\gamma \in \Gamma$ and $\xi \in H$. Thus $u \in R_{\sigma, \pi}$, as required. \square

A primary representation σ will be called *type I*, *II* or *III* according to if R_σ is a factor of types I, II or III; see [1]. Following further standard terminology, we will say that a primary representation is *finite* if the identity is a finite projection, *semifinite* if it is type I or II, and *purely infinite* if it is type III. We shall also need the following fundamental fact (see, e.g., [1, III.2.5.8]); we also refer the reader to [14] for an overview of the basic notions of von Neumann algebra theory.

Theorem 2.7. *Every finite von Neumann factor M acting on a separable Hilbert space admits a faithful trace $\tau : M \rightarrow \mathbb{C}$ such that $\tau(1) = 1$, which is weakly continuous on the unit ball of M .*

²Our definition of \perp is made so that Theorem 2.9 can be stated without making the additional assumption that the representations are non-degenerate.

2.3. Direct integral theory. We now review the direct integral theory of unitary representations. The main reference for this is [11].

Definition 2.8. Let $\sigma \in \text{Rep}(\Gamma, H)$. By a *direct integral decomposition* of σ into *primary representations* we mean a pair $((X, \mu), \sigma_x \curvearrowright H_x)$, where (X, μ) is a standard measure space and $\sigma_x \in \text{Rep}(\Gamma, H_x)$ is a measurable assignment of representations σ_x acting on Hilbert spaces H_x , and the following holds:

- (a) σ_x is a primary representation for all $x \in X$;
- (b) σ is isomorphic to

$$\int \sigma_x d\mu(x);$$

- (c) $\mu(\{x : (\forall y \in X) y \neq x \implies \sigma_x \perp \sigma_y\}) = 1$.

Without any real impact, we could of course replace (c) with

- (c') For all $x, y \in X$, if $x \neq y$ then $\sigma_x \perp \sigma_y$.

Using the direct integral decomposition of R_σ into factors, one obtains (see [1, III.5.1.14] and [11, Ch. 3]):

Theorem 2.9. *Every $\sigma \in \text{Rep}(\Gamma, H)$ admits a direct integral decomposition $((X, \mu), \sigma_x \curvearrowright H_x)$ into primary representations. Moreover, this decomposition is essentially unique in the following sense: If $((Y, \nu), \sigma_y \curvearrowright H_y)$ is any other direct integral decomposition of σ , then there is a measure-class preserving bijection $\phi : X \rightarrow Y$ such that $\sigma_x \simeq \sigma_{\phi(x)}$ for almost all $x \in X$.*

Not only do representations admit direct integral decompositions, but so do intertwiners. If $\sigma \in \text{Rep}(\Gamma, H)$ and $((X, \mu), \sigma_x \curvearrowright H_x)$ is a decomposition into primary representations, then any $T \in R_\sigma$ can be written as $T = \int T_x d\mu(x)$ essentially uniquely, where $T_x \in R_{\sigma_x}$. We define the *central support* of T to be

$$\text{supp}(T) = \{x \in X : T_x \neq 0\}.$$

This is well-defined modulo a μ -null set. From [11, Theorem 12.4] we further have:

Lemma 2.10. *Let $\sigma_i \in \text{Rep}(\Gamma, H)$, $i = 0, 1$, and let $((X_i, \mu_i), \sigma_{i,x} \curvearrowright H_{i,x})$ be the decomposition of σ_i into factor representations. Suppose $u \in R_{\sigma_0, \sigma_1}^{\text{p.i.}}$. Then there is a measure class preserving bijection $\phi : \text{supp}(u^*u) \rightarrow \text{supp}(uu^*)$ and partial isometries $u_x : H_{0,x} \rightarrow H_{1,\phi(x)}$ such that $u_x \in R_{\sigma_{0,x}, \sigma_{1,\phi(x)}}^{\text{p.i.}}$ and*

$$u = \int \theta(x)^{\frac{1}{2}} u_x d\mu_0(x),$$

i.e., $(u\xi)(\phi(x)) = \theta(x)^{\frac{1}{2}} u_x(\xi(x))$, and where $\theta = \frac{d\mu_1}{d\phi[\mu_0]}$ is the Radon-Nikodym derivative.

3. The proof

We will now give the proof of Theorems 1.1 and 1.2. First, we define a certain $F_{\sigma\delta}$ relation $S_0 \in \text{Rep}(\Gamma, H)^2$, along with the auxiliary relations S_1 and S_2 . The set of finite sequences in H is denoted by $H^{<\mathbb{N}}$.

Definition 3.1. The relation $S_2 \subseteq \text{Rep}(\Gamma, H) \times \text{Rep}(\Gamma, H) \times \mathbb{Q}_+ \times H^{<\mathbb{N}} \times \mathcal{B}_1(H)$ is defined as

$$S_2(\sigma, \pi, \varepsilon, \vec{\xi}, T) \iff R^1(\sigma, \pi, T) \wedge (\forall i < \text{lh}(\vec{\xi})) |\langle \vec{\xi}_i, \vec{\xi}_i \rangle - \langle T(\vec{\xi}_i), T(\vec{\xi}_i) \rangle| < \varepsilon,$$

and the relation $S_1 \subseteq \text{Rep}(\Gamma, H)^2 \times \mathbb{Q}_+ \times H^{<\mathbb{N}}$ is defined as

$$S_1(\sigma, \pi, \varepsilon, \vec{\xi}) \iff (\exists T) S_2(\sigma, \pi, \varepsilon, \vec{\xi}, T).$$

Finally, the relation $S_0 \subseteq \text{Rep}(\Gamma, H)^2$ is defined as

$$S_0(\sigma, \pi) \iff (\forall \varepsilon \in \mathbb{Q}_+) (\forall \vec{\xi}) S_1(\sigma, \pi, \varepsilon, \vec{\xi}).$$

Lemma 3.2. Let Γ and H be as above, and give $\mathcal{B}^1(H)$ the weak topology. Then

- (i) The relation S_2 is F_σ (i.e., Σ_2^0 .)
- (ii) The relation S_1 is F_σ .
- (iii) The relation S_0 is $F_{\sigma\delta}$ (i.e., Π_3^0 .)

The proof of Lemma 3.2 needs the following easy (and well-known) topological fact.

Lemma 3.3. Let X be a Polish space, Y a compact Polish space, and $A \subseteq X \times Y$. Then:

- (1) If A is closed then $\text{proj}_X(A) = \{x \in X : (\exists y \in Y)(x, y) \in A\}$ is closed.
- (2) If A is F_σ then $\text{proj}_X(A)$ is F_σ .

Proof of Lemma 3.2. (i) Follows directly from Lemma 2.1 (c).

(ii) Directly from Lemma 3.3 (2).

(iii) Clear, since it suffices to quantify over sequences in a countable dense subset of H .

□

We will prove Theorems 1.1 and 1.2 by proving

Theorem 3.4. Let $\sigma, \pi \in \text{Rep}(\Gamma, H)$. Then $\sigma \simeq \pi$ if and only if $S_0(\sigma, \pi)$ and $S_0(\pi, \sigma)$.

We first prove Theorem 3.4 for σ and π primary. Note that when σ, π are primary it follows by an easy application of Lemma 2.6 that if $S_0(\sigma, \pi)$ holds and σ is type III, then so is π , and $\sigma \simeq \pi$, since all non-zero projections in a type III factor are equivalent. This leaves us to deal with the semifinite case.

Lemma 3.5. Let $T \in R_{\sigma, \pi}^1$ and let $T = u|T|$ be the polar decomposition, and let $\vec{\xi}$ be a sequence of vectors in H^1 . Let $p = u^*u$. Then for $0 < \varepsilon < 1$:

- (1) If $S_2(\sigma, \pi, \varepsilon^2, \vec{\xi}, T)$ then $\|\vec{\xi}_i - p\vec{\xi}_i\| < \varepsilon$ for all $i < \text{lh}(\vec{\xi})$.
- (2) If $\|p\vec{\xi}_i - \vec{\xi}_i\| < \varepsilon$ for all $i < \text{lh}(\vec{\xi})$ then $S_2(\sigma, \pi, \varepsilon, \vec{\xi}, u)$ holds.

In particular, the following are equivalent:

- (I) $S_0(\sigma, \pi)$.
- (II) For all $\varepsilon > 0$ and $\vec{\xi} \in H^{<\mathbb{N}}$ there is a partial isometry $u \in R_{\sigma, \pi}^1$ such that $\|u^*u\vec{\xi}_i - \vec{\xi}_i\| < \varepsilon$ for all $i < \text{lh}(\vec{\xi})$.
- (III) For some (any) orthonormal basis (e_n) for H it holds that for all $\varepsilon > 0$ and $n \in \mathbb{N}$ there is a partial unitary $u \in R_{\sigma, \pi}^1$ such that $\|u^*ue_i - e_i\| < \varepsilon$, for all $i \leq n$.

Proof. (1) From

$$|\langle \vec{\xi}_i, \vec{\xi}_i \rangle - \langle T\vec{\xi}_i, T\vec{\xi}_i \rangle| = |\langle \vec{\xi}_i, \vec{\xi}_i \rangle - \langle T|p\xi_i, T|p\xi_i \rangle| < \varepsilon^2$$

and that $\|T\| \leq 1$, it follows that $1 - \|p\xi_i\|^2 < \varepsilon^2$, from which $\|\vec{\xi}_i - p\xi_i\| < \varepsilon$ follows, since p is an orthogonal projection.

(2) We have

$$|\langle \vec{\xi}_i, \vec{\xi}_i \rangle - \langle u\xi_i, u\xi_i \rangle| = |\langle \vec{\xi}_i, \vec{\xi}_i \rangle - \langle u^*u\xi_i, \xi_i \rangle| = |\langle \vec{\xi}_i - p\xi_i, \vec{\xi}_i \rangle| \leq \|\vec{\xi}_i - p\xi_i\| < \varepsilon.$$

Finally, the equivalence of (I) and (II) is clear from (1) and (2), and the easy proof that (II) is equivalent to (III) is left to the reader. \square

Before proceeding, we make the following easy observation about rearrangement of projections that we use below several times: suppose M is a semifinite factor, and that $p_i \in M$ are finite projections such that $p_i \precsim p_{i+1}$ for all $i \in \mathbb{N}$. Then we can find projections $q_i \in M$ such that $q_i \sim p_i$ and $q_i \leq q_{i+1}$ for all $i \in \mathbb{N}$. To prove this, let $q_1 = p_1$ and let $u \in M$ be a partial unitary such that $p_1 = u^*u$ and $p'_1 = uu^* \precsim p_2$. Since we must have $1 - p_1 \sim 1 - p'_1$, choose $v \in M$ witnessing this. Since u and v have orthogonal domain and range projections, $w = u + v \in M$ is unitary, and an easy calculation shows that $wp_1w^* = p'_1$. Let $q_2 = w^*p_2w$, and proceed inductively.

Lemma 3.6. *Let $\sigma, \pi \in \text{Rep}(\Gamma, H)$ be primary representations. Then $S_0(\sigma, \pi)$ implies $\sigma \precsim \pi$.*

Proof. As observed after the statement of Theorem 3.4, it suffices to consider the case when R_σ is semifinite.

Assume first that R_σ is a finite factor, and let $\tau : R_\sigma \rightarrow \mathbb{C}$ be the normalized trace. Since $S_0(\sigma, \pi)$ holds, we can find partial isometries $u_i \in R_{\sigma, \pi}$ such that $p_i = u_i^*u_i \rightarrow 1$ strongly, and so $\tau(p_i) \rightarrow 1$. Thus, after possibly going to a subsequence, we can assume that $p_i \precsim p_{i+1}$ for all $i \in \mathbb{N}$. By the observation preceding the Lemma, we can then find projections $q_i \in R_\sigma$ such that $q_i \sim p_i$ and $q_i \leq q_{i+1}$ for all $i \in \mathbb{N}$. Note that if $u \in R_{\sigma, \pi}^{\text{p.i.}}$ is such that $q_i = u^*u$ and $p_i = uu^*$, then $u_i u \in R_{\sigma, \pi}^{\text{p.i.}}$ witnesses the equivalence of q_i and $\bar{p}_i = u_i u_i^* \in R_\pi$.

Clearly $\bar{p}_i \precsim \bar{p}_{i+1}$, and so we can again find $\bar{q}_i \in R_\pi$ such that $\bar{p}_i \sim \bar{q}_i$ and $\bar{q}_i \leq \bar{q}_{i+1}$ for all $i \in \mathbb{N}$.

Now let $r_1 = q_1$, $\bar{r}_1 = \bar{q}_1$, and for $i > 1$, let $r_i = q_i - q_{i-1}$, $\bar{r}_i = \bar{q}_i - \bar{q}_{i-1}$. Then (r_i) and (\bar{r}_i) are sequences of mutually orthogonal projections, and since $\sum_{i=1}^n r_i = q_n$, we have $\sum_{i=1}^\infty r_i = 1$. Furthermore, $r_i \sim \bar{r}_i$ for all $i \in \mathbb{N}$ (see [1, III.1.3.8]), so fix $w_i \in R_{\sigma, \pi}^{\text{p.i.}}$ such that $r_i = w_i^*w_i$ and $\bar{r}_i = w_i w_i^*$. Then $\sum_{i=1}^\infty w_i \in R_{\sigma, \pi}$ is an isometry, whence $\sigma \precsim \pi$.

Dropping the assumption that R_σ is finite, then it follows from the above that $\sigma|_p \precsim \pi$ for all finite projections $p \in R_\sigma$. So if we take p_i to be any increasing sequence of finite projections such that $p_i \rightarrow 1$, then we can make exactly the same argument as above to obtain that $\sigma \precsim \pi$. \square

Proof of Theorem 3.4 for σ, π primary. We may assume that σ and π are semifinite primary representations. By the previous lemma we have $\sigma \precsim \pi$, and so if σ is properly infinite semifinite (i.e. type I_∞ or II_∞), then so is π , and since it then follows that π is isomorphic to its restriction to any infinite projection in R_π , we have that $\sigma \simeq \pi$. So we may assume that σ and π are finite. Let $U : H \rightarrow H$ and $V : H \rightarrow H$ be isometries

witnessing that $\sigma \lesssim \pi$ and $\pi \lesssim \sigma$. Then, since σ is not equivalent to its restriction to any proper subprojection of 1, it follows that $VU = 1_H$, and symmetrically, that $UV = 1_H$. Thus $U^{-1} = V$, and U witnesses that $\sigma \simeq \pi$. \square

We now proceed to consider the case when σ and π are not necessarily primary representations.

Lemma 3.7. *Let $\sigma_i \in \text{Rep}(\Gamma, H_i)$, $i \in \{0, 1\}$, and let $((X_i, \mu_i), \sigma_{i,x} \curvearrowright H_{i,x})$ be the corresponding decompositions into primary representations. Suppose $S_0(\sigma_0, \sigma_1)$ holds. Then there is a measure class preserving injection $\phi : X_0 \rightarrow X_1$ such that*

$$(3.1) \quad (\forall^{\mu_0} x \in X_0) S_0(\sigma_{0,x}, \sigma_{1,\phi(x)}).$$

In particular, $\sigma_0 \lesssim \sigma_1$.

Proof. Without loss of generality, $\mu_0(X_0) = 1$.

By Lemma 2.10, for a partial isometry $u \in R_{\sigma_0, \sigma_1}$ we have a measure class preserving injection $\phi_u : \text{supp}(u^*u) \rightarrow X_1$ and an assignment $x \mapsto u_x \in R_{\sigma_{0,x}, \sigma_{1,\phi(x)}}$ of partial isometries such that

$$u = \int \theta(x)^{\frac{1}{2}} u_x d\mu_0(x),$$

where θ is the Radon–Nikodym derivative. Note that if $u, v \in R_{\sigma_0, \sigma_1}$ are partial isometries, then $\phi_u(x) = \phi_v(x)$ for almost all $x \in \text{supp}(u) \cap \text{supp}(v)$. To see this, note that u_x witnesses that $\sigma_{0,x} \not\leq \sigma_{1,\phi_u(x)}$ and v_x witnesses that $\sigma_{0,x} \not\leq \sigma_{1,\phi_v(x)}$, and for primary representations the relation $\not\leq$ is transitive. Hence $\phi_u(x) = \phi_v(x)$ a.e. where both are defined, since $\sigma_{1,x} \perp \sigma_{1,x'}$ whenever $x \neq x'$.

Since $S_0(\sigma_0, \sigma_1)$ holds we can find a sequence $(u_n)_{n \in \mathbb{N}}$ in $R_{\sigma_0, \sigma_1}^{\text{p.i.}}$ such that $\mu(\text{supp}(u_n^* u_n)) > 1 - \frac{1}{n}$. Defining

$$\phi = \bigcup_{n \in \mathbb{N}} \phi_{u_n},$$

it is then clear that ϕ defines a measure class preserving injection almost everywhere on X_0 . We claim that this ϕ works.

To see this, let $x \mapsto e_{n,x} \in H_{0,x}$ be a measurable assignment of orthonormal bases³, and let $e_n = \int e_{n,x} d\mu_0(x)$. By Lemma 3.5 it suffices to show that for all $\varepsilon > 0$ and $n \in \mathbb{N}$ we have

$$\alpha_{\varepsilon, n} = \mu_0(\{x \in X_0 : (\exists u \in R_{\sigma_{0,x}, \sigma_{1,\phi(x)}}^{\text{p.i.}})(\forall i \leq n) \|u^* u e_{i,x} - e_{i,x}\| < \varepsilon\}) > 1 - \varepsilon.$$

For this, let $u \in R_{\sigma_0, \sigma_1}^{\text{p.i.}}$ be a partial isometry such that $\|u^* u e_i - e_i\| < \varepsilon^3$ for all $i \leq n$. Decompose u as $u = \int \theta(x)^{\frac{1}{2}} u_x d\mu_0(x)$ for some assignment $x \mapsto u_x \in R_{\sigma_{0,x}, \sigma_{1,\phi(x)}}$. Then since

$$\|u^* u e_i - e_i\|^2 = \int \|u_x^* u_x e_{i,x} - e_{i,x}\|^2 d\mu_0(x) < \varepsilon^3,$$

it follows that $(1 - \alpha_{\varepsilon, n})\varepsilon^2 < \varepsilon^3$, and so $\alpha_{\varepsilon, n} > 1 - \varepsilon$. \square

³For simplicity, we assume that $\dim(H_{i,x}) = \infty$ a.e.

Proof of Theorem 3.4. Fix direct integral decompositions of σ_0 and σ_1 as in Lemma 3.7. Applying Lemma 3.7, there are measure class preserving injections $\phi : X_0 \rightarrow X_1$ and $\psi : X_1 \rightarrow X_0$ such that $\sigma_{0,x} \simeq \sigma_{1,\phi(x)}$ and $\sigma_{1,y} \simeq \sigma_{0,\psi(y)}$. Then we must have $\psi = \phi^{-1}$, from which $\sigma_0 \simeq \sigma_1$ follows. \square

Remark 3.8. Theorem 3.4 is best possible in general, since we have the following:

Proposition 3.9. *If Γ has uncountably many disjoint primary representations then \simeq is a Π_3^0 -complete subset of $\text{Rep}(\Gamma, H)^2$ (in the sense of [10, 21.13].)*

For the proof we need the following: Define as in [7, 2.22] an equivalence relation on $2^{\mathbb{N} \times \mathbb{N}}$ by

$$xF_2y \iff (\forall n)(\exists m)(\forall k)x(n, k) = y(m, k) \wedge (\forall m)(\exists n)(\forall k)x(n, k) = y(m, k).$$

If we let $x(n, \cdot) \in 2^{\mathbb{N}}$ denote the n th “row” in $x \in 2^{\mathbb{N} \times \mathbb{N}}$, then this means that

$$xF_2y \iff \{x(n, \cdot) : n \in \mathbb{N}\} = \{y(n, \cdot) : n \in \mathbb{N}\}.$$

Then we have

Lemma 3.10 (Folklore). *The relation F_2 is a Π_3^0 -complete subset of $2^{\mathbb{N} \times \mathbb{N}} \times 2^{\mathbb{N} \times \mathbb{N}}$.*

Proof. Recall from [10, 23.A] that the set

$$P_3 = \{x \in 2^{\mathbb{N} \times \mathbb{N}} : (\forall n)(\exists k)(\forall m \geq k)x(n, m) = 0\}$$

is Π_3^0 -complete. For $x, y \in 2^{\mathbb{N} \times \mathbb{N}}$, let $x \oplus y \in 2^{\mathbb{N} \times \mathbb{N}}$ be defined by $(x \oplus y)(2k - 1, \cdot) = x(k, \cdot)$ and $(x \oplus y)(2k, \cdot) = y(k, \cdot)$, and note that $(x, y) \mapsto x \oplus y$ is continuous. Fix $z \in 2^{\mathbb{N} \times \mathbb{N}}$ such that $z(n, \cdot)$ enumerates all eventually zero sequences in $2^{\mathbb{N}}$. Then the map

$$f : 2^{\mathbb{N} \times \mathbb{N}} \rightarrow 2^{\mathbb{N} \times \mathbb{N}} \times 2^{\mathbb{N} \times \mathbb{N}} : x \mapsto (x \oplus z, z)$$

is a continuous reduction of P_3 to F_2 since $x \in P_3$, iff $(x \oplus z)F_2z$. \square

Proof of Proposition 3.9. It suffices to show that F_2 is continuously reducible to \simeq in $\text{Rep}(\Gamma, H)$, i.e., there is a continuous $g : 2^{\mathbb{N} \times \mathbb{N}} \rightarrow \text{Rep}(\Gamma, H)$ such that xF_2y if and only if $g(x) \simeq g(y)$.

Let $\text{vN}(H)$ denote the set of von Neumann algebras acting on H , and equip this with the Effros Borel structure, as defined in [2]. Then it is easy to see that the map $\text{Rep}(\Gamma, H) \rightarrow \text{vN}(H) : \sigma \mapsto R_\sigma$ is Borel. It was shown in [2] that the set of factors in $\text{vN}(H)$ is a Borel set, and so it follows that the set $\mathcal{F} \subseteq \text{Rep}(\Gamma, H)$ of factor representations is Borel. Consider the relation $\not\sim$ in \mathcal{F} . By Remark 2.5, $\not\sim$ is a Borel equivalence relation in \mathcal{F} . By assumption, $\not\sim$ has uncountably many classes in \mathcal{F} , and so it follows by Silver’s dichotomy theorem (see [15] or [5]) that there is a continuous $f : 2^{\mathbb{N}} \rightarrow \mathcal{F}$ such that $f(x) \perp f(y)$ whenever $x \neq y$. Further, we can clearly arrange that $f(x) \simeq \bigoplus_{n=1}^{\infty} f(x)$, the infinite direct sum of countably many copies of $f(x)$. Now define $g : 2^{\mathbb{N} \times \mathbb{N}} \rightarrow \text{Rep}(\Gamma, \bigoplus_{n=1}^{\infty} H)$ by

$$g(x) = \bigoplus_{n=1}^{\infty} f(x(n, \cdot)).$$

Then g is easily seen to be a continuous reduction of F_2 to \simeq in $\text{Rep}(\Gamma, \bigoplus_{n=1}^{\infty} H)$. \square

Corollary 3.11. *For any countable discrete group Γ , the conjugacy relation in $\text{Rep}(\Gamma, H)$ is Π_3^0 -complete.*

Proof. If Γ is type I, it follows from [16] that it is Abelian by finite, and so it has uncountably many spectrally disjoint unitary representations, and the above argument applies. If Γ is not type I, then it has uncountably many non-isomorphic irreducible unitary representations (see [6]) and so Proposition 3.9 applies. \square

Remark 3.12. Simon Thomas has pointed out the following interesting consequence of Proposition 3.9:

Corollary 3.13. *If Γ has uncountably many disjoint primary representations then the conjugacy relation in $\text{Rep}(\Gamma, H)$ is not Borel reducible to the conjugacy relation on irreducible unitary representations of Γ .*

Proof. We refer to [8] for the relevant notions. The conjugacy relation for irreducible unitary representations is F_σ (see e.g. [3]), hence it is a pinned equivalence relation by [8, Theorem 17.1.3]. However, by that same theorem the equivalence relation F_2 is not pinned and not Borel reducible to any pinned equivalence relation. \square

That the conclusion of Corollary 3.13 holds when Γ is a discrete countable group of type I is well known. That this also holds when Γ is not type I seems not to have been known.

Note from the second author

Greg Hjorth, who was my dear friend, colleague, and PhD thesis advisor (at UCLA, 2001–2005), died unexpectedly on January 13, 2011, while we were finishing work on this paper.

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