

## THE REGULARITY PROBLEM FOR ELLIPTIC OPERATORS WITH BOUNDARY DATA IN HARDY–SOBOLEV SPACE $HS^1$

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ABSTRACT. Let  $\Omega$  be a Lipschitz domain in  $\mathbb{R}^n, n \geq 3$ , and  $L = \operatorname{div}A\nabla$  be a second order elliptic operator in divergence form. We will establish that the solvability of the Dirichlet regularity problem for boundary data in Hardy–Sobolev space  $HS^1$  is equivalent to the solvability of the Dirichlet regularity problem for boundary data in  $H^{1,p}$  for some  $1 < p < \infty$ . This is a “dual result” to a theorem in [7], where it has been shown that the solvability of the Dirichlet problem with boundary data in BMO is equivalent to the solvability for boundary data in  $L^p(\partial\Omega)$  for some  $1 < p < \infty$ .

### 1. Introduction

We<sup>1</sup> shall prove an equivalence between solvability of certain end-point Dirichlet regularity problem in  $HS^1$  for second order elliptic operators and the solvability of the Dirichlet regularity problem with boundary data in  $H^{1,p}$  for some  $1 < p < \infty$ . The space  $HS^1$  is defined in section 2.

To be more precise, we study the regularity problem for elliptic operators in divergence form  $L = \operatorname{div}A\nabla$  on a Lipschitz domain  $\Omega \subset \mathbb{R}^n, n \geq 3$ . The matrix  $A = (a_{ij}(X))$  has real, bounded measurable coefficients such that there exists  $\lambda > 0$  with  $\lambda|\xi|^2 \leq \sum_{ij} a_{ij}(X)\xi_i\xi_j$  for all  $\xi \in \mathbb{R}^n$  and all  $X \in \Omega$ .

For these elliptic operators the Lax–Milgram Theorem implies that for every  $f \in H^{\frac{1}{2},2}(\partial\Omega)$  there exists a unique weak solution  $u \in H^{1,2}(\Omega)$ , i.e.,

$$\int_{\Omega} A\nabla u \cdot \nabla \varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega)$$

with  $u \equiv f$  on  $\partial\Omega$ , which means that the Dirichlet problem

$$Lu = 0 \text{ in } \Omega, \quad \operatorname{Tr} u \equiv f \text{ on } \partial\Omega$$

is solvable for boundary data in  $H^{\frac{1}{2},2}(\partial\Omega)$ . The question, if solvability still holds for other classes of boundary values, was extensively studied. In [18] it was shown that the continuous Dirichlet problem is solvable for these elliptic operators, i.e., for every  $f \in C^0(\partial\Omega)$  there exists a unique  $u \in W_{\text{loc}}^{1,2}(\Omega) \cap C^0(\bar{\Omega})$  such that  $Lu = 0$  in  $\Omega$  and  $u \equiv f$  on  $\partial\Omega$ .

Historically the study of the Dirichlet problem with boundary data in  $L^p$  for elliptic operators of the form  $L = \operatorname{div}A\nabla$  was initiated by B.E.J. Dahlberg in [5], where the Laplacian on Lipschitz domains was considered (the pullback of the Laplacian on a

Received by the editors October 26, 2011. Revision received April 30, 2012.

1991 *Mathematics Subject Classification.* 35J25, 42B37.

*Key words and phrases.* second order elliptic equations, boundary value problems, Hardy-Sobolev spaces.

<sup>1</sup>The first author is partially supported by EPSRC EP/J017450/1 grant.

Lipschitz domain leads to an operator of the form  $L = \operatorname{div}A\nabla$  for  $A$  elliptic with bounded, measurable coefficients).

Apart from the Dirichlet boundary value problem with data in  $L^p$  of great interests are also other boundary value problems in particular the  $L^p$  Neumann problem and Dirichlet regularity problem (or just Regularity problem) where the data are in

$$H^{1,p}(\partial\Omega) = \{f \in L^p(\partial\Omega); \nabla_T f \in L^p(\partial\Omega)\}.$$

Our result is motivated by a recent result [7] that established that the Dirichlet problem with boundary data in  $L^p(\partial\Omega)$  is solvable (abbreviated  $(D)_p$ ) for some  $1 < p < \infty$  if and only the Dirichlet problem with boundary data is solvable in the endpoint BMO space (abbreviated  $(D)_{BMO}$ ).

By the theory of Muckenaupt's  $B_p$ -weights it is well known that  $(D)_p$  implies  $(D)_q$  for  $q \in (p - \varepsilon, \infty)$  and some  $\varepsilon > 0$ , i.e., solvability is open with respect to  $p$  on  $(1, \infty)$ . The result in [7] establishes that this "extrapolation property" also holds at the endpoint where the correct endpoint is  $(D)_{BMO}$ . Furthermore the  $(D)_{BMO}$  solvability is also equivalent to the fact that the harmonic measure for the operator  $L$  is an  $A_\infty(d\sigma)$  weight with respect to the surface measure. A condition on coefficients that guarantees  $A_\infty$  was explored in [16].

The most classical method for solving these types of boundary value problems (at least for *symmetric* operators with coefficients of sufficient smoothness) is the method of layer potentials [9] for the Laplacian in  $\mathbb{R}^n$  and [19–21] for variable coefficients operators. What has been observed are intriguing relationships between various boundary value problems. Of particular note is the duality between the  $L^p$  Dirichlet boundary value problem and  $H^{1,p'}$  Regularity problem ( $\frac{1}{p} + \frac{1}{p'} = 1$ ). It turns out that the  $L^p$  Dirichlet boundary value problem is solvable if and only if the  $H^{1,p'}$  Regularity problem is solvable for the same operator (assuming *symmetry* of the operator).

We note that our assumptions do not allow to use the method of layer potentials, but this informal duality led us to hypothesize and later prove that the result from [7] does have a corresponding dual result. We observed that the dual of the Hardy space is the BMO space and this leads to hypothesis that the correct endpoint space for the Regularity problem is the atomic Hardy space.

It should be also noted that our result is the end-point (for  $p = 1$ ) of the perturbation result in [15] that established  $(R)_p \implies (R)_{p+\varepsilon}$  for all  $p \in (1, \infty)$  and some  $\varepsilon > 0$ . Here  $(R)_p$  denotes that the Regularity problem is solvable with data in  $H^{1,p}(\partial\Omega)$  (cf. Definition 1.3).

Before we formulate our main result precisely we introduce few necessary definitions. The study of boundary data in  $L^p(\partial\Omega)$  is related to the study of the non-tangential maximal function, see for example [5].

**Definition 1.1.** For  $\kappa > 1$  we define the cone-like family of non-tangential approach regions  $\{\Gamma_\kappa(Q)\}_{Q \in \partial\Omega}$  by

$$\Gamma_\kappa(Q) = \{X \in \Omega : |X - Q| < \kappa\delta(X)\}.$$

Here and throughout the paper  $\delta(X) = \operatorname{dist}(X, \partial\Omega)$ . We will omit the index  $\kappa$  and write  $\Gamma(Q)$ , if no confusion can arise. The non-tangential maximal function for the

non-tangential approach region  $\{\Gamma(Q)\}_{Q \in \partial\Omega}$  is defined by

$$u^*(Q) = \sup_{X \in \Gamma(Q)} |u(X)|.$$

The truncation at height  $h$  of the non-tangential maximal function is defined by  $(u)_h^*(Q) = \sup_{X \in \Gamma(Q) \cap B_h(Q)} |u(X)|$ .

Moreover we define the following variant of the non-tangential maximal function:

$$(1) \quad N(h)(Q) = \sup_{X \in \Gamma(Q)} \left( \int_{B_{\frac{\delta(X)}{2}}(X)} |h(Y)|^2 dY \right)^{\frac{1}{2}} \quad h \in L^2_{loc}(\Omega).$$

**Definition 1.2.** The Dirichlet problem with boundary data in  $L^p(\partial\Omega)$ ,  $1 < p < \infty$ , is solvable (abbreviated  $(D)_p$ ), if there exists a constant  $C > 0$  such that for every  $f \in C^0(\partial\Omega)$  the corresponding unique weak solution  $u \in W^{1,2}_{loc}(\Omega) \cap C^0(\Omega)$  satisfies

$$\|u^*\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega)}.$$

**Definition 1.3.** The regularity problem with boundary data in  $H^{1,p}(\partial\Omega)$ ,  $1 < p < \infty$ , is solvable (abbreviated  $(R)_p$ ), if for every  $f \in H^{1,p}(\partial\Omega) \cap C^0(\partial\Omega)$  the weak solution  $u$  to the problem

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f & \text{on } \partial\Omega \end{cases}$$

satisfies

$$\|N(\nabla u)\|_{L^p(\partial\Omega)} + \|u\|_{L^p(\Omega)} \leq C \|f\|_{H^{1,p}(\partial\Omega)}$$

for a constant  $C$  independent of  $f$ . Similarly, we say that the regularity problem with boundary data in  $HS^1(\partial\Omega)$  (abbreviated  $(R)_{HS^1}$ ) if for every  $f \in HS^1(\partial\Omega) \cap C^0(\partial\Omega)$  the solution  $u$  satisfies the estimate

$$(2) \quad \|N(\nabla u)\|_{L^1(\partial\Omega)} + \|u\|_{L^1(\Omega)} \leq C \|f\|_{HS^1(\partial\Omega)}.$$

We define  $(D^*)_{p'} = (D)_{p'}^{L^*}$  for  $L^* = \text{div} A^T \nabla$ . Now we can formulate the main result of this paper:

**Theorem 1.1.** *Let  $L$  be a divergence form elliptic operator satisfying the ellipticity condition on a Lipschitz domain  $\Omega$ . Then the following two statements hold:*

- *If  $(R)_{HS^1}$  is solvable then  $(D^*)_{BMO}$  and  $(R)_p$  are also solvable for some  $1 < p < \infty$ . Moreover, under this assumption*  
 $(R)_p$  *is solvable if and only if  $(D^*)_{p'}$  is solvable for  $p' = p/(p - 1)$ .*
- *If  $(R)_p$  is solvable for some  $1 < p < \infty$  so is  $(R)_{HS^1}$ .*

We note that the new result is the first part of this Theorem. The second part of this statement is not new and appears in [15] (at least for symmetric operators). Simultaneously, the first part of this statement improves the result of Shen [23] (again only stated for symmetric operators). Shen has established that the statement

$$(R)_p \text{ is solvable if and only if } (D^*)_{p'} \text{ is solvable for } p' = p/(p - 1),$$

holds provided  $(R)_q$  is solvable for at least one  $q \in (1, \infty)$ . In our statement this can be replaced by the  $(R)_{HS^1}$  solvability as follows from the first part of the Theorem.

### 2. Lipschitz domains and the Hardy–Sobolev space $HS^1$

In this section we will follow [2] to introduce the Hardy–Sobolev space  $HS^1$  on the boundary of a Lipschitz domain.

**Definition 2.1.**  $\mathcal{Z} \subset \mathbb{R}^n$  is an  $M$ -cylinder of diameter  $d$  if there exists a coordinate system  $(x, t)$  such that

$$\mathcal{Z} = \{(x, t) : |x| \leq d, -2Md \leq t \leq 2Md\}$$

and for  $s > 0$ ,

$$s\mathcal{Z} = \{(x, t) : |x| < sd, -2Md \leq t \leq 2Md\}.$$

**Definition 2.2.**  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain with Lipschitz ‘character’  $(M, N, C_0)$  if there exists a positive scale  $r_0$  and at most  $N$  cylinders  $\{\mathcal{Z}_j\}_{j=1}^N$  of diameter  $d$ , with  $\frac{r_0}{C_0} \leq d \leq C_0 r_0$  such that

- (i)  $s\mathcal{Z}_j \cap \partial\Omega$  is the graph of a Lipschitz function  $\phi_j$ ,  $\|\phi_j\|_\infty \leq M$ ,  $\phi_j(0) = 0$ ,
- (ii)  $\partial\Omega = \bigcup_j (\mathcal{Z}_j \cap \partial\Omega)$ ,
- (iii)  $\mathcal{Z}_j \cap \Omega \supset \left\{ (x, t) \in \Omega : |x| < d, \text{dist}((x, t), \partial\Omega) \leq \frac{d}{2} \right\}$ .

Throughout the whole paper, we will assume that  $\Omega$  is a Lipschitz domain in  $\mathbb{R}^n$  for  $n \geq 3$ . By definition  $\Omega$  is locally the region above a Lipschitz graph  $\varphi$  and so for  $Q = (x', \varphi(x')) \in \partial\Omega$  we define  $A_R(Q) = (x', \varphi(x') + R)$  and for  $X \in \Omega$  we define  $\hat{X} \in \partial\Omega$  such that  $A_R(\hat{X}) = X$  for an appropriate  $R$ . Thus  $A_R(Q)$  and  $\hat{X}$  are well defined in each  $\mathcal{Z}_j \cap \Omega$ . This means that  $A_R(Q)$  and  $\hat{X}$  depend on  $j$ , but we will omit the index  $j$  to maintain an easy readable notation. If we speak about an  $A_R(Q)$  for  $R$  so large it falls outside  $\mathcal{Z}_j$  we mean an appropriate point (which will be clear by the context) in  $\Omega$ , which has distance to  $\partial\Omega$  comparable to  $R$ . The radius of a ball  $B$  will be denoted by  $r(B)$  and for  $Q \in \partial\Omega, X \in \Omega$  and  $R > 0$  we write:

$$\begin{aligned} \Delta_R(Q) &= \partial\Omega \cap B_R(Q), & T_R(Q) &= \Omega \cap B_R(Q), \\ \delta(X) &= \text{dist}(X, \partial\Omega), & (\partial\Omega)_\beta &= \{X \in \Omega : \delta(X) < \beta\}, \\ \Omega_\beta &= \Omega \setminus (\partial\Omega)_\beta. \end{aligned}$$

**Definition 2.3.** Let  $\Omega$  be a Lipschitz domain in  $\mathbb{R}^n$ . For  $0 < q \leq 1$  and  $f \in L^q_{loc}(\partial\Omega)$ , we define the maximal function  $f^b_q$  by

$$f^b_q(x) = \sup_{B \ni x} \inf_{c \in \mathbb{R}^n} \frac{1}{r(B)} \left( \int_B |f - c|^q \right)^{\frac{1}{q}},$$

where the supremum is taken over all balls  $B$ , which are contained in  $\partial\Omega$  and contain  $x$ . The space  $\mathcal{C}^q$  is defined as all  $f \in L^q_{loc}(\partial\Omega)$  such that the norm

$$\|f\|_{\mathcal{C}^q} = \|f^b_q\|_{L^q(\partial\Omega)} + \|f\|_{L^q(\partial\Omega)}$$

is finite.

For  $q = 1$  we see that  $f_1^b(x) = \sup_{B \ni x} \frac{1}{|B|^{\frac{1}{n}}} \int_B |f - f_B|$  with  $f_B = \int_B f = \frac{1}{|B|} \int_B f$ , whereas for  $q < 1$  the function  $f$  might not be locally integrable and so  $f_B$  might not be defined. To simplify the notation we will write  $Nf = f_1^b$ , keeping the same notation in [2]. This type of maximal functions is motivated by the paper [6].

In [13] (see (6) in [2] as well) it was proved for  $f \in \mathcal{C}^1$ ,  $\frac{s}{s+1} \leq q < 1$ , where  $s$  is a constant larger than 2, which depends on the doubling property of the underlying metric space, and  $q^* = \frac{sq}{s-q}$  that

$$(3) \quad \left( \int_{B_r} |f - f_{B_r}|^{q^*} \right)^{\frac{1}{q^*}} \leq Cr \left( \int_{\lambda B_r} |Nf|^q \right)^{\frac{1}{q}}$$

for some  $\lambda > 1$ , which is independent of  $f$  and  $r$ . We define

$$\mathcal{M}_q f(x) = \sup_{B \ni x} \left( \int_B |f|^q \right)^{\frac{1}{q}},$$

where the supremum is taken over all balls containing  $x$ .

In [2] Badr and Dafni proved a relationship between the Hardy–Sobolev space and the space  $\mathcal{C}^1$  on complete Euclidian manifolds  $M$  with  $\mu(M) = \infty$  and  $\mu$  a doubling measure. Since we would like to apply this result later on to boundary data on  $\partial\Omega$  for  $\Omega$  a Lipschitz domain, we will not work in such a general setting. Our domain will be  $\partial\Omega$  for  $\Omega$  a Lipschitz domain, where the surface measure is the underlying measure. Therefore our domain is bounded and has a finite doubling measure. We will not write  $\partial\Omega$ , if there is no confusion possible, which domain is meant. Similar to Definition 2.11 and Definition 4.3 in [2] and [1] we define

**Definition 2.4.** For  $1 < t \leq \infty$  we say that a function  $a$  is a Hardy–Sobolev  $(1, t)$ -atom, if

- $a$  is supported in a ball  $B$
- $\|a\|_{L^t} + \|\nabla_T a\|_{L^t} \leq \frac{1}{|B|^{\frac{1}{t'}}$

For this  $a$  we will use the terminology that  $a$  is a Hardy–Sobolev  $(1, t)$ -atom corresponding to the ball  $B$ . Here  $\nabla_T$  is the derivative in the tangential direction on  $\partial\Omega$ .

We define the space  $HS_{t,ato}^1$  as follows:  $f \in HS_{t,ato}^1$ , if there exists a family of Hardy–Sobolev  $(1, t)$ -atoms  $\{a_j\}_j$  such that  $f$  can be decomposed as

$$f = \sum_j \lambda_j a_j \quad \text{almost everywhere}$$

with  $\sum_j |\lambda_j| < \infty$ . We equip  $HS_{t,ato}^1$  with the norm  $\|f\|_{HS_{t,ato}^1} = \inf \sum_j |\lambda_j|$ , where the infimum is taken over all possible decompositions.

The following three claims that we state for completeness of our presentation follow directly from results of [2]. (See also [22]).

**Lemma 2.1.** *Let  $a$  be a Hardy–Sobolev  $(1, t)$ -atom, then*

$$\|a\|_{\mathcal{C}^1} \leq C_t.$$

*Thus  $HS_{t,ato}^1 \subset \mathcal{C}^1$  with  $\|f\|_{\mathcal{C}^1} \leq C_t \|f\|_{HS_{t,ato}^1}$ .*

**Theorem 2.2.** *Let  $f \in C^1$  and  $q$  and  $s$  be as in (3). Then for every  $\alpha \geq \alpha_0 = C_\Omega \|f\|_{C^1}$  with  $C_\Omega$  a constant depending on the domain  $\Omega$ , one can find balls  $\{B_i\}_i \subset \partial\Omega$ , functions  $b_i \in W^{1,1}$  and  $g \in W^{1,\infty}$  such that*

- $f = g + \sum_i b_i$ ;
- $|g| + |\nabla g| \leq C\alpha$  almost everywhere;
- $\text{supp } b_i \subset B_i, \|b_i\|_1 \leq Cr_i\alpha|B_i|, \|b_i\|_q + \|\nabla b_i\|_q \leq C\alpha|B_i|^{\frac{1}{q}}$ ;
- $\sum_i |B_i| \leq \frac{C}{\alpha} \int (Mf + Nf)$ ;
- $\sum_i \chi_{B_i} \leq C$ .

**Theorem 2.3.** *Let  $f, q$  and  $s$  be as in Theorem 2.2 and  $q^* = \frac{sq}{s-q} (>1)$ . There exists a family of Hardy–Sobolev  $(1, q^*)$ -atoms  $\{a_j\}_j$  such that*

$$f = \sum \lambda_j a_j \quad \text{and} \quad \sum_j |\lambda_j| \leq C \|f\|_{C^1}.$$

Thus,  $C^1 \subset HS_{t,ato}^1$  for  $1 < t \leq q^*$ .

**Remark.** The major difference to the proof of Proposition 4.7 in [2] is the fact that our domain is bounded. Let  $\alpha_0$  be as in the proof of Theorem 2.2. Then for every  $j \geq j_0$  with  $j_0$  the smallest integer such that  $2^{j_0} > \alpha_0$ , we apply Theorem 2.2 to get

$$f = g^j + \sum_i b_i^j.$$

Following the proof in [2], we see that we can write

$$f = \sum_{j \geq j_0} (g^{j+1} - g^j) + g^{j_0}$$

in the  $W^{1,1}$  sense. The terms  $(g^{j+1} - g^j)$  are treated as in [2]. The term  $g^{j_0}$  is seen after a normalization as an atom for  $\partial\Omega$ .

**Remark 2.1.** From the construction of the atoms  $a_j$  we see that if  $f \in C^0(\partial\Omega)$ , then the  $a_j$  are in  $C^0(\partial\Omega)$ .

Since in our setting Poincaré’s inequality on  $L^1$  holds and every  $(1, t)$ -atom can be decomposed in a  $(1, \infty)$ -atom and a  $(1, t)$ -atom that satisfies the cancellation condition, Theorem 0.1 in [1] gives:

**Theorem 2.4.**  *$HS_{t_1,ato}^1 = HS_{t_2,ato}^1$  for all  $1 < t_1, t_2 \leq \infty$ . The norms are comparable, where the implicit constant depends on  $t_1$  and  $t_2$ .*

Thus we can define  $HS^1 = HS_{t,ato}^1$  for any  $1 < t \leq \infty$  and we will impose the norm of  $HS_{\infty,ato}^1$  on  $HS^1$ .

We finish this section with a result in the spirit of [4] about the  $C^q$ -norm, which is equivalent to the  $HS^1$ -norm in the  $q = 1$  case. In order to keep the notation simple, we assume that we work on  $\mathbb{R}^n$  instead of  $\partial\Omega$ .

**Lemma 2.5.** *Fix  $0 < R$  and  $0 < q \leq 1$ . Let  $\varphi \in C_0^\infty(\mathbb{R}^n)$  be supported in  $B_{2R}(0)$  with values in  $[0, 1]$ ,  $\varphi \equiv 1$  on  $B_R(0)$  and  $|\nabla\varphi| \leq \frac{C}{R}$ . Assume that  $f \in C_q^1 \cap C^0$  and let  $C_R = \int_{B_{2R}(0)} f$ . Then there exists and  $C_0$  independent of  $f$  such that*

$$\|\varphi(f - C_R)\|_{C_q}^q \leq C_q R^n M[M(|\nabla f|)^q](x) + C_q R^n R^q M(|\nabla f|)(x)^q$$

for any  $x \in B_{C_0 R}(0)$ .

*Proof.* First we claim that for  $x \in B_{2R}(0)$  one has  $(\varphi[f - C_R])_q^b(x) \leq CM(|\nabla f|)(x)$ . For  $x \in B_{2R}(0)$  Hölder's inequality implies

$$\begin{aligned} (\varphi[f - C_R])_q^b(x) &= \sup_{B \ni x} \inf_{c \in \mathbb{R}} \frac{1}{|B|^{\frac{1}{n}}} \left( \int_B |\varphi(f - C_R) - c|^q \right)^{\frac{1}{q}} \\ &\leq \sup_{B \ni x} \frac{1}{|B|^{\frac{1}{n}}} \int_B |\varphi(f - C_R) - (\varphi[f - C_R])_B| \leq C \sup_{B \ni x} \int_B |\nabla(\varphi[f - C_R])| \\ &\leq C \sup_{B \ni x} \int_B |\nabla\varphi| |f - C_R| + \sup_{B \ni x} \int_B \varphi |\nabla f| \\ &\leq \frac{C}{R} \sup_{\substack{B \ni x \\ r(B) > R}} \int_B \chi_{B_{2R}} |f - C_R| + \frac{C}{R} \sup_{\substack{B \ni x \\ r(B) \leq R}} \int_B \chi_{B_{2R}} |f - C_R| + M(|\nabla f|)(x). \end{aligned}$$

We denote the first two terms in the last line by  $I$  and  $II$ . For  $I$  observe that  $B \cap B_{2R}(0) \neq \emptyset$  implies  $B_{2R}(0) \subset 5B$  and so

$$I \leq \frac{C}{R} \sup_{\substack{B \ni x \\ r(B) > R}} \frac{1}{|B|} \int_{B_{2R}} |f - C_R| \leq \frac{C}{|B_R|} \int_{B_{2R}} |\nabla f| \leq M(|\nabla f|)(x).$$

For  $II$ , we first use the fact that the uncentered maximal function is dominated by  $c_n$  times the centered dyadic maximal function. Hence it is enough to consider for the supremum balls of the form  $B_j = B(x, R2^{-j+1}), j \geq 0$ :

$$\begin{aligned} II &\leq \frac{C}{R} \sup_{j \geq 0} \int_{B_j} |f - C_R| \leq \frac{C}{R} \sup_{j \geq 0} \int_{B_j} |f - f_{B_j}| + \frac{C}{R} \sum_{j \geq 0} |f_{B_{j+1}} - f_{B_j}| \\ &\leq \frac{C}{R} \sup_{j \geq 0} 2^{-j} R \int_{B_j} |\nabla f| + \frac{C}{R} \sum_{j \geq 0} \int_{B_{j+1}} |f - f_{B_j}| \\ &\leq CM(|\nabla f|)(x) + \frac{C}{R} \sum_{j \geq 0} 2^{-j} R \int_{B_j} |\nabla f| \leq CM(|\nabla f|)(x) \end{aligned}$$

i.e., the claim is proved. To use the claim we write

$$\|(\varphi[f - C_R])_q^b\|_q^q = \int_{B_{2R}(0)} ((\varphi[f - C_R])_q^b)^q + \int_{B_{2R}(0)^c} ((\varphi[f - C_R])_q^b)^q.$$

By the previous claim the first term is bounded by  $CR^n M[M(|\nabla f|)^q](x)$  for any  $x \in B_{C_0R}(0)$ . For the second term we will use the fact that if  $x \in B$  and  $|x| \approx 2^j R$  then for  $B \cap B_{2R} \neq \emptyset$  one needs  $r(B) \geq C2^j R$ . Thus we have

$$\begin{aligned} &\int_{B_{2R}(0)^c} ((\varphi[f - C_R])_q^b)^q \\ &= \sum_{j \geq 1} \int_{\{|x| \approx 2^j R\}} \left[ \sup_{B \ni x} \inf_{c \in \mathbb{R}} \frac{1}{|B|^{\frac{1}{n}}} \left( \int_B |\varphi(f - C_R) - c|^q \right)^{\frac{1}{q}} \right]^q dx \end{aligned}$$

$$\begin{aligned}
 \{\text{choose } c = 0\} &\leq \sum_{j \geq 1} \int_{\{|x| \approx 2^j R\}} \left[ \sup_{B \ni x} \frac{1}{|B|^{\frac{1}{n}}} \left( \int_B \chi_{B_{2R}} |f - C_R|^q \right)^{\frac{1}{q}} \right]^q \\
 &\leq C \sum_{j \geq 1} \int_{\{|x| \approx 2^j R\}} \left[ \frac{1}{2^j R} \left( \frac{1}{(2^j R)^n} \int_{B_{2R}} |f - C_R|^q \right)^{\frac{1}{q}} \right]^q dx \\
 &\leq C \sum_{j \geq 1} (2^j R)^n \frac{1}{(2^j R)^q} \frac{1}{(2^j R)^n} \left( \int_{B_{2R}} |f - C_R|^q \right) \\
 &\leq C \sum_{j \geq 1} \frac{1}{(2^j R)^q} \left( \int_{B_{2R}} |f - C_R| \right)^q |B_{2R}|^{1-q} \\
 &\leq C_q \left( \int_{B_{2R}} |\nabla f| \right)^q R^{n(1-q)} \leq C_q R^n M(|\nabla f|)(x)^q
 \end{aligned}$$

for any  $x \in B_{C_0 R}(0)$ .

To deal with the  $L^q$ -norm of  $\varphi(f - C_R)$  one applies Hölder’s inequality and Poincaré’s inequality to get  $\|\varphi(f - C_R)\|_{L^q} \leq C R^n R^q M(|\nabla f|)(x)^q$  for any  $x \in B_{C_0 R}$ . Thus the proof of the Lemma is complete.  $\square$

### 3. The Regularity Problem for boundary data in $HS^1$

We start this section by adjusting some results from [15] to the  $(R)_{HS^1}$ -case. By the proof of Theorem 3.1 in [15] and the Vitali–Hahn–Soks Theorem (see for example [8], p.155) we get for

$$(\nabla_T u)_r(Q) = \int_{B_{r/2}(A_r(Q))} \nabla u(X) \cdot \vec{T}(Q) dX$$

**Theorem 3.1.** *Assume that  $u \in W_{loc}^{1,2}(\Omega)$  solves  $Lu = 0$  and that  $\|N(\nabla u)\|_{L^1(\partial\Omega)} + \|u\|_{L^1(\Omega)} < \infty$  then*

- $u$  converges non-tangentially almost everywhere to a function  $f$  with  $f \in W^{1,1}(\partial\Omega)$ .
- If  $f = 0$  almost everywhere, then  $u \equiv 0$ .
- There exists a sequence  $r_j \rightarrow 0$  such that  $(\nabla_T u)_{r_j}$  converges in the weak\* topology of  $(L^\infty(\partial\Omega))^*$  to  $\nabla_T f$ .

We first observe that the solvability of  $(R)_{HS^1}$  can be reduced to proving the estimate (2) for smooth atoms.

**Lemma 3.2.** *Assume that (2) holds for smooth<sup>2</sup> Hardy–Sobolev atoms, then  $(R)_{HS^1}$  holds.*

*Proof.* We first claim that if (2) holds for all continuous Hardy–Sobolev atoms then  $(R)_{HS^1}$  holds. Indeed, let  $f \in HS^1 \cap C^0(\partial\Omega)$ . Then by the remark below Theorem 2.3 there exist continuous atoms  $a_j$  and scalars  $\lambda_j$  such that  $f = \sum \lambda_j a_j$ . Thus if  $u$  is the

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<sup>2</sup>We call an atom smooth if it is a restriction of an  $C^\infty(\mathbb{R}^n)$  function to  $\partial\Omega$



solution for  $f$  and  $u_j$  for  $a_j$  we have

$$\begin{aligned} \|N(\nabla u)\|_{L^1(\partial\Omega)} &\leq \sum_j |\lambda_j| \|N(\nabla u_j)\|_{L^1(\partial\Omega)} \leq C \sum_j |\lambda_j|, \\ \|u\|_{L^1(\Omega)} &\leq \sum_j |\lambda_j| \|u_j\|_{L^1(\Omega)} \leq C \sum_j |\lambda_j|. \end{aligned}$$

Since this holds for all decompositions we get  $\|N(\nabla u)\|_{L^1(\partial\Omega)} + \|u\|_{L^1(\Omega)} \leq C\|f\|_{HS^1}$  and so the claim holds. Hence it is enough to prove (2) for continuous Hardy–Sobolev atoms  $a$  under the assumption that (2) holds for smooth Hardy–Sobolev atoms.

Every continuous Hardy–Sobolev atom  $a$  can be uniformly approximated in  $HS^1$  by smooth Hardy–Sobolev atoms  $a_j$  (by the use of mollifiers). We call the corresponding weak solutions  $u$  and  $u_j$ . The maximum principle implies that  $u_j$  converges uniformly to  $u$  on  $\bar{\Omega}$ , hence  $\|u\|_{L^1(\Omega)} \leq \lim_j \|u_j\|_{L^1(\Omega)} \leq C \lim_j \|a_j\|_{HS^1} \leq C\|a\|_{HS^1}$ . Let

$$N_\varepsilon(h)(Q) = \sup_{\substack{X \in \Gamma(Q) \\ \delta(X) \geq \varepsilon}} \left( \int_{B(X, \delta(X)/2)} |h|^2 \right)^{\frac{1}{2}}$$

be the truncated below maximal function. Cacciopoli’s inequality and the uniform convergence of  $u_j$  to  $u$  imply  $N_\varepsilon(\nabla u_j - \nabla u) \rightarrow 0$  uniformly on  $\partial\Omega$ . Therefore

$$\int_{\partial\Omega} N_\varepsilon(\nabla u) \leq \lim_{j \rightarrow \infty} \int_{\partial\Omega} N_\varepsilon(\nabla u_j) \leq C \lim_j \|a_j\|_{HS^1} \leq C\|a\|_{HS^1}.$$

Since  $N_\varepsilon$  increases to  $N$  the monotone convergence theorem completes the proof.  $\square$

Recall that when we defined the  $(R)_{HS^1}$  solvability we only did it for data in  $HS^1(\partial\Omega) \cap C^0(\partial\Omega)$ . The following theorem states that this is sufficient and that it implies existence of a unique solution for any data in  $HS^1(\partial\Omega)$ . For brevity of the presentation we omit the proof as it standard.

**Theorem 3.3.** *Assume that  $(R)_{HS^1}$  holds. Given  $f \in HS^1$ , there exists a unique  $u \in L^1(\Omega)$  with  $N(\nabla u) \in L^1(\partial\Omega)$  such that  $Lu = 0$  in  $\Omega$  and  $u$  converges non-tangentially almost everywhere to  $f$ . Moreover  $(\nabla_T u)_{r_j}$  ( $r_j \rightarrow 0$ ) converges in the weak\* topology of  $(L^\infty(\Omega))^*$  to  $\nabla_T f$ .*

**3.1.  $(R)_{HS^1}$  implies  $(D^*)_{BMO}$ .** In this subsection, we explore the relation between the  $(R)_{HS^1}$  and the elliptic measure of the adjoint operator  $L^*$ .

Let us recall the definition of the elliptic measure. In [18] it was proved that for every  $g \in C^0(\partial\Omega)$  there exists a unique  $u \in W_{loc}^{1,2}(\Omega) \cap C^0(\bar{\Omega})$  such that  $L^*u = 0$  in  $\Omega$  and  $u = g$  on  $\partial\Omega$ . By the maximum principle we have  $\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}$ . Thus for every fixed  $X \in \Omega$  the map defined by

$$C^0(\partial\Omega) \ni g \mapsto u(X)$$

is a bounded linear functional on  $C^0(\partial\Omega)$ . The Riesz Representation Theorem implies the existence of a unique regular Borel measure  $\omega^X$  such that

$$u(X) = \int_{\partial\Omega} g(Q) d\omega^X(Q).$$

We will write  $\omega$  instead of  $\omega^{X_0}$  if we speak about a fixed  $X_0$ . The reverse Hölder class  $B_q$ ,  $q > 1$ , is defined as the class of all non-negative functions  $k \in L^1_{loc}$  such that

$$\left( \int_Q k^q \right)^{\frac{1}{q}} \leq C \int_Q k$$

for all cubes  $Q$ , where  $\int_Q k = \frac{1}{|Q|} \int_Q k$ . Using for example Lemma 1.4.2 in [14] one sees that

$$(D^*)_p \Leftrightarrow \omega \in B_{p'}(d\sigma).$$

By the result of [7] we also have

$$(D^*)_{BMO} \Leftrightarrow \omega \in A_\infty(d\sigma) = \bigcup_{p' > 1} B_{p'}(d\sigma).$$

Let us recall a variant of the non-tangential maximal function from [15]. For any  $h : \Omega \rightarrow \mathbb{R}$ ,  $Q \in \partial\Omega$  we consider  $S_{\varepsilon,R}(Q) = T_R(Q) \cap (\partial\Omega)_{\varepsilon R}$  and define

$$N^\varepsilon(h)(Q) = \sup_{X \in \Gamma(Q)} \left( \int_{T_\delta(X) \setminus S_{\varepsilon,\delta(X)}(\hat{X})} |\nabla h(Z)|^2 dZ \right)^{\frac{1}{2}}.$$

**Lemma 3.4.** *For all  $0 < p < \infty$  there exists  $C_1, C_2$  depending only on  $\varepsilon, p$  and  $\Omega$  such that*

$$C_1 \|N^\varepsilon(h)\|_{L^p(\partial\Omega)} \leq \|N(h)\|_{L^p(\partial\Omega)} \leq C_2 \|N^\varepsilon(h)\|_{L^p(\partial\Omega)}.$$

*Proof.* As it is stated in [15], the proof can be found in [10], Lemma 1, Section 7.  $\square$

**Lemma 3.5** ([Lemma 5.8 and Lemma 5.13 in [15]]). *Let  $0 < R < \frac{1}{4}R'$  and  $Q \in \partial\Omega$ . Assume that  $u$  is a non-negative weak solution, which vanishes on  $\Delta_{R'}(Q)$ , then there exists an  $\varepsilon > 0$  such that*

$$\int_{T_R(Q)} |\nabla u|^2 \leq C \int_{T_R(Q) \setminus S_{\varepsilon,R}(Q)} |\nabla u|^2.$$

Moreover for  $X \in T_{\frac{1}{4}R'}(Q)$  and  $\delta(X) = R$  we have

$$\frac{u(X)}{\delta(X)} \approx \left( \int_{T_R(Q) \setminus S_{\varepsilon,R}(Q)} |\nabla u|^2 \right)^{\frac{1}{2}}.$$

**Theorem 3.6.**  *$(R)_{HS^1}$  implies  $(D^*)_{BMO}$  (and also  $(D^*)_p$  for some  $1 < p < \infty$ ).*

*Proof.* We use the methods and ideas from the proof in [15] and change them a bit to suit the  $(R)_{HS^1}$  condition. Let  $\omega$  be the elliptic measure for  $L^*$ . By [7] it suffices to prove that  $\omega$  is absolutely continuous with respect to the surface measure and that  $\omega \in A_\infty(d\sigma)$ .

Choose  $R \leq \frac{1}{5}R_0$  and  $Q_0 \in \partial\Omega$ . Let  $f \in C^\infty(\partial\Omega)$  be non-negative with  $0 \leq f \leq 1$ ,  $|\nabla f| \leq \frac{C}{R}$  and

$$\begin{cases} f \equiv 0 & \text{on } \Delta_R = \Delta_R(Q_0), \\ f \equiv 1 & \text{on } \Delta_{4R} \setminus \Delta_{2R}, \\ f \equiv 0 & \text{on } \partial\Omega \setminus \Delta_{5R}. \end{cases}$$

Clearly,  $\|f\|_{L^\infty(\partial\Omega)} \leq 1$  and  $\|\nabla f\|_{L^\infty(\partial\Omega)} \leq \frac{C}{R}$ , thus  $\frac{C}{R^{n-2}}f$  is a Hardy–Sobolev  $(1, \infty)$ -atom. It follows that  $\|f\|_{HS^1} \leq CR^{n-2}$ .

Let  $u$  be the weak solution with boundary data  $f$ . Then  $C \leq u(A_R(Q_0)) \leq 1$ . By the comparison principle and Lemma 2.2 in [3] we have for  $X \in T_{R/2}(Q_0)$ :

$$\frac{u(X)}{G(X, 0)} \approx \frac{u(A_R(Q_0))}{G(A_R(Q_0), 0)} \approx \frac{1}{G^*(0, A_R(Q_0))} \approx \frac{R^{n-2}}{\omega(\Delta_R)}.$$

Lemma 3.5 and Lemma 2.2 in [3] imply

$$\left( \int_{T_{\delta(X)}(\hat{X}) \setminus S_{\varepsilon, \delta(X)}(\hat{X})} |\nabla u|^2 \right)^{\frac{1}{2}} \approx \frac{u(X)}{\delta(X)} \approx \frac{G(X, 0)}{\delta(X)} \frac{R^{n-2}}{\omega(\Delta_R)} \approx \frac{\omega(\Delta_{\delta(X)}(\hat{X}))}{\delta(X)^{n-1}} \frac{R^{n-2}}{\omega(\Delta_R)}$$

and so for  $P = \hat{X}$  we have

$$\frac{\omega(\Delta_{\delta(X)}(P))}{\delta(X)^{n-1}} \approx \frac{\omega(\Delta_R)}{R^{n-2}} \left( \int_{T_{\delta(X)}(\hat{X}) \setminus S_{\varepsilon, \delta(X)}} |\nabla u|^2 \right)^{\frac{1}{2}} \leq \frac{C\omega(\Delta_R)}{R^{n-2}} N^\varepsilon(\nabla u)(P).$$

Hence if we define  $h(P) = \sup_{0 < s < \frac{R}{2}} \frac{\omega(\Delta_s(P))}{s^{n-1}}$ , the estimate above gives that  $h(P) \leq \frac{C\omega(\Delta_R)}{R^{n-2}} N^\varepsilon(\nabla u)(P)$ . By Lemma 3.5, the assumption that  $(R)_{HS^1}$  holds and the doubling property of  $\omega$  we see that  $\omega$  is absolutely continuous with respect to  $d\sigma$ , i.e.,  $\omega = k \, d\sigma$  for some  $k \in L^1(d\sigma)$ .

In order to show that  $\omega \in A_\infty(d\sigma)$  it is enough to show that  $\|\omega\|_{L(\log L)(d\tilde{\sigma})} \leq C\|\omega\|_{L^1(d\tilde{\sigma})}$  for all  $d\tilde{\sigma} = \frac{\chi_\Delta}{|\Delta|} d\sigma$  (see for example [11]), where we can assume without losing generality that  $r(\Delta) \leq R_0$ . We have by [24] that

$$\|k\|_{L(\log L)(d\tilde{\sigma})} \leq C\|M_\Delta k\|_{L^1(d\tilde{\sigma})},$$

where  $M_\Delta$  denotes the Hardy–Littlewood maximal function over all balls contained in  $\Delta$ . By the doubling property of  $\omega$  we see that

$$\begin{aligned} \|M_\Delta k\|_{L^1(d\tilde{\sigma})} &\leq C \int_{\Delta_{R/2}} h(P) \, d\sigma(P) \leq \frac{C\omega(\Delta_R)}{R^{n-2}} \int_{\Delta_{R/2}} N^\varepsilon(\nabla u)(P) \, d\sigma \\ &\leq \frac{C\omega(\Delta_R)}{R^{n-2}} \frac{1}{R^{n-1}} R^{n-2} = C\|\omega\|_{L^1(d\tilde{\sigma})}, \end{aligned}$$

which concludes that  $\omega \in A_\infty(d\sigma)$  proving our claim. □

**3.2. A new proof for:  $(R)_p$  implies  $(R)_{HS^1}$ .** In [15] Kenig and Pipher used localization argument to prove the implication that  $(R)_p$  implies  $(R)_{HS^1}$ . In order to prove the same result without the localization theorem of [15] we need the following:

**Lemma 3.7 ([Lemma 2.5 in [23]]).** *Let  $u$  be a weak solution for  $L$  in  $\Omega$ , which vanishes on  $\Delta_{5R}(Q)$ . Then for any  $X \in T_{2R}(Q)$  we have*

$$|u(X)| \approx \frac{G(X, 0)}{G(A_R(Q), 0)} \left( \int_{T_{4R}(Q)} |u|^2 \right)^{\frac{1}{2}}.$$

The next Lemma is part of the proof of Theorem 2.9 in [23]:

**Lemma 3.8.** *Assume that  $\omega \in A_\infty(d\sigma)$ . Then for  $u$  and  $R$  as in Lemma 3.7 we get*

$$\left( \int_{\Delta_R(Q)} \left( \frac{u}{\delta} \right)_R^* \right) \leq \frac{C}{R} \left( \int_{T_{4R}(Q)} |u|^2 \right)^{\frac{1}{2}}.$$

*Proof.* By Lemma 3.7 we have for any  $P \in \Delta_R(Q)$

$$\left( \frac{u}{\delta} \right)_R^*(P) \leq C \frac{1}{G(A_R, 0)} \left( \int_{T_{4R}(Q)} |u|^2 \right)^{\frac{1}{2}} \left( \frac{G(\cdot, 0)}{\delta(\cdot)} \right)_R^*(P).$$

Lemma 2.2 in [3] and (1.3) Theorem in [12] imply

$$\frac{G(X, 0)}{\delta(X)} \approx \frac{\omega(\Delta_{\delta(X)}(\hat{X}))}{\delta(X)^{n-1}}.$$

Thus for  $h_R(Q) = \sup_{X \in \Gamma(Q) \cap B_R(Q)} \frac{\omega(\Delta_{\delta(X)}(\hat{X}))}{\delta(X)^{n-1}}$  we get

$$\begin{aligned} \left( \int_{\Delta_R(Q)} \left( \frac{u}{\delta} \right)^* \right) &\leq \frac{C}{G(A_R(Q), 0)} \left( \int_{T_{4R}(Q)} |u|^2 \right)^{\frac{1}{2}} \left( \int_{\Delta_R(Q)} h_R \right) \\ &\leq \frac{C}{G(A_R(Q), 0)} \left( \int_{T_{4R}(Q)} |u|^2 \right)^{\frac{1}{2}} \frac{\omega(\Delta_R(Q))}{R^{n-1}}, \end{aligned}$$

where for the last step we used the  $A_\infty(d\sigma)$  condition. Thus

$$\left( \int_{\Delta_R(Q)} \left( \frac{u}{\delta} \right)^* \right) \leq \frac{C}{R} \left( \int_{T_{4R}(Q)} |u|^2 \right)^{\frac{1}{2}}. \quad \square$$

The result below takes care of the estimate for non-tangential maximal function away from the support of an  $(1, \infty)$ -atom.

**Theorem 3.9.** *Assume that  $\omega \in A_\infty(d\sigma)$ , where  $\omega$  is the elliptic measure of the operator  $L^*$ . Let  $f$  be a smooth Hardy–Sobolev  $(1, \infty)$ -atom corresponding to the surface ball  $\Delta_R(Q_0)$ . Then  $u$  the weak solution for  $L$  with boundary data  $f$  satisfies*

$$\|N(\nabla u)\|_{L^1(\partial\Omega \setminus \Delta_{8R}(Q_0))} \leq C$$

for a constant  $C$  independent of  $f$  and  $R$ .

*Proof.* Without loosing generality we can assume that  $R \leq R_0$  and that  $f$  is non-negative. Since  $f$  is a smooth Hardy–Sobolev  $(1, \infty)$ -atom for  $\Delta_R(Q)$  we have  $|f| \leq \frac{C}{R^{n-2}}$ . Thus for  $X \in \Omega \setminus T_{2R}(Q)$  Lemma 2.2 in [3] and Theorem 1.8 in [12] imply

$$(4) \quad u(X) \leq CR^{2-n} w^X(\Delta_R(Q_0)) \approx G(X, A_R(Q_0)) \leq C \frac{R^\alpha}{|X - Q_0|^{n+\alpha-2}}.$$

Define  $R_j = \{Q \in \partial\Omega : |Q - Q_0| \approx 2^j R\}$  for  $j \geq 3$ . For  $Q \in R_j$  and  $X \in \Gamma(Q)$  with  $|X - Q| \geq 2^j R$  we have by (4) and Cacciopoli's inequality

$$\begin{aligned} \left( \int_{B_{\frac{\delta(X)}{2}}(X)} |\nabla u|^2 \right)^{\frac{1}{2}} &\leq \frac{C}{\delta(X)} \left( \int_{B_{\frac{\delta(X)}{2}}(X)} u^2 \right)^{\frac{1}{2}} \\ &\leq \frac{C}{\delta(X)} \frac{R^\alpha}{(2^j R)^{n+\alpha-2}} \leq \frac{C}{2^{j\alpha}} \frac{1}{(2^j R)^{n-1}}. \end{aligned}$$

Therefore

$$\int_{R_j} N(\nabla u)(Q) \, d\sigma(Q) \leq \int_{R_j} N_{2^j R}(\nabla u)(Q) \, d\sigma(Q) + \frac{C}{2^{j\alpha}},$$

where  $N_{2^j R}$  is as before the truncated non-tangential maximal function at the height  $2^j R$ . By Cacciopoli inequality in the interior we get

$$\int_{R_j} N_{2^j R}(\nabla u)(Q) \, d\sigma(Q) \leq C \int_{R_j} \left(\frac{u}{\delta}\right)^*(Q) \, d\sigma(Q).$$

Thus if we cover  $R_j$  with finite many balls  $\Delta_\alpha^j$  with radius comparable to  $2^j R$  and apply Lemma 3.8 to each of the balls we get

$$\begin{aligned} \int_{R_j} N_{2^j R}(\nabla u)(Q) \, d\sigma(Q) &\leq C(2^j R)^{n-1} \sum_\alpha \int_{\Delta_\alpha^j} \left(\frac{u}{\delta}\right)^* \\ &\leq \frac{C(2^j R)^{n-1}}{2^j R} \sum_\alpha \left( \int_{T_{\Delta_\alpha^j}} u^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where  $T_{\Delta_\alpha^j} = T_{r_\alpha^j}(Q_\alpha^j)$  for  $r_\alpha^j = r(\Delta_\alpha^j)$  and  $Q_\alpha^j$  the center of  $\Delta_\alpha^j$ . Inequality (4) implies that each term is bounded by  $\frac{R^\alpha}{(2^j R)^{n+\alpha-2}}$ , thus

$$\int_{R_j} N_{2^j R}(\nabla u)(Q) \, d\sigma(Q) \leq C(2^j R)^{n-2} \frac{R^\alpha}{(2^j R)^{n+\alpha-2}} \leq \frac{C}{2^{j\alpha}}.$$

Therefore  $\int_{R_j} N(\nabla u) \, d\sigma \leq \frac{C}{2^{j\alpha}}$ , which means that we can take the sum in  $j$  to get

$$\int_{\partial\Omega \setminus \Delta_{8R}(Q)} N(\nabla u) \, d\sigma \leq C. \quad \square$$

Thanks to Theorem 3.9 we now can reprove Theorem 5.2 of [15].

**Theorem 3.10.**  $(R)_p$  implies  $(R)_{HS^1}$ .

*Proof.* By Lemma 3.2 it is enough to show that (2) holds for smooth Hardy–Sobolev  $(1, \infty)$ -atoms. Let  $f$  be a smooth Hardy–Sobolev  $(1, \infty)$ -atom corresponding to  $\Delta_R(Q)$  and  $u$  the weak solution for  $f$ . Without losing generality we can assume that  $R \leq R_0$ . Either by slightly adjusting proof for Theorem 3.6 (or by Theorem 5.4 [15]) we know that  $(D^*)_p$  holds and therefore the elliptic measure  $\omega$  of the operator  $L^*$  belongs to  $A_\infty(d\sigma)$ . From this by Theorem 3.9 we obtain

$$\|N(\nabla u)\|_{L^1(\partial\Omega \setminus \Delta_{8R}(Q))} \leq C.$$

For the  $\Delta_{8R}(Q)$  part, we use Hölder’s inequality and the  $(R)_p$  condition to get

$$\begin{aligned} \|N(\nabla u)\|_{L^1(\Delta_{8R}(Q))} &\leq C|\Delta_R(Q)|^{\frac{1}{p'}} \|N(\nabla u)\|_{L^p(\Delta_{8R}(Q))} \\ &\leq C|\Delta_R(Q)|^{\frac{1}{p'}} \|f\|_{H^{1,p}(\partial\Omega)} \leq C, \end{aligned}$$

since  $f$  is a  $(1, \infty)$ -atom for  $\Delta_R(Q)$ .

It remains to show that  $\|u\|_{L^1(\Omega)} \leq C$ . From (4) we see that for  $X \in \Omega \setminus T_{2R}(Q)$  we have

$$u(X) \leq CG(X, A_R(Q))$$

and so

$$\|u\|_{L^1(\Omega)} \leq C\|u\|_{L^1(\Omega_{R_0})} + \|N(\nabla u)\|_{L^1(\partial\Omega)} \leq C,$$

which completes the proof. □

**3.3.  $(R)_{HS^1}$  implies  $(R)_p$  for some  $1 < p < \infty$ .** We are now ready to establish the main result of this paper, namely the implication that  $(R)_{HS^1}$  implies  $(R)_p$  for some  $1 < p < \infty$ . In the course of thinking about this problem we discovered that there are two possible ways to establish this result. One is to adapt the proof in [15] where for  $(R)_p$  implies  $(R)_{p+\varepsilon}$  was established. The other way is motivated by the proof of the main Theorem in [23] (adjusted with the aid of Lemma 2.5). We decided we prefer the second method as it avoids the use of a localization theorem and real variable techniques with rather lengthy proofs. We present this method here.

We define

$$E(\lambda) = \{P \in \partial\Omega : M(N(\nabla u))(P) > \lambda\}.$$

**Theorem 3.11.** *Assume that  $(R)_{HS^1}$  holds. Choose any  $p \in (1, \infty)$  for which the  $(D^*)_{p'}$  holds. Let  $f \in C^\infty(\partial\Omega)$  and  $u$  be the corresponding weak solution of the Dirichlet problem. Then there exist positive constants  $\varepsilon, \eta, C_0$  such that*

$$(5) \quad |E(A\lambda)| \leq \varepsilon^{1+\eta}|E(\lambda)| + |\{P \in \partial\Omega : M(M(|\nabla f|)) > \gamma\lambda\}|$$

for all  $\lambda > \lambda_0 = C_0 \int_{\partial\Omega} N(\nabla u) d\sigma$ ,  $\gamma = \gamma(\varepsilon)$  and  $A = \varepsilon^{-\frac{1}{p}}$ .

*Proof.* This proof for the  $(R)_p$  case can be found in Lemma 3.4 in [23]. The weak  $(1, 1)$  inequality for the Hardy Littlewood maximal function implies

$$E(\lambda) \leq \frac{C}{\lambda} \int_{\partial\Omega} N(\nabla u) d\sigma \leq \frac{C}{\lambda} \frac{\lambda_0}{C_0}.$$

Thus by choosing  $C_0 = C_0(\Omega)$  sufficiently large we can ensure that  $E(\lambda) \leq \frac{1}{2}|\Delta_{R_0/4}|$ , where  $\Delta_{R_0/4}$  is any surface ball with radius  $R_0/4$ . Thus  $E(\lambda)^c \cap \Delta_{R_0/4} \neq \emptyset$  for  $\lambda > \lambda_0$ .

Let  $\{Q_k\}$  be a Whitney decomposition of  $E(\lambda)$ , i.e.,

- $E(\lambda) = \bigcup_k Q_k$ ,
- $\sum_k \chi_{Q_k} \leq K$ ,
- $3Q_k \cap E(\lambda)^c \neq \emptyset$ .

To prove the lemma it is sufficient to prove that

$$(6) \quad Q_k \cap \{M(M(|\nabla f|)) \leq \gamma\lambda\} \neq \emptyset \text{ implies } |E(A\lambda) \cap Q_k| \leq \varepsilon^{1+\eta}|Q_k|.$$

Indeed, since  $E(A\lambda) \subset E(\lambda)$  it follows that for  $\varepsilon$  small enough such that  $K\varepsilon^{1+\eta} \leq \varepsilon^{1+\frac{\eta}{2}}$  we have

$$\begin{aligned} |E(A\lambda)| &\leq \sum_{\{k: Q_k \cap \{M_{R_0/2}(M_{R_0}(|\nabla f|)) \leq \gamma\lambda\} \neq \emptyset\}} |E(A\lambda) \cap Q_k| + |\{M(M(|\nabla f|)) \geq \gamma\lambda\}| \\ &\leq \varepsilon^{1+\frac{\eta}{2}} |E(\lambda)| + |\{M(M(|\nabla f|)) \geq \gamma\lambda\}|, \end{aligned}$$

which is the statement of our theorem.

Hence we focus on establishing (6). By the properties imposed from the Whitney decomposition on  $Q_k$  we have for  $P \in Q_k$ :

$$M(N(\nabla u))(P) \leq \max\{M_{5Q_k}(N(\nabla u)), C_1\lambda\}$$

for some  $C_1 = C_1(\Omega)$  depending only on the geometry of our domain. Here  $M_Q$  is a modified version of the maximal function

$$M_Q(f)(P) = \sup_{\substack{\tilde{Q} \ni P \\ \tilde{Q} \subset Q}} \int_{\tilde{Q}} |f|.$$

Take now  $A$  larger than  $C_1$  we see by the properties of the Whitney decomposition on  $Q_k$  that

$$(7) \quad |Q_k \cap E(A\lambda)| \leq |\{P \in Q_k : M_{5Q_k}(N(\nabla u))(P) > A\lambda\}|.$$

Let  $v$  be a weak solution to the Dirichlet problem for the operator  $L$  in the domain  $\Omega$  with boundary data  $\varphi(f - \alpha)$ , where  $\varphi \in C^\infty(\partial\Omega)$  with  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 1$  on  $6Q_k$ ,  $\text{supp } \varphi \subset 10Q_k$  and  $\alpha = \int_{10Q_k} f$ . Then

$$\begin{aligned} |Q_k \cap E(A\lambda)| &\leq |\{P \in Q_k : M_{5Q_k}[N(\nabla(u - v))] > \frac{A\lambda}{2}\}| \\ &\quad + |\{P \in Q_k : M_{5Q_k}[N(\nabla v)] > \frac{A\lambda}{2}\}| \\ &\leq \frac{C}{(A\lambda)^{\bar{p}}} \int_{5Q_k} N(\nabla(u - v))^{\bar{p}} d\sigma + \frac{C}{A\lambda} \int_{5Q_k} N(\nabla v) d\sigma = I + II \end{aligned}$$

by the weak  $(\bar{p}, \bar{p})$  and the weak  $(1, 1)$  inequality. We choose  $\bar{p} > p$  so that  $(D^*)_{\bar{p}'}$  still holds. Since  $(R)_{HS^1}$  holds, Lemma 2.5 for  $q = 1$  implies for the second term

$$II \leq \frac{C}{A\lambda} \|\varphi(f - \alpha)\|_{HS^1} \leq \frac{C}{A\lambda} |Q_k| M(M(|\nabla f|))(Q)$$

for any  $Q \in 5Q_k$ . Thus we can choose a  $Q$  from  $Q_k \cap \{M(M(|\nabla f|)) \leq \gamma\lambda\}$  to get  $II \leq \frac{C\gamma}{A} |Q_k|$ .

For  $I$  observe that  $u - v - \alpha$  is a weak solution with vanishing boundary data on  $6Q_k$ . For this term we use the Main Lemma of [23], namely the reverse Hölder inequality for  $N(\nabla u)$ .

**Lemma 3.12 (Theorem 2.9 in [23]).** *Assume that  $(D^*)_{p'}$  holds. Let  $w$  be a weak solution which vanishes on  $\Delta_{4R}(Q)$ . Then*

$$\left( \int_{\Delta_R(Q_0)} N(\nabla w)^p d\sigma \right)^{\frac{1}{p}} \leq \int_{\Delta_{4R}(Q_0)} N(\nabla w) d\sigma.$$

Hence it follows that

$$\begin{aligned} I &\leq \frac{C}{(A\lambda)^{\bar{p}}} |Q_k| \left( \int_{6Q_k} N(\nabla(u-v)) \, d\sigma \right)^{\bar{p}} \\ &\leq \frac{C}{(A\lambda)^{\bar{p}}} |Q_k| \left[ \left( \int_{6Q_k} N(\nabla u) \, d\sigma \right)^{\bar{p}} + \left( \int_{6Q_k} N(\nabla v) \, d\sigma \right)^{\bar{p}} \right] \\ &\leq \frac{C}{(A\lambda)^{\bar{p}}} [\lambda^{\bar{p}} + (\gamma\lambda)^{\bar{p}}] |Q_k| \leq \frac{C}{A^{\bar{p}}} |Q_k|. \end{aligned}$$

To get the last line we have used the facts that  $3Q_k \cap E(\lambda)^c \neq \emptyset$  as well as  $Q_k \cap \{M(M(|\nabla f|)) \leq \gamma\lambda\} \neq \emptyset$  and that  $(R)_{HS^1}$  holds. In the last step we hid  $\gamma$  into a generic constant  $C$ , we can do this since  $\gamma > 0$  will be chosen small in the next step. Collecting all estimates together we see that

$$\begin{aligned} |Q_k \cap E(A\lambda)| &\leq |Q_k| \left( \frac{C\gamma}{A} + \frac{C}{A^{\bar{p}}} \right) \\ &= |Q_k| (C\gamma\varepsilon^{\frac{1}{p}} + C\varepsilon^{\bar{p}/p}) \\ &= |Q_k| \varepsilon^{1+\eta} (C\gamma\varepsilon^{\frac{1}{p}-1-\eta} + C\varepsilon^\eta), \end{aligned}$$

for  $\eta = \frac{1}{2}(\bar{p}/p - 1) > 0$ . We now choose  $\varepsilon > 0$  small enough to make the second term less than  $\frac{1}{2}$  and then choose  $\gamma$  such that the first term is smaller than  $\frac{1}{2}$ . Therefore

$$|Q_k \cap E(A\lambda)| \leq \varepsilon^{1+\eta} |Q_k|,$$

which finishes the proof. □

With (5) established the proof of the Main Theorem in [23] implies the our main result. For completeness we include the proof.

**Theorem 3.13.** *There exists  $1 < p < \infty$  such that  $(R)_{HS^1}$  implies  $(R)_p$ .*

*Proof.* By Theorem 3.6 there exists  $1 < p < \infty$  such that  $(D^*)_{p'}$  holds. We multiply (5) both sides with  $\lambda^{p-1}$  and integrate then over  $(\lambda_0, \Lambda)$  to get

$$\int_{\lambda_0}^{\Lambda} |E(A\lambda)| \lambda^{p-1} \, d\lambda \leq \varepsilon^{1+\eta} \int_{\lambda_0}^{\Lambda} |E(\lambda)| \lambda^{p-1} \, d\lambda + C \int |\nabla f|^p \, d\sigma,$$

where for the last term we used the boundedness of the Hardy–Littlewood maximal function on  $L^p$  twice. Using the change of variables  $A\lambda \mapsto \lambda$  we get

$$\int_{A\lambda_0}^{A\Lambda} |E(\lambda)| \lambda^{p-1} A^{1-p} A^{-1} \, d\lambda \leq \varepsilon^{1+\eta} \int_{\lambda_0}^{\Lambda} |E(\lambda)| \lambda^{p-1} \, d\lambda + C \int |\nabla f|^p \, d\sigma.$$

By the definition of  $A$  we have  $A^{1-p} A^{-1} = \varepsilon$ . Therefore the previous inequality simplifies to

$$\int_{A\lambda_0}^{A\Lambda} |E(\lambda)| \lambda^{p-1} \, d\lambda \leq \varepsilon^\eta \int_{\lambda_0}^{\Lambda} |E(\lambda)| \lambda^{p-1} \, d\lambda + C(\varepsilon) \int |\nabla f|^p \, d\sigma.$$



For  $\varepsilon$  small enough such that  $\varepsilon^\eta \leq \frac{1}{2}$  and  $\Lambda$  large enough such that  $\Lambda \geq A\lambda_0$ , we can hide the part  $\varepsilon^\eta \int_{A\lambda_0}^\Lambda |E(\lambda)|\lambda^{p-1} d\lambda$  on the left hand side to get

$$\int_{A\lambda_0}^{A\Lambda} |E(\lambda)|\lambda^{p-1} d\lambda \leq C \int_{\lambda_0}^{A\lambda_0} |E(\lambda)|\lambda^{p-1} d\lambda + C \int |\nabla f|^p d\sigma.$$

By adding  $\int_0^{A\lambda_0} |E(\lambda)|\lambda^{p-1} d\lambda$  on both sides we end up with

$$(8) \quad \int_0^{A\Lambda} |E(\lambda)|\lambda^{p-1} d\lambda \leq C \int_0^{A\lambda_0} |E(\lambda)|\lambda^{p-1} d\lambda + C \int |\nabla f|^p d\sigma.$$

By the definition of  $\lambda_0$ , the  $(R)_{HS^1}$ -condition and Hölder's inequality the first term of the right hand side is bounded by

$$C \left( \int_{\partial\Omega} N(\nabla u) d\sigma \right)^p \leq C \|f\|_{HS^1}^p \leq C \|f\|_{H^{1,p}(\partial\Omega)}^p.$$

Thus sending  $\Lambda \rightarrow \infty$  in (8) gives  $\int_{\partial\Omega} (M(N(\nabla u)))^p \leq C \|f\|_{H^{1,p}(\partial\Omega)}^p$ , i.e.,

$$(9) \quad \|N(\nabla u)\|_{L^p(\partial\Omega)} \leq C \|f\|_{H^{1,p}(\partial\Omega)}.$$

It remains to check that  $\|u\|_{L^p(\Omega)} \leq C \|f\|_{H^{1,p}(\partial\Omega)}$ . By the usual splitting into the positive and negative part, we can without losing generality assume that  $f$  is non-negative. We have

$$\begin{aligned} \|u\|_{L^p(\Omega)} &\leq C \|N(\nabla u)\|_{L^p(\partial\Omega)} + C \|u\|_{L^1(\Omega_{R_0})} \\ &\leq C \|f\|_{H^{1,p}(\partial\Omega)} + \|f\|_{HS^1} \leq C \|f\|_{H^{1,p}(\partial\Omega)}. \end{aligned} \quad \square$$

The  $p$  in Theorem 3.13 was determined by the  $p'$  for which  $(D^*)_{p'}$  holds. Thus Theorem 3.13 allows to conclude the following:

**Corollary 3.14.** *Let  $L$  be an elliptic operator with the elliptic measure of the adjoint  $L^*$  operator in  $A_\infty(d\sigma)$ . Then either*

$$\begin{cases} (i)_a (D^*)_{p'} \text{ implies } (R)_p \text{ for all } p \in (1, \infty) \text{ for which } (D^*)_{p'} \text{ holds} \\ (i)_b (D^*)_{BMO} \text{ implies } (R)_{HS^1} \end{cases}$$

or

$$\begin{cases} (ii)_a (R)_p \text{ is not solvable for any } p \in (1, \infty) \\ (ii)_b (R)_{HS^1} \text{ is not solvable.} \end{cases}$$

It remains an open question whether the second alternative in Corollary 3.14 does happen or whether  $(D^*)_{p'}$  always implies  $(R)_p$ . By Corollary 3.14, Theorem 3.9, part of the proof of Theorem 3.10 regarding the  $\|u\|_{L^1(\Omega)}$  norm and Lemma 3.2, we get the following:

**Corollary 3.15.** *Assume that for all smooth Hardy–Sobolev  $(1, \infty)$ -atoms  $f$  the weak solution  $u$  of the equation  $Lu = 0$  with Dirichlet data  $f$  satisfies*

$$\int_{8\Delta_R(Q)} N(\nabla u) \leq C,$$

where  $\Delta_R(Q)$  is a surface ball on which the atom  $f$  is supported and  $C$  is a constant independent of  $f$ . Then

$$(D^*)_{p'} \quad \text{implies} \quad (R)_p.$$

### Acknowledgments

The second author thanks his Ph.D supervisor Martin Dindoš for the encouragement and for many helpful discussions and suggestions. The second author is also grateful to Jill Pipher for her help concerning the endpoint of Gehring's lemma. The results in this paper also appear in the PhD thesis [17], where amongst other things the same problem for elliptic operators with small drift terms is considered.

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