C^1-Boundary Regularity of Planar Infinity Harmonic Functions

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Abstract. We prove that if \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with \( C^2 \)-boundary and \( g \in C^2(\mathbb{R}^2) \), then any viscosity solution \( u \in C(\bar{\Omega}) \) of the infinity Laplacian equation (1.1) is \( C^1(\bar{\Omega}) \). The interior \( C^1 \) and \( C^{1,\alpha} \)-regularity of \( u \) in dimension two has been proved by Savin [20], and Evans and Savin [15], respectively. We also show that for any \( n \geq 3 \), if \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( C^1 \)-boundary and \( g \in C^1(\mathbb{R}^n) \), then the solution \( u \) of equation (1.1) is differentiable on \( \partial \Omega \). This can be viewed as a supplementary result to the much deeper interior differentiability theorem by Evans and Smart [16,17].

1. Introduction

In 1960s, Aronsson [3] introduced the notion of the absolutely minimizing Lipschitz extension. Namely, \( u \in W^{1,\infty}(\Omega) \) is said to be an absolutely minimizing Lipschitz extension in some bounded open subset \( \Omega \subset \mathbb{R}^n \) if for any open set \( V \subset \Omega \), we have that

\[
\sup_{x \neq y \in \partial V} \frac{|u(x) - u(y)|}{|x - y|} = \sup_{x \neq y \in V} \frac{|u(x) - u(y)|}{|x - y|}.
\]

The results of Crandall et al. [13] imply that the above definition is equivalent to saying that for any open set \( V \subset \Omega \) and \( v \in W^{1,\infty}(V) \),

\[
u\in \partial V = v\in \partial V \Rightarrow \|Du\|_{L^\infty(V)} \leq \|Dv\|_{L^\infty(V)}.
\]

Jensen proved in [18] that \( u \in W^{1,\infty}(\Omega) \) is an absolutely minimizing Lipschitz extension with a given Lipschitz continuous boundary data \( g \) iff \( u \) is a viscosity solution of the infinity Laplacian equation:

\[
\Delta_\infty u := \sum_{1 \leq i, j \leq n} u_{x_i} u_{x_j} u_{x_i} u_{x_j} = 0 \quad \text{in } \Omega,
\]

\[
u = g \quad \text{on } \partial \Omega.
\]

Moreover, (1.1) has a unique viscosity solution with any given continuous boundary data. The reader can refer to Armstrong and Smart [2] for a nice new proof of Jensen’s uniqueness theorem. After Jensen’s celebrated work, there has been an explosion of interest in the infinity Laplacian equation and its generalizations. Two natural extensions include: (i) absolute minimal Lipschitz extensions with respect to more general metrics on \( \mathbb{R}^n \) (see, e.g., [7]); and (ii) absolute minimizers of quasiconvex functions of the gradient (see, e.g., [1,4–6,9,10]). We would like to mention beautiful connections between the infinity harmonic functions and the differential game theory first discovered by Peres et al. [19] and later by Barron et al. [8] for Aronsson’s equations.

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Viscosity solutions of the infinity Laplacian equation (1.1) are also called *infinity harmonic functions.* One of the most important problems concerning infinity harmonic function is its $C^1$-regularity. When $n = 2$, this has been proved by Savin [20], and the $C^{1,\alpha}$-regularity was subsequently obtained by Evans and Savin [15]. Very recently, Evans and Smart [16,17] made a breakthrough in dimensions $n \geq 3$ by showing that any infinity harmonic function is differentiable everywhere. While the continuity of gradient of $u$ remains an open question.

In this short article, we will study the boundary regularity of infinity harmonic functions. We are able to prove

**Theorem 1.1.** Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain with $\partial \Omega \in C^2$. Assume that $g \in C^2(\mathbb{R}^2)$ and $u \in C(\overline{\Omega})$ is the viscosity solution of the infinity Laplacian equation (1.1). Then $u \in C^1(\overline{\Omega})$. Moreover, for any $\delta > 0$, there exists $\epsilon_\delta > 0$ depending only on $||g||_{C^2(\mathbb{R}^2)}$ and $||\partial \Omega||_{C^2}$ such that for $x, y \in \Omega$,

\begin{equation}
|x - y| \leq \epsilon_\delta \Rightarrow |Du(x) - Du(y)| \leq \delta.
\end{equation}

Here $||\partial \Omega||_{C^2}$ is understood as follows: We say that $||\partial \Omega||_{C^2} \leq C < +\infty$, if there exist $0 < r_C < R_C < +\infty$ such that $\Omega \subset B_{R_C}(O)$ and for any $x = (x_1, x_2) \in \partial \Omega$, after suitable rotation, there exists $f(x)(t) \in C^2(\mathbb{R})$ such that $||f(x)||_{C^2(\mathbb{R})} \leq C$, $f(x)(0) = \frac{d}{dt}f(x)(0) = 0$ and for all $r \in (0, r_C)$

$$B_r(x) \cap \partial \Omega = \{x\} + (B_r(O) \cap \{y = (y_1, y_2) \mid y_2 > f(x)(y_1)\})$$

and

$$B_r(x) \cap \partial \Omega = \{x\} + (B_r(O) \cap \{y = (y_1, y_2) \mid y_2 > f(x)(y_1)\}).$$

The $C^2$ assumption can actually be relaxed to $C^{1,1}$ and the above definition of norm is equivalent to saying that $\Omega$ has a uniform interior and exterior ball condition.

**Sketch of the ideas of proof of Theorem 1.1:** The $C^2$-regularities of both $\partial \Omega$ and $g$ assure the existence of classical solutions of the eikonal equation: $|Du| = \text{constant}$ near $\partial \Omega$, which serve as barrier functions. Using interior estimate established in [20] and routine scaling arguments, to prove Theorem 1.1, it suffices to show that $u$ locally lies between two barrier functions that are $C^1$-close. One side bound comes easily from the method of characteristics. The proof for the other side bound is more tricky and we utilize some ideas of [20], but is simpler than [20]. The $C^2$-regularity assumption is necessary to implement the method of characteristics. It remains an interesting question whether Theorem 1.1 holds when $g$ and $\partial \Omega$ are assumed to be $C^1$, a more natural assumption. It is also an interesting question to ask whether the $C^{1,\alpha}$-interior regularity by Evans and Savin [15] holds up to the boundary for infinity harmonic functions.

Using the tool of *comparison with cones* by Crandall et al. [13], we also establish the differentiability of infinity harmonic functions on the boundary in all dimensions.

**Theorem 1.2.** For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial \Omega \in C^1$ and $g \in C^1(\mathbb{R}^n)$. Assume that $u$ is the viscosity solution of the infinity Laplacian equation (1.1). Then $u$ is differentiable on the boundary, i.e., for any $x_0 \in \partial \Omega$, there exists $Du(x_0) \in \mathbb{R}^n$ such that

$$u(x) = u(x_0) + Du(x_0) \cdot (x - x_0) + o(|x - x_0|) \quad \text{for all } x \in \overline{\Omega}.$$
Remark 1.1. The interior differentiability of infinity harmonic functions in all dimensions has been proved by Evans and Smart [16]. It is not clear to us whether the $C^1$ assumption of $g$ and $\partial \Omega$ in Theorem 1.2 can be relaxed to be everywhere differentiable. We need the continuity of the gradient of $g$ and $\partial \Omega$ to derive (2.1) in the next section.

2. Boundary differentiability and proof of Theorem 1.2

In this section, we will assume that $\partial \Omega \in C^1$ and $g \in C^1(\mathbb{R}^n)$ and $u \in C(\overline{\Omega})$ is a viscosity solution of (1.1). We will prove the boundary differentiability Theorem 1.2.

For $x \in \Omega$ and $r > 0$, we define

$$S_r^+(x) = \max_{y \in \partial(B_r(x) \cap \Omega) \setminus \{x\}} \frac{u(y) - u(x)}{|y - x|},$$

and

$$S_r^-(x) = \max_{y \in \partial(B_r(x) \cap \Omega) \setminus \{x\}} \frac{u(x) - u(y)}{|y - x|}.$$

By the comparison principle with cones as in [12,13], it is readily seen that both $S_r^+$ and $S_r^-$ are monotone increasing functions of $r > 0$. Hence, for any $x \in \Omega$, we have that

$$S^+(x) = \lim_{r \to 0} S_r^+(x) \quad \text{and} \quad S^-(x) = \lim_{r \to 0} S_r^-(x)$$

exist. Let

$$S(x) = \max \{S^+(x), S^-(x)\}.$$

Then it is standard that the following properties of $S(x)$ hold, whose proof is left to the readers. Note that by Evan and Smart [16,17], $Du(x)$ exists for all $x \in \Omega$.

Lemma 2.1. (i) For $x \in \Omega$,

$$S^+(x) = S^-(x) = S(x) = |Du(x)|.$$

(ii) For $x \in \partial \Omega$,

$$\min \{S^+(x), S^-(x)\} \geq |DTg(x)|,$$

where $DTg$ denotes the tangential gradient of $g$ on $\partial \Omega$.

(iii) $S(x)$ is upper-semicontinuous, i.e.,

$$\limsup_{y \to x} S(y) \leq S(x) \quad \forall x \in \overline{\Omega}.$$

We first prove Aronsson’s tightness property for infinity harmonic functions in $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n \geq 0\}$, such a property was first proved by Crandall and Evans [13] for infinity harmonic functions in $\mathbb{R}^n$.

Lemma 2.2. Suppose $w = w(x', x_n) \in W^{1,\infty}(\mathbb{R}^n_+)$ and

$$|Dw(x)| \leq 1 \quad \text{a.e. } x \in \mathbb{R}^n_+.$$

Let $e = (e', e_n) \in \mathbb{R}^n$ be a unit vector with $e_n \geq 0$. Assume that $w(x', 0) = e' \cdot x'$ for all $x' \in \mathbb{R}^{n-1}$ and for $t > 0$ $w(te) = t$. Then $w(x) = e \cdot x$ for $x \in \mathbb{R}^n_+$. 

Proof. For \( t > 0 \) and \( x = (x', x_n) \in \mathbb{R}^n_+ \), we have that
\[
 w(te) - w(x) \leq |te - x| 
\]
so that
\[
 w(x) \geq t - |te - x| = \frac{2e \cdot x - t^{-1}|x|^2}{1 + |e - t^{-1}x|},
\]
This, after taking \( t \to +\infty \), implies
\[
 w(x) \geq e \cdot x, \quad \forall x \in \mathbb{R}^n_+.
\]
It remains to show
\[
 w(x) \leq e \cdot x, \quad \forall x \in \mathbb{R}^n_+.
\]
Case 1: \( e_n = 0 \). Then we have \( -te \in \mathbb{R}^n_+ \) and
\[
 w(x) \leq w(-te) + |x + te| = -t + |x + te|.
\]
Hence
\[
 -w(x) \geq t - |x + te| = \frac{-2e \cdot x - t^{-1}|x|^2}{1 + |e + t^{-1}x|},
\]
so that (2.2) follows by taking \( t \to +\infty \).
Case 2: \( e_n > 0 \). Then we have that for any \( x \in \mathbb{R}^n_+ \),
\[
 w(x) \leq w \left( x' - \frac{x_n}{e_n} e', 0 \right) + \left| \left( \frac{x_n}{e_n} e', x_n \right) \right| = e' \cdot x' - \frac{x_n}{e_n} |e'|^2 + \frac{x_n}{e_n} = e \cdot x.
\]
This completes the proof. \( \square \)

Proof of Theorem 1.2. Since \( \partial \Omega \in C^1 \), by suitable rotations and translations we may assume that \( x_0 = 0 \in \partial \Omega \) and for some \( r > 0 \)
\[
 \Omega \cap B_r(0) = \{(x', x_n) \in B_r(0) \mid x_n > f(x')\},
\]
where \( f \in C^1(\mathbb{R}^{n-1}) \), \( f(0) = 0 \) and \( Df(0) = 0 \). Without loss of generality, we may assume that
\[
 S^+(0) \geq S^-(0)
\]
so that \( S(0) = \max\{S^+(0), S^-(0)\} = S^+(0) \). Our goal is to show that
\[
 D\lambda(x) = p_0 := \left( D_T g(0), \sqrt{S^2(0) - |D_T g(0)|^2} \right).
\]
Here \( D_T g(0) = (\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \ldots, \frac{\partial g}{\partial x_{n-1}})(0) \) is the tangential gradient of \( g \) at \( 0 \in \partial \Omega \). If \( S(0) = 0 \), this follows immediately from Lemma 2.1. So we may assume after scalings that \( S(0) = 1 \). For \( \lim_{m \to +\infty} \lambda_m = 0 \), set \( \Omega_m = \lambda_m^{-1} \Omega \) and define
\[
 u_m(x) = \frac{u(\lambda_m x) - g(0)}{\lambda_m}, \quad x \in \Omega_m.
\]
Since \( u_m(0) = 0 \) and \( u \) is an absolute minimal Lipschitz extension, we have
\[
 \|D u_m\|_{L^\infty(\Omega_m)} = \|D u\|_{L^\infty(\Omega)} \leq \|D g\|_{L^\infty(\Omega)}
\]
so that\(^1\)
\[
 \|u_m\|_{L^\infty(\Omega_m \cap B_R)} + \|D u_m\|_{L^\infty(\Omega_m)} \leq (1 + R)\|D g\|_{L^\infty(\Omega)}, \quad \forall R > 0.
\]
\(^1\)If \( Dg = 0 \) (i.e., \( g \) is constant), then \( u \) is constant so that \( u_m \equiv 0 \). Hence we may assume \( Dg \neq 0 \).
Since \( \lim_{m \to \infty} \Omega_m = \mathbb{R}_+^n \), we may assume that \( u_m \to w \) locally uniformly in \( \mathbb{R}_+^n \). It is clear that

- \( w \in W^{1,\infty}(\mathbb{R}_+^n) \) is an infinity harmonic function in \( \mathbb{R}^{n-1} \times (0, +\infty) \),
- \( w(x', 0) = DTg(0) \cdot x' \) for \( x' \in \mathbb{R}^{n-1} \),

\[(2.4)\]
\[|Dw|(x) \leq S(0) = 1 \text{ a.e. } x \in \mathbb{R}_+^n.\]

We need to verify that

\[(2.5)\]
\[w(x) = p_0 \cdot x, \quad \forall x = (x', x_n) \in \mathbb{R}_+^n,\]

with \( p_0 \) given by (2.3).

Since \( g \in C^1 \), by the definition of \( S^+ \) there exists \( r_0 > 0 \) such that for any \( 0 < r \leq r_0 \) there exists \( x_r \in \partial B_r \cap \Omega \) such that

\[ \lim_{r \to 0} \frac{u(x_r) - g(0)}{r} = S^+(0) = 1. \]

Note that if \( |DTg(0)| < 1 \), we may in fact choose \( x_r \in \partial B_r \cap \Omega \) satisfying

\[ \frac{u(x_r) - g(0)}{r} = S^+_r(0). \]

We now claim that for each \( k \in \mathbb{N} \), there exists a unit vector \( e_k = (e'_k, (e_k)_n) \) with \( (e_k)_n \geq 0 \) such that

\[(2.6)\]
\[w(te_k) = t \quad \text{for } t \in [0, k].\]

In fact, taking possible subsequences, we may assume that (for \( r = k\lambda_m \))

\[ \lim_{m \to +\infty} \frac{x_{k\lambda_m}}{k\lambda_m} = e_k. \]

Then \( ke_k = \frac{x_{k\lambda_m}}{\lambda_m} + o(1) \) for \( \lim_{m \to +\infty} o(1) = 0 \). Hence

\[ w(ke_k) = \lim_{m \to +\infty} \frac{u(x_{k\lambda_m}) - g(0)}{\lambda_m} = k. \]

This and (2.4) yield (2.6). After taking a subsequence if necessary, we assume that

\[ \lim_{k \to +\infty} e_k = e \]

for a unit vector \( e = (e', e_n) \) with \( e_n \geq 0 \). By (2.6), it is clear that

\[ w(te) = t, \quad \forall t > 0. \]

Hence Lemma 2.2 implies \( w(x) = e \cdot x \). Since \( w(x', 0) = DTg(0) \cdot x' \), we have \( e' = DTg(0) \). Combining with \( e_n \geq 0 \) and \( |e| = 1 \), we conclude that \( e_n = \sqrt{1 - |DTg(0)|^2} \) and hence (2.3) holds. This completes the proof. \( \square \)
3. $C^1$-boundary regularity and proof of Theorem 1.1

In this section, we will assume that $n = 2$, $\partial \Omega \in C^2$, $g \in C^2(\mathbb{R}^2)$, and $u \in C(\overline{\Omega})$ is a viscosity solution of (1.1). We will prove the $C^1$-boundary regularity Theorem 1.1.

Write $e = (e_1, e_2)$. Assume that $|e| = 1$ and $e_2 = \tau > 0$. For $\mu, \nu > 0$, let $B_{\mu,\nu}$ denote the parallelogram

$$B_{\mu,\nu} = \left\{ te + (s,0) \mid t \in \left[ -\frac{1}{4}, \mu \right], s \in [-\nu, \nu] \right\}.$$ 

We assume that

$$\Omega = B_{1,1} \cap \{(x_1, x_2) \mid x_2 > f(x_1)\}, \quad \Gamma = \partial \Omega \cap \{(x_1, x_2) \in B_{1,1} \mid x_2 = f(x_1)\}$$

for a function $f \in C^2(\mathbb{R})$ and $f(0) = f'(0) = 0$. Let $O = (0,0) \in \Gamma$. See figure 1 below.

**Lemma 3.1.** Assume $|f'| \leq \epsilon$ and $e_2 = \tau > 0$. Suppose that $u \in C(\overline{\Omega})$ is infinity harmonic function in $\Omega$ satisfying that

(i) $u = g$ on $\Gamma$;

(ii) $|u(x) - e \cdot x| \leq \epsilon$ in $\overline{\Omega}$.

Assume that $w \in C^1(\Omega) \cap C(\overline{\Omega})$ is a solution of

$$\begin{cases} |Dw| = 1 - \delta & \text{in } \Omega, \\ w = g & \text{on } \Gamma. \end{cases}$$

For any fixed $\delta, \tau > 0$, if $\epsilon$ is sufficiently small then we have that

$$u(x) \geq w(x) \quad \text{for } x \in \overline{\Omega} \cap B_{1,\frac{1}{4}}.$$

![Figure 1. Proof of Lemma 3.1.](image-url)
Proof. We argue by contradiction. Suppose that there exists \( x_0 \in \Omega \cap B_{1/4} \) such that \( u(x_0) < w(x_0) \). Note that when \( \epsilon \) is small, within \( B_{1,1} \), each line \( x + te \) intersects the curve \( \{ x_2 = f(x_1) \} \) exactly once. Denote \( U \) as the connected component of \( \{ u < w \} \) containing \( x_0 \). Since \( |w(te + x) - g(x)| \leq (1 - \delta)t \) for \( x \in \Gamma \) and \( x + te \in \Omega \), it is clear that if \( \epsilon \) is sufficiently small then

\[
U \subset \Omega \cap B_{1/4,1}.
\]

See figure 1 above. Also, \( U \) should stretch all the way to \( \partial \Omega \setminus \Gamma \) although \( \partial U \cap \Gamma \) might not be empty. Without loss of generality, we assume

\[
\partial U \cap \left\{ te + (1,0) \mid t \in \left[-\frac{1}{4}, \frac{1}{4}\right] \right\} \neq \emptyset.
\]

Let \( K \) be the line segment \( \left\{ \left( \frac{3}{10}, 0 \right) + \lambda e : \lambda \in [\frac{1}{4}, \frac{1}{2}] \right\} \). According to (ii), if \( \epsilon \) is small enough, then there must exist \( \bar{x} \in K \) such that

\[
|Du(\bar{x})| > 1 - 10\epsilon.
\]

Let \( \xi(t) : (-T, 0] \to \Omega \) be a backward generalized gradient flow from \( \bar{x} \), i.e., \( \xi(0) = \bar{x} \), \( \xi(-T) \in \partial \Omega \),

\[
|Du(\xi(t))| \geq |Du(\bar{x})| \geq 1 - 10\epsilon, \quad -T \leq t \leq 0
\]

and

\[
u(\bar{x}) - u(\xi(t)) \geq \int_{t}^{0} |\xi'(s)| ds \geq (1 - 10\epsilon)|\bar{x} - \xi(t)|, \quad -T \leq t \leq 0.
\]

See [11] for the construction of \( \xi \). Let \( S \) denote the strip bounded by two lines \( L_1 = \frac{1}{4} + \lambda e \) and \( L_2 = \frac{1}{2} + \lambda e \). According to (ii), when \( \epsilon \) is small enough, the whole curve \( \bar{\xi} \) must lie within the strip \( S \) and \( \xi(-T) \in \Gamma \). Hence, there exists \( t_0 \in (-T, 0) \) such that \( \xi(t_0) \in S \cap U \). This leads a contradiction if we are able to establish the following claim.

**Claim.** If \( \epsilon \) is sufficiently small, then

\[
\sup_{x \in U \cap S} |Du(x)| \leq 1 - 12\epsilon.
\]

In fact, we again argue by contradiction. Assume that there is a \( \tilde{x} \in U \cap S \) such that

\[
|Du(\tilde{x})| > 1 - 12\epsilon.
\]

Let \( \tilde{\xi}(t) : (-\tilde{T}, 0] \to U \) be a backward gradient flow from \( \tilde{x} \) such that \( \xi(-\tilde{T}) \in \partial U \). Since

\[
u(\tilde{x}) - u(\tilde{\xi}(-\tilde{T})) \geq (1 - 12\epsilon) \int_{-\tilde{T}}^{0} |\tilde{\xi}'(s)| ds,
\]

we have that \( u(\tilde{\xi}(-\tilde{T})) < w(\tilde{\xi}(-\tilde{T})) \) provided that \( 12\epsilon < \delta \). Hence \( \tilde{\xi}(-\tilde{T}) \in \left\{ te + (1,0) | t \in [-\frac{1}{4}, \frac{1}{4}] \right\} \). Then by (ii),

\[
e \cdot (\tilde{x} - \tilde{\xi}(-\tilde{T})) \geq (1 - 12\epsilon)|\tilde{x} - \tilde{\xi}(-\tilde{T})| - 2\epsilon.
\]

This is impossible provided that \( \epsilon \) is small enough. \( \square \)
Let $f$ be the same function as in the statement of Lemma 3.1. Denote

$$\Sigma_t = B_t(O) \cap \{(x_1, x_2) | x_2 > f(x_1)\}.$$ 

and

$$\Gamma_t = \overline{B_t(O)} \cap \{(x_1, x_2) | x_2 = f(x_1)\}.$$ 

See figure 2 below.

**Lemma 3.2.** Assume $|f'| \leq \epsilon$, $|f''| \leq 1$ and $|g|_{C^2(\mathbb{R}^2)} \leq 1$. Suppose that $u$ is infinity harmonic in $\Sigma_1$ and $u = g$ on $\Gamma_1$. Assume that

$$\max_{x \in \Sigma_1} |u - e \cdot x| \leq \epsilon$$

and

$$\max_{x \in \Gamma_1} |(Dg - e)_T| \leq \epsilon.$$ 

Here $(Dg - e)_T$ denotes the tangential component of $(Dg - e)$ along the boundary $\Gamma_1$. Then for any $\tau > 0$, there exists $\epsilon_{e, \tau} > 0$ depending only on $\epsilon$ and $\tau$ such that when $\epsilon \leq \epsilon_{e, \tau}$,

$$|Du(x) - e| \leq \tau \quad \text{for all} \ x \in \overline{\Sigma_{1/2}}.$$ 

**Proof:** When $\epsilon > 0$ is sufficiently small, $\partial B_t(O) \cap \{(x_1, x_2) | x_2 = f(x_1)\}$ contains exactly two points, for $t \in (0, 1]$. Due to (3.1) and $|f'| \leq \epsilon$, by comparison with cones (first on the boundary and then in the interior), it is easy to prove that

$$\sup_{\overline{\Sigma_{1/2}}} |Du(x)| \leq |e| + C\epsilon.$$ 

If $|e| = 0$, then (3.2) follows from (3.3) immediately. Now we assume $|e| = \mu > 0$.

**Claim:** Given $\delta > 0$, when $\epsilon(\leq \min\{\delta, \frac{\mu}{2}\})$ is small enough, there exists a positive constant $\hat{r} \in (0, \frac{1}{6})$ depending only on $e$ and $\delta$ such that for any point $x \in \Gamma_{\frac{1}{4}}$, we can find two barrier functions $w^\pm_x(y) \in C^1(B_{\hat{r}}(x))$ satisfying

$$w^-_x(y) \leq u(y) \leq w^+_x(y) \quad \text{in} \ B_{\hat{r}}(x) \cap \overline{\Sigma_1}$$

and

$$\max\{|Dw^+_x(y) - e|, |Dw^-_x(y) - e|\} \leq 2\delta \quad \text{in} \ B_{\hat{r}}(x).$$

For simplicity, we will only prove this claim for $x = O = (0, 0)$ (the proof for other points can be done similarly). Since $f'(0) = 0$, $D_T g(O) = g_{x_1}(0)$. Denote $g_{x_1}(0) = s$ and $e = (e_1, e_2)$. Then by (3.1), $|s - e_1| \leq \epsilon$.

\[\text{Figure 2. Uniform control.}\]
Case 1: $e_2 = 0$. Then $|e_1| = \mu$. Choose $\epsilon$ small enough such that by (3.3),
\begin{equation}
\sup_{\Sigma_4} |Du(x)| \leq \sqrt{s^2 + \delta^2}.
\end{equation}
Using the method of characteristics (see [14] Chapter 3 for instance), there exist a simply connected open set $V$ containing $O$ such that $V^+ := V \cap \{x_2 > f(x_1)\} \subset \Sigma_4$ and two barrier functions $w^\pm \in C^2(V)$ that are classical solutions of the eikonal equation:
\begin{equation}
\begin{cases}
|Dw^\pm| = \sqrt{s^2 + \delta^2} & \text{in } V, \\
w^\pm = g & \text{on } V \cap \Gamma_1
\end{cases}
\end{equation}
subject to the condition: $Dw^\pm(O) = (g_{x_1}(O), \pm \delta) = (s, \pm \delta)$. Since $|s - e_1| \leq \epsilon$, $|s| \leq \mu + \delta$. We may choose $r_2 > 0$ depending only on $\mu$ and $\delta$ such that $B_{r_2}(O) \subset V$. From the constructions of $w^\pm$, we have that
\begin{equation}
w^-(x) \leq u(x) \leq w^+(x) \quad \text{for } x \in B_{r_2}(O) \cap \Sigma_1.
\end{equation}
We will indicate the proof of the second inequality in (3.7) (the first inequality in (3.7) can be proved similarly). According to the method of characteristics, for any $x \in B_{r_2}(O) \cap \Sigma_1$, there exists a unique $y_x \in V \cap \Gamma_3$ and $t_x > 0$ such that
\[\xi(t_x) = x, \quad \xi(0) = y_x\]
and the characteristics $\xi : (0, t_x] \to V^+$ satisfies that
\[\dot{\xi}(t) = \frac{Dw^+(\xi(t))}{\sqrt{s^2 + \delta^2}}.
\]
Hence, by (3.6), we have
\[\frac{d}{dt}\left(u(\xi(t)) - w^+(\xi(t))\right) = \frac{Du(\xi(t)) \cdot Dw^+(\xi(t))}{\sqrt{s^2 + \delta^2}} - \sqrt{s^2 + \delta^2} \leq 0, \quad 0 \leq t \leq t_x.
\]
This implies $u(x) \leq w^+(x)$. We would like to point out that $\xi$ is actually a straight line and
\[Dw^+(\xi(t)) \equiv DTg(y_x)\tau(y_x) + n(y_x)\sqrt{s^2 + \delta^2 - D_T^2g(y_x)}.
\]
Here $\tau(y_x) = \frac{(1, f'(y_{x_1}))}{\sqrt{1 + (f'(y_{x_1}))^2}}$ is the unit tangential direction of $\Gamma_1$ at $y_x = (y_{x_1}, y_{x_2})$, $n(y_x) = \frac{(-f'(y_{x_1}), 1)}{\sqrt{1 + (f'(y_{x_1}))^2}}$ is the inward normal vector of $\Gamma_1$ at $y_x$, and $D_Tg(y_x) = Dg(y_x) \cdot \tau(y_x)$.

Case 2: $e_2 \neq 0$. Without loss of generality, we assume that $e_2 > 0$. For otherwise, we can consider $-u$ and $-e$. Let $0 < \delta < \frac{e_2}{2}$. When $\epsilon$ is small enough, by (3.3) we have
\[\sup_{\Gamma_4} |Du(x)| \leq \sqrt{s^2 + (e_2 + \delta)^2}
\]
and
\[\sqrt{s^2 + (e_2 - \delta)^2} \leq \sqrt{|e|^2 - \delta^2}.
\]
Using the method of characteristics, there exist a simply connected open set $V$ containing $\Omega$ such that $V^+ := V \cap \{x_2 > f(x_1)\} \subset \Sigma_{\frac{1}{4}}$ and two barrier functions $w^\pm$ on $V$ which are classical solutions of
\[
\begin{align*}
|Dw^\pm| &= \sqrt{s^2 + (e_2 \pm \delta)^2} \quad \text{in } V, \\
w^\pm &= g \quad \text{on } V \cap \Gamma_1
\end{align*}
\]
subject to the condition: $Dw^\pm(\Omega) = (g_{x_1}(\Omega), e_2 \pm \delta) = (s, e_2 \pm \delta)$. Since $|s| \leq |e_1| + \epsilon \leq \mu + \delta$, we may choose $r_2 > 0$ depending only on $\epsilon$ and $\delta$ such that $B_{r_2}(\Omega) \subset V$. From the construction of $w^+$, we have that
\[
\begin{align*}
u(x) &\leq w^+(x) \quad \text{for } x \in B_{r_2}(\Omega) \cap \Sigma_1. \\
\end{align*}
\]
The proof is similar to that of (3.7). Moreover, let $\lambda \in (0, 1)$ such that $B_{1,1} \subset B_{\frac{r_2}{2}}(\Omega)$ (see the definition of $B_{1,1}$ at the begin of this section), and consider $u_\lambda(x) = \frac{u(\lambda x) - u(\Omega)}{\lambda}$, $x \in B_{1,1}$. Apply Lemma 3.1 to $u_\lambda$, $f_\lambda(t) = \frac{f(\lambda t)}{\lambda}$, $g_\lambda(x) = \frac{g(\lambda x) - g(\Omega)}{\lambda}$, and $w_\lambda(x) = \frac{w^-(\lambda x) - w^-(\Omega)}{\lambda}$, we conclude that when $\epsilon$ is small enough, there exists $0 < r_3 = \alpha r_2$ for some $\alpha \in (0, 1)$ depending only on $\epsilon$ and $\delta$ such that
\[
\begin{align*}
u(x) &\geq w^-(x) \quad \text{for } x \in B_{r_3}(\Omega) \cap \Sigma_1. \\
\end{align*}
\]
Hence
\[
\begin{align*}
w^-(x) &\leq \nu(x) \leq w^+(x) \quad \text{for } x \in B_{r_3}(\Omega) \cap \Sigma_1. \\
\end{align*}
\]
Note that $|Dw^\pm(\Omega) - e| \leq \epsilon + \delta$. Also, the module of continuity of $Dw^\pm$ depends only on $\delta$ and $\epsilon$. Hence, we may choose $\hat{r} > 0$ depending only on $\delta$ and $\epsilon$ such that the Claim holds.

Next let $W = \left\{x \in \Sigma_{\frac{1}{4}} \mid d(x, \Gamma_\frac{1}{2}) \leq \frac{\hat{r}}{2}\right\}$. When $x \in W$, (3.2) can be derived from our claim and Savin’s interior estimate (see [20] Proposition 2) through routine scaling argument. For reader’s convenience, we sketch it here. Fix $x_0 \in W$. Choose $y_0 \in \partial\Omega$ such that $|x_0 - y_0| = d(x_0, \partial\Omega) = r_0 < \frac{\hat{r}}{2} \leq \frac{1}{12}$. Clearly, $y_0 \in \Gamma_{\frac{1}{3}}$. Denote
\[
v(y) = \frac{u(y_0 + r_0(y - y_0)) - u(y_0)}{r_0}, \quad y \in B_1(\bar{x}_0).
\]
Then $v$ is an infinity harmonic function in $B_1(\bar{x}_0)$, here $\bar{x}_0 = y_0 + \frac{x_0 - y_0}{r_0}$. By (3.4) and (3.5), we have
\[
|v(y) - e \cdot (y - y_0)| \leq 4\delta \quad \text{for } y \in B_1(\bar{x}_0).
\]

**Figure 3.** Rescaling argument along the boundary.
Let \( \tilde{v}(z) = v(x_0 + z) + e \cdot y_0 - e \cdot \tilde{x}_0 \) for \( z \in B_1(O) \). Then we have

\[
|\tilde{v}(z) - e \cdot z| \leq 4\delta, \quad z \in B_1(O).
\]

By Savin’s interior estimate ([20] Proposition 2), for any given \( \tau > 0 \), if \( \delta \) is chosen to be sufficiently small, we have that

\[
|Du(x_0) - e| = |Dv(x_0) - e| = |D\tilde{v}(0) - e| \leq \tau.
\]

If \( x \in \Sigma_{\frac{1}{2}} \setminus W \), (3.2) follows immediately from Savin’s interior estimate ([20] Proposition 2).

**Proof of Theorem 1.1.** It suffices to prove (1.2). We argue by contradiction. If it were false, then there would exist \( \tau > 0 \), a sequence of \( C^2 \) bounded domains \( \Omega_m \), boundary values \( g_m \in C^2(\mathbb{R}^2) \), and infinity harmonic functions \( u_m \in C(\overline{\Omega}_m) \), and two sequences of points \( \{ x_m \} \) and \( \{ y_m \} \) in \( \overline{\Omega}_m \) such that

\[
||g_m||_{C^2(\mathbb{R}^2)} \leq 1, \quad ||\Omega_m||_{C^2} \leq C,
\]

(3.8)

\[
|x_m - y_m| \leq \frac{1}{m} \quad \text{and} \quad |Du_m(x_m) - Du_m(y_m)| \geq 4\tau.
\]

(3.9)

Upon taking possible subsequences, we may assume that there exist a bounded \( C^{1,1} \) domain \( \Omega \) (i.e., \( \partial \Omega \in C^{1,1} \)) and \( g \in C^{1,1}(\mathbb{R}^2) \) such that \( \Omega_m \to \Omega \) and \( g_m \to g \) in \( C^1 \) as \( m \to +\infty \). Due to Savin’s interior estimate [20] or the \( C^{1,\alpha} \) regularity in [15], \( x_m \) and \( y_m \) must converge to a point on \( \partial \Omega \). Let us assume that

\[
\lim_{m \to +\infty} x_m = \lim_{m \to +\infty} y_m = (0, 0) = O \in \partial \Omega.
\]

By suitable translations and rotations, we may assume that \( O \in \partial \Omega_m \) and there exists some \( r > 0 \) such that for all \( m \geq 1 \)

\[
\Omega_m \cap B_r(O) = \{(y_1, y_2) \in B_r(O) \mid y_2 > f_m(y_1)\},
\]

for some \( f_m \in C^2(\mathbb{R}) \), \( f_m(0) = 0, f_m'(0) = 0 \) and ||\( f_m ||_{C^2(\mathbb{R})} \leq C \). Next, we suppose as \( m \to \infty \),

\[
u_m \to u \quad \text{uniformly in} \ C(\overline{\Omega}).
\]

Here \( u \in C(\overline{\Omega}) \) is the infinity harmonic function satisfying \( u = g \) on \( \partial \Omega \). According to Theorem 1.2, \( u \) is differentiable at \( O \). Denote \( e = Du(O) \). For \( \tau \) and \( \epsilon \), let \( \epsilon = \epsilon_{e, \tau} \) be the same number as in Lemma 3.2. Choose a positive number \( \lambda_\epsilon < \min\{r, \epsilon\} \) such that

\[
\left| \frac{u(\lambda_\epsilon x) - u(O)}{\lambda_\epsilon} - e \cdot x \right| \leq \frac{\epsilon}{2} \quad \text{for} \ x \in \lambda_\epsilon^{-1}(B_{\lambda_\epsilon}(O) \cap \Omega)
\]

and

\[
|Dg - e)_T| \leq \frac{\epsilon}{2} \quad \text{for} \ x \in B_{\lambda_\epsilon}(O) \cap \partial \Omega.
\]

Hence when \( m \) is large enough,

\[
\left| \frac{u_m(\lambda_\epsilon x) - u_m(O)}{\lambda_\epsilon} - e \cdot x \right| \leq \epsilon \quad \text{for} \ x \in \lambda_\epsilon^{-1}(B_{\lambda_\epsilon}(O) \cap \Omega_m)
\]

and

\[
|Dg_m - e)_T| \leq \epsilon \quad \text{for} \ x \in B_{\lambda_\epsilon}(O) \cap \partial \Omega_m.
\]
Set $v_m(x) = \frac{u_m(\lambda x) - u_m(O)}{\lambda}$. Apply Lemma 3.2 to $\tilde{u} = v_m$, $\tilde{f}(t) = f_m(\lambda t)$ and $\tilde{g} = \frac{g_m(\lambda x) - g_m(O)}{\lambda}$, we have that

$$|Du_m(\lambda x) - e| = |Dv_m(x) - e| \leq \tau \text{ in } x \in \lambda^{-1}B_{\frac{\lambda}{2}}(O) \cap \Omega_m.$$  

This contradicts to (3.9) when $m$ is sufficiently large. The proof is no complete. □

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