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ENDPOINT BOUNDS FOR MULTILINEAR FRACTIONAL INTEGRALS

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Abstract. We prove that the multilinear fractional integral operator $I_\alpha(f_1,\ldots, f_k)(x) = \int_{\mathbb{R}^n} f_1(x-\theta_1 y)\ldots f_k(x-\theta_k y)|y|^{\alpha-n}dy$, where $\theta_j,\ j=1,\ldots,k$ are distinct and nonzero, (due to Grafakos [G]) has the endpoint weak-type boundedness into $L^{r,\infty}$ when $r=\frac{n}{2n-\alpha}$. Hence, we obtain by the multilinear interpolation theorem that I_α is bounded into L^r for all $r>\frac{n}{2n-\alpha}$. Moreover, We also prove that I_α is not bounded into L^r for any $r<\frac{n}{2n-\alpha}$ under some conditions on θ_j's. Similarly, we show that the multilinear Hilbert transform $H(f,g,h_1,\ldots,h_k)(x) = \text{ p.v.} \int f(x+t)g(x-t)\prod_{j=1}^k h_j(x-\theta_j t)\frac{dt}{t}$, where $\theta_j \neq \pm 1$ are distinct and nonzero, is not bounded into L^r for any $r<\frac{1}{2}$ under some conditions on θ_j's.

1. Introduction

The purpose of this note is to describe the endpoint boundedness result on the higher-order multilinear fractional integral operator I_α acting on functions of $\mathbb{R}^n$ defined as follows

$$I_\alpha(f_1,\ldots,f_k)(x) = \int_{\mathbb{R}^n} f_1(x-\theta_1 t)\cdots f_k(x-\theta_k t)|t|^{\alpha-n}\,dt, \tag{1.1}$$

where $0<\alpha<n$, k denotes an integer ≥ 3, and θ_j, $j=1,\ldots,k$, are fixed, distinct, nonzero real numbers. In 1992, Grafakos [G] proved that I_α is bounded from $L^{p_1}\times\cdots\times L^{p_k}$ to L^r, where $\frac{1}{r}=\frac{1}{p_1}+\cdots+\frac{1}{p_k}-\frac{\alpha}{n}$, $p_1,\ldots,p_k>1$, under the crucial restriction $1\leq r<\infty$. Kenig and Stein [KS] treated another type $\mathcal{I}_\alpha$ of multilinear operator of fractional integration defined by

$$\begin{aligned}&\mathcal{I}_\alpha(f_1,f_2,\ldots,f_{k+1})(x)\\ &\quad= \int_{\mathbb{R}^{nk}} f_1(l_1)f_2(l_2)\cdots f_{k+1}(l_{k+1})K(x_1,\ldots,x_k)dx_1\cdots dx_k,\end{aligned} \tag{1.2}$$

where $K(x_1,\ldots,x_k) = |(x_1,\ldots,x_k)|^{-nk+\alpha}$, $x_j\in\mathbb{R}^n, 1\leq j\leq k$, $x\in\mathbb{R}^n$, and $l_j = l_j(x_1,\ldots,x_k,x), 1\leq j\leq k+1$, are linear mappings from $\mathbb{R}^{n(k+1)}$ to $\mathbb{R}^n$ satisfying appropriate assumptions, see [KS].

The operators I_α and $\mathcal{I}_\alpha$ could be regarded as modified multilinear versions of the bilinear fractional integration

$$B_\alpha(f,g)(x) = \int_{\mathbb{R}^n} f(x-t)g(x+t)|t|^{\alpha-n}dt.$$

In [GK, KS], it is shown by very elementary considerations that B_α is bounded from $L^{p_1}\times L^{p_2}$ to L^r for the full range $1<p_i\leq\infty$, $i=1,2$, and $\frac{1}{r}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$, with

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$r < \infty$ instead of the crucial condition $1 \le r < \infty$ for boundedness of I_α. Viewing the bilinear fractional integration B_α as a special case of I_α with $k = 2$, we would like to investigate the property of I_α for the range $r < 1$. As a boundedness result, the endpoint weak boundedness is established in Section 2. The unboundedness results are given in Section 3. To be more specific, we show that I_α is not bounded into L^r for any $r < \frac{n}{2n-\alpha}$ under some conditions on θ_j's. Similarly, we show that the multilinear Hilbert transform $H(f, g, h_1, \ldots, h_k)(x) = \text{ p.v. } \int f(x+t)g(x-t)\prod_{j=1}^{k} h_j(x-\theta_j t)\frac{dt}{t}$, where $\theta_j \neq \pm 1$ are distinct and nonzero, is not bounded into L^r for any $r < \frac{1}{2}$ under some conditions on θ_j's.

2. The endpoint weak-type estimate of I_α

Let us first treat the following preliminary result.

Lemma 2.1. *Let*

$$I(f_1, \ldots, f_k)(x) = \int_{\mathbb{R}^n} f_1(x-\theta_1 t)\cdots f_k(x-\theta_k t)dt,$$
$$I_i(f_1, \ldots, f_k)(x) = \int_{|t|\le 2^i} f_1(x-\theta_1 t)\cdots f_k(x-\theta_k t)dt,$$

for $i \in \mathbb{Z}$. Assume that $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 2,\ p_j \ge 1,\ j = 1, \ldots, k$. Then

(i) $\|I(f_1, \ldots, f_k)\|_{L^1} \le C \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}$.

(ii) $\|I_i(f_1, \ldots, f_k)\|_{L^{\frac{1}{2}}} \le C2^{in} \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}$.

We adapt the argument of the proof of Lemma 5 in [KS].

Proof. Without loss of generality, we may assume that $f_1, \ldots, f_k \ge 0$. We begin with proving (i).

$$\|I(f_1, \ldots, f_k)\|_{L^1} = \int_{\mathbb{R}^n} I(f_1, \ldots, f_k)(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\prod_{j=1}^{k} f_j(x-\theta_j t)\,dt\,dx$$
$$\le C\|f_1\|_{L^1}\|f_2\|_{L^1}\prod_{j=3}^{k}\|f_j\|_{L^\infty}$$

where the last inequality follows from using the change of variables and Fubini's theorem. Thus, I is bounded from $L^1 \times L^1 \times L^\infty \times \cdots \times L^\infty$ to L^1. We write it as $I : \overbrace{(1, 1, 0, \ldots, 0)}^{k} \to L^1$. Moreover, we have $I : \overbrace{(1/p_1, \ldots, 1/p_k)}^{k} \to L^1$, where the two among $\{p_j : 1 \le j \le k\}$ are 1 and the other else are ∞, by interchanging the roles of the functions f_j in an arbitrary way. By multilinear (Riesz–Thorin) interpolation theorem, (i) follows. We next show (ii) when $i = 0$. For $\vec{a} \in \mathbb{Z}^n$, we let $Q_{\vec{a}}$ be the unit size cube in $\mathbb{R}^n$, whose bottom left coordinate is $\vec{a}$. Then

$$\begin{aligned}\int_{Q_{\vec{a}}} I_0(f_1,\dots,f_k)(x)dx &= \int_{Q_{\vec{a}}}\int_{|t|\le 1} f_1(x-\theta_1 t)\cdots f_k(x-\theta_k t)\,dt\,dx\\ &\le C\int_{Q^*_{\vec{a}}} f_1\cdot\int_{Q^*_{\vec{a}}} f_2\cdot\prod_{j=3}^k \|f_j\|_{L^\infty(Q^*_{\vec{a}})},\end{aligned}$$

for some sufficiently expansion (depending only on $\theta_j,\ j=1,\dots,k$) $Q^*_{\vec{a}}$ of $Q_{\vec{a}}$. Thus,

$$\|I_0(f_1,\dots,f_k)\|_{L^1(Q_{\vec{a}})} \le C\|f_1\|_{L^1(Q^*_{\vec{a}})}\|f_2\|_{L^1(Q^*_{\vec{a}})}\prod_{j=3}^k \|f_j\|_{L^\infty(Q^*_{\vec{a}})}.$$

Moveover, we have the same boundedness of I_0 by interchanging the roles of the functions f_j in an arbitrary way:

$$\|I_0(f_1,\dots,f_k)\|_{L^1(Q_{\vec{a}})} \le C\prod_{j=1}^k \|f_j\|_{L^{p_j}(Q^*_{\vec{a}})},$$

where the two among $\{p_j : 1\le j\le k\}$ are 1 and the other else are ∞. By multilinear (Riesz–Thorin) interpolation theorem as proving (i), we have

$$\|I_0(f_1,\dots,f_k)\|_{L^1(Q_{\vec{a}})} \le C\prod_{j=1}^k \|f_j\|_{L^{p_j}(Q^*_{\vec{a}})}, \tag{2.1}$$

for $2=\frac{1}{p_1}+\cdots+\frac{1}{p_k}$. Hence

$$\begin{aligned}\|I_0(f_1,\dots,f_k)\|_{L^{\frac12}} &= \left(\sum_{\vec{a}\in\mathbb{Z}^n}\int_{Q_{\vec{a}}} I_0(f_1,\dots,f_k)(x)^{\frac12}\,dx\right)^2\\ &\le \left\{\sum_{\vec{a}\in\mathbb{Z}^n}\left(\int_{Q_{\vec{a}}} I_0(f_1,\dots,f_k)(x)dx\right)^{\frac12}\right\}^2\\ &\le C\prod_{j=1}^k\left(\sum_{\vec{a}\in\mathbb{Z}^n}\left\{\left(\int_{Q_{\vec{a}^*}} f_j^{p_j}\right)^{\frac{1}{p_j}}\right\}^{p_j}\right)^{\frac{1}{p_j}}\\ &= C\prod_{j=1}^k\left(\sum_{\vec{a}\in\mathbb{Z}^n}\int_{Q^*_{\vec{a}}} f_j^{p_j}\right)^{\frac{1}{p_j}},\end{aligned}$$

where the first inequality holds by Jensen's inequality and the second inequality by Hölder inequality for sequences for (2.1). Finally, since the $\{Q^*_{\vec{a}}\}$ have finite overlap, $(\sum_{\vec{a}}\int_{Q^*_{\vec{a}}} f_j^{p_j})^{\frac{1}{p_j}} \le C\|f_j\|_{L^{p_j}}$, for each $j=1,\dots,k$. Hence, we have shown that (ii) follows when $i=0$. For the remainder cases $i\ne 0$ else, (ii) follows from the case $i=0$ by scaling. Lemma 2.1 is established. □

Our main result is the following.

Theorem 2.2. *I_α is bounded from $L^{p_1}\times\cdots\times L^{p_k}$ to $L^{\frac{n}{2n-\alpha},\infty}$ for $\frac{1}{p_1}+\cdots+\frac{1}{p_k}=2$, $p_j\ge 1$, $j=1,\dots,k$.*

Proof. This proof is the adaptation of Example 5.6. in [GK]. However, we expose the details with amending the misprint of Example 5.6. in [GK] for the convenience of the reader.

We may assume that $f_1, \ldots, f_k \geq 0$. We note that

$$I_\alpha(f_1, \ldots, f_k) \leq C \sum_{i \in \mathbb{Z}} 2^{i(\alpha-n)} I_i(f_1, \ldots, f_k).$$

Observe that for any measurable set E with finite measure

$$\begin{aligned} \int_E (I_i(f_1, \ldots, f_k)(x))^{1/2} dx &\leq \left(\int_E I_i(f_1, \ldots, f_k)(x) dx \right)^{1/2} |E|^{1/2} \\ &\leq C \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}^{1/2} |E|^{1/2}, \end{aligned}$$

where the first inequality follows from Hölder inequality and the second from Lemma 2.1 (i). This observation and Lemma 2.1 (ii) implies that for any measurable set E with finite measure we have

$$\int_E (I_i(f_1, \ldots, f_k)(x))^{1/2} dx \leq C \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}^{1/2} \min(2^{in}, |E|)^{1/2}. \tag{2.2}$$

Let K be a compact subset of $E_\lambda = \{x : |I_\alpha(f_1, \ldots, f_k)(x)| > \lambda\}$. Then applying Chebychev's inequality and (2.2), we obtain

$$\begin{aligned} \lambda^{1/2}|K| &\leq \int_K \left| \sum_{i \in \mathbb{Z}} 2^{i(\alpha-n)} I_i(f_1, \ldots, f_k)(x) \right|^{1/2} dx \\ &\leq \sum_{i \in \mathbb{Z}} 2^{i(\alpha-n)/2} \int_K |I_i(f_1, \ldots, f_k)(x)|^{1/2} dx \\ &\leq C \sum_{i \in \mathbb{Z}} 2^{i(\alpha-n)/2} \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}^{1/2} \ \min(2^{in}, |K|)^{1/2} \\ &\leq C \left(\sum_{2^{in} < |K|} 2^{i(\alpha-n)/2} 2^{in/2} + \sum_{|K| \leq 2^{in}} 2^{i(\alpha-n)/2} |K|^{1/2} \right) \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}^{1/2} \\ &\leq C \left(|K|^{\alpha/2n} + |K|^{(\alpha-n)/2n} |K|^{1/2} \right) \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}^{1/2} \\ &= C|K|^{\alpha/2n} \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}^{1/2}. \end{aligned}$$

This means that

$$\lambda |K|^{2-\frac{\alpha}{n}} \leq C \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}.[1]$$

[1]It has a pedagogic meaning as an analysis technique to consider a compact subset K of E_λ, because of having divided $|K|^{\frac{\alpha}{n}}$ here.

Taking the supremum over all compact $K \subset E_\lambda$ and using the inner regularity of Lebesque measure, we obtain that

$$\lambda |E_\lambda|^{2-\frac{\alpha}{n}} \leq C \prod_{j=1}^{k} \|f_j\|_{L^{p_j}},$$

which is the required weak-type estimate $L^{p_1} \times \cdots \times L^{p_k} \to L^{\frac{n}{2n-\alpha},\infty}$. Theorem 2.2 is now completely proved. $\square$

Remark 2.3. Substituting $f_1 = \delta_0$, we have $I_\alpha(f_1, \ldots, f_k)(x) = f_2(x - \theta_2 \frac{x}{\theta_1}) f_3(x - \theta_3 \frac{x}{\theta_1}) \cdots f_k(x - \theta_k \frac{x}{\theta_1}) |\frac{x}{\theta_1}|^{\alpha - n}$. This shows that in Theorem 2.2, the strong-type boundedness of I_α for $1 + \frac{1}{p_2} + \frac{1}{p_3} + \cdots + \frac{1}{p_k} = 2,\ p_2, \ldots, p_k \geq 1$ is not established.

We find the usefulness in the following multilinear Marcinkiewicz interpolation theorem for Lorentz spaces, which is introduced from Theorem 3 in [KS].

Theorem 2.4 (Theorem 3 (Janson [J]), [KS])**.** *Suppose that an k-linear operator $T : L^{p_{1_j},1} \times \cdots \times L^{p_{k_j},1} \to L^{q_j,\infty}$, where $0 < p_{i_j} \leq \infty$, $0 < q_j \leq \infty$, for $k+1$ points $(\frac{1}{p_{1_j}}, \ldots, \frac{1}{p_{k_j}})$, $1 \leq j \leq k+1$ in $\mathbb{R}^k$, that do not lie on the same hyperplane. Suppose further that there are real numbers $\alpha_0, \alpha_1, \ldots, \alpha_k$ with $\alpha_i > 0$ for $i = 1, \ldots, k$, so that $\frac{1}{q_j} = \alpha_0 + \sum_{i=1}^{k} \frac{\alpha_i}{p_{i_j}}$, for $j = 1, \ldots, k+1$. Then*

$$T : L^{p_1,s_1} \times \cdots \times L^{p_k,s_k} \to L^{q,s},$$

where $1 \leq s_i \leq \infty$, $\frac{1}{s} = \frac{1}{s_1} + \cdots + \frac{1}{s_k}$, and $(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_k}, \frac{1}{q})$ lies in the open convex hull of $(\frac{1}{p_{1_j}}, \frac{1}{p_{2_j}}, \ldots, \frac{1}{p_{k_j}}, \frac{1}{q_j})$.

Remark 2.5. Instead of the above multilinear interpolation theorem, we may use a multilinear interpolation theorem of [GLLZ] to pay attention when the case $s_i = \infty$ in proving Theorem 2.6 in the following. Thus, the proof of Theorem 2.6 may be a sketch of the proof of Theorem 2.6.

We improve Theorem 1 of [G] by applying the preceding theorems.

Theorem 2.6. *I_α is bounded from $L^{p_1} \times \cdots \times L^{p_k}$ to L^r for $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_k} - \frac{\alpha}{n}$, $\frac{n}{2n-\alpha} < r < \infty$, $p_i > 1$, $i = 1, \ldots, k$. (In fact, I_α is bounded from $L^{p_1} \times \cdots \times L^{p_k}$ to $L^{r,s}$, where $\frac{1}{s} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}$, for $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_k} - \frac{\alpha}{n}$, $\frac{n}{2n-\alpha} < r < \infty$, $p_i > 1$, $i = 1, \ldots, k$.)*

Proof. By applying Theorem 2.4 (multilinear interpolation theorem), this follows from Theorem 2.2 in this paper and Theorem 1 in [G], and Remark 4 in [KS]. Indeed, since $L^p \supseteq L^{p,1}$ for $p \geq 1$, we have by Theorem 2.2 that I^α is bounded from $L^{p_{1_j},1} \times \cdots \times L^{p_{k_j},1}$ to $L^{r_j,\infty}$, for $r_j = \frac{n}{2n-\alpha}$, $\frac{1}{p_{1_j}} + \cdots + \frac{1}{p_{k_j}} = 2$, $p_{i_j} \geq 1$, $i = 1, \ldots, k$ when $j = 1$. Since I^α is bounded from $L^{p_1} \times \cdots \times L^{p_k}$ to L^r, for $\frac{1}{p_1} + \cdots + \frac{1}{p_k} - \frac{\alpha}{n} = \frac{1}{r}$, $r \geq 1$, $p_i > 1$, $i = 1, \ldots, k$ (Theorem 1, [G]), and $L^{r,\infty} \supseteq L^r$, we have that I^α is bounded from $L^{p_{1_j},1} \times \cdots \times L^{p_{k_j},1}$ to $L^{r_j,\infty}$, where $\frac{1}{p_{1_j}} + \cdots + \frac{1}{p_{k_j}} - \frac{\alpha}{n} = \frac{1}{r_j}$, $r_j \geq 1$, $p_{i_j} > 1$, $i = 1, \ldots, k$, when $j = 2, \ldots, k+1$, satisfying $k+1$ points $\left(\frac{1}{p_{1_j}}, \ldots, \frac{1}{p_{k_j}}\right) \in \mathbb{R}^k$, $j = 1, \ldots, k+1$ that do not lie on the same hyperplane. Now by applying Theorem 2.4 with considering the open convex hull consisting of $\left(\frac{1}{p_{1_j}}, \frac{1}{p_{2_j}}, \ldots, \frac{1}{p_{k_j}}, \frac{1}{r_j}\right)$, $j = 1, \ldots, k+1$, we obtain that $I_\alpha : L^{p_1,s_1} \times \cdots \times L^{p_k,s_k} \to L^{r,s}$, where $1 \leq s_i \leq \infty$, $\frac{1}{s} = \frac{1}{s_1} + \cdots + \frac{1}{s_k}$, for $\frac{1}{p_1} + \cdots + \frac{1}{p_k} - \frac{\alpha}{n} = \frac{1}{r}$, $\frac{n}{2n-\alpha} < r < \infty$, $p_i > 1$, $i = 1, \ldots, k$. Taking $s_i = p_i$, and then $\frac{1}{s} \geq \frac{1}{r}$, so that $L^{r,s} \subset L^r$, we obtain that $I_\alpha : L^{p_1} \times \cdots \times L^{p_k} \to L^r$ (Remark 4, [KS]). This completes the proof. $\square$

Remark 2.7. In conclusion of this section, the question whether there exist results when $r < 1$ for I_α raised in [KS] is solved manifestly in this paper. (the range $r < \frac{n}{2n-\alpha}$ is dealt in the next section 3 as the unboundedness of I_α, see Theorem 3.2.)

3. The unboundedness of multilinear fractional integration and Hilbert transform

Observing Remark 2.3, the strong-type boundedness of I_α does not hold for the hyper-line $p_i = 1$ for either $i = 1, \ldots, k$ included in the hyperplane $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 2$, $p_1, \ldots, p_k \geq 1$. However, the failure of the strong-type boundedness of I_α for the open hyperplane $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 2$, $p_1, \ldots, p_k > 1$ is unknown. Hence, considering the multilinear interpolation theorem, the boundedness of I_α beyond the hyperplane $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 2$, $p_1, \ldots, p_k > 1$ is yet to be known. Thus, we investigate whether I_α is bounded for $\frac{1}{p_1} + \cdots + \frac{1}{p_k} > 2$, $p_1, \ldots, p_k > 1$ in this section.

Let us first state the Theorem of M. Christ [C] which is used to obtain our results.

Theorem 3.1 (Theorem 1, [C]). *Suppose that $\{S_j\}$ is nondegenerate.*

If $\{S_j\}$ is not rationally commensurate, then for any $p < 3/2$, there exist nonnegative functions $f_j \in L^p$ and a set $E \subset \mathbb{R}^d$ of positive Lebesgue measure such that

$$T(f_1, f_2, f_3)(x) = \int_{|t|\leq 1} \prod_{j=1}^{3} f_j(S_j(x,t))dt = +\infty,$$

for all $x \in E$.

Here, we would mention the reference [C], p.44 about the definition of *rationally commensurate* and *nondegenerate* of $\{S_j\}$. To put it briefly, it may be said that the above trilinear operator $T(f_1, f_2, f_3)$ has a *rationally commensurate* $\{S_j : 1 \leq j \leq 3\}$ if $T(f_1, f_2, f_3)$ is reduced to a canonical form $\int f_1(x+t)f_2(x-t)f_3(x-\theta t)dt$ for some parameter $\theta \in \mathbb{Q}\backslash\{0, -1, +1\}$, or to $\int f_1(x+t)f_2(x-t)f_3(t)dt$ by simple symmetries invariant on the boundedness of the operator, as explained in [C], p. 45. Thus, we may regard $\int \prod_{j=1}^{k} f_j(S_j(x,t))dt$ in the above Theorem 3.1 as a generalization of $\int f_1(x-\theta_1 t)\ldots f_k(x-\theta_k t)dt$, where θ_j's are nonzero and distinct.

Before stating our theorem in the following, we may note that homogeneity considerations imply that $I_\alpha(f_1, \ldots, f_k)$ can map $L^{p_1} \times \cdots \times L^{p_k} \to L^r$ only when

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} - \frac{\alpha}{n} = \frac{1}{r}.$$

Theorem 3.2. *In some cases of θ_j's, I_α is not bounded from $L^{p_1} \times \cdots \times L^{p_k}$ to L^r for any $r < \frac{n}{2n-\alpha}$, equivalently $\frac{1}{p_1} + \cdots + \frac{1}{p_k} > 2$, $p_1, \ldots, p_k > 1$.*

Proof. By referring to Section 7 Remarks in [C], we have a higher-order multilinear analogue of Theorem 3.1 (ii). Given finitely many distinct coefficients $\theta_j \neq \pm 1$, consider the multilinear operator

$$T(f_1, \dots, f_k)(x) = \int_{|t|\le 1} f_1(x+t) f_2(x-t) \prod_{j=3}^{k} f_j(x-\theta_j t)\, dt.$$

Suppose $\theta_3 \notin \mathbb{Q}$. Define $\gamma_j = (1+\theta_j)/(1-\theta_j)$ for $j = 3, \dots, k$. If all γ_j/γ_3 are rational, then for any p_i, $i = 1, \dots, k$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_k} > 2$, $p_1, \dots, p_k > 1$, there exist nonnegative $f_i \in L^{p_i}$, $i = 1, \dots, k$ for which $T(f_1, \dots, f_k)(x) = +\infty$ for all x in a set of positive measure.[2] Since

$$\int_{\mathbb{R}^n} f_1(x+t) f_2(x-t) \prod_{j=3}^{k} f_j(x-\theta_j t) \frac{dt}{|t|^{n-\alpha}} \ge \int_{|t|\le 1} f_1(x+t) f_2(x-t) \prod_{j=3}^{k} f_j(x-\theta_j t) dt,$$

the proof is complete. $\square$

Viewing the boundedness and the unboundedness of I_α in the preceding, we regard that the ranges $r < \frac{n}{2n-\alpha}$ and $\frac{n}{2n-\alpha} < r < \infty$ of the boundedness and the unboundedness of B_α, respectively, are the same with those of I_α. We may note that B_α and I_α have the same integral over $\mathbb{R}^n$, comparing that $\mathcal{I}_\alpha$ in (1.2) have the integral over $\mathbb{R}^{nk}$. In contrast to the boundedness of I_α, we may also observe that it is shown in [KS] that $\mathcal{I}_\alpha$ is bounded into L^r for the full range $r < \infty$.

Let us consider the multilinear Hilbert transform as following: given distinct and nonzero $\theta_j \neq \pm 1$,

$$H(f, g, h_1, \dots, h_k) = \text{p.v.} \int f(x+1) g(x-1) \prod_{j=1}^{k} h_j(x-\theta_j t) \frac{dt}{t}. \tag{3.1}$$

We may note that it follows from homogeneity considerations that $H(f, g, h_1, \dots, h_k)$ maps from $L^{p_1} \times \cdots \times L^{p_{k+2}}$ to L^r appears only when

$$\frac{1}{p_1} + \cdots + \frac{1}{p_{k+2}} = \frac{1}{r}.$$

We assume in Theorem 3.3 below that $p_i > 1$, $i = 1, \dots, k+2$. The case $p_i \le 1$, for any $i = 1, \dots, k+2$, is impossible for the boundedness of H, which is seen easily by the indirect proof using the multilinear interpolation theorem since H is not of strong-type bounded on $L^{p_i} \times L^\infty \times \cdots \times L^\infty$ (the order of $\{p_i, \infty, \dots, \infty\}$ of the product space is arbitrary) when $p_i = 1$.

The bilinear case of $H(f, g, h_1, \dots, h_k)$ is well-known as the bilinear Hilbert transform $H(f, g)$. We see that $H(f, g)$ is not bounded from $L^{p_1} \times L^{p_2}$ to L^r for any $r < 1/2$ since p_1, p_2 ought to be > 1 for the boundedness of $H(f, g)$. Referring to our results in the preceding that I_α and B_α have the same range for the unboundedness, we question whether the multilinear Hilbert transform $H(f, g, h_1, \dots, h_k)$ and the bilinear Hilbert transform $H(f, g)$ have the same unboundedness range. The following theorem states that the answer about it is indeed positive. However, the authors would mention that the following theorem is nothing else but to change the location of functions f, g of

[2] For detail about it, the readers may refer to the proof of Theorem 3.3.

the operator T in Section 2 of [C], because we use almost identically the argument of [C] to prove the following theorem for the completeness of the proof.

Theorem 3.3. *In some cases of θ_j's, $H(f, g, h_1, \ldots, h_k)$ is not bounded into L^r for any $r < 1/2$. In particular, for given irrational θ with $\frac{-1-\theta}{1-\theta} > 0$, the trilinear Hilbert transform $H(f, g, h) = p.v. \int f(x+t)g(x-t)h(x-\theta t)\frac{dt}{t}$ is not bounded to L^r for any $r < 1/2$.*

Remark 3.4. After this work, we learn the result of C. Demeter [D] about the unboundedness of the trilinear Hilbert transform:

$$H(f, g, h) = \text{p.v.} \int f(x+t)g(x+2t)h(x+3t)\frac{dt}{t} \tag{3.2}$$

is not bounded into L^r for any $r < \frac{1}{3}(1 + \frac{\log_6 2}{1+\log_6 2})$. Observe $\frac{1}{3}(1 + \frac{\log_6 2}{1+\log_6 2}) < 1/2$. We may note that the case of $\{S_j\}$ in (3.2) is rationally commensurate, comparing that the case of $\{S_j\}$ in the above Theorem 3.3 is irrationally commensurate.

Proof. Let us denote $\gamma_j = \frac{-1-\theta_j}{1-\theta_j}$, for each $j = 1, \ldots, k$. Let θ_1 be irrational with $\gamma_1 = \frac{-1-\theta_1}{1-\theta_1} > 0$. Then there exist rational approximations that there are sequences $\{c_n\}$ and $\{d_n\}$ of positive integers tending to ∞ such that c_n, d_n are relatively prime for each n, and so that

$$\left| \frac{-1-\theta_1}{1-\theta_1} - \frac{c_n}{d_n} \right| < \frac{1}{d_n^2}. \tag{3.3}$$

Let θ_j, $j = 2, \ldots, k$ be such that γ_j/γ_1 is positive rational. Then for each $j = 2, \ldots, k$ there exist integers a_j, b_j such that for c_n, d_n in (3.3)

$$\left| \frac{-1-\theta_j}{1-\theta_j} - \frac{b_j c_n}{a_j d_n} \right| < \frac{1}{d_n^2}.$$

For simplicity, we drop the subscript n, and write c, d for c_n, d_n. Set $N = d$ and $\delta = C(N)^{-2}$, for sufficiently small C. Let us consider the sets

$$F = \bigcup_{i=\frac{d}{2}}^{d} \{x : |x - id^{-1}| < \delta\},$$

$$G = \bigcup_{m=1}^{\frac{c}{2}} \{x : |x + mc^{-1}| < \delta\},$$

$$H_j = \bigcup_{l=a_j\frac{d}{2}+b_j}^{a_j d + b_j \frac{c}{2}} \{x : |x - ly| < C_{2,j}\delta\}, \quad j = 1, \ldots, k,$$

where $y = (1-\theta_j)/(2a_j d)$, $a_1 = b_1 = 1$, and $C_{2,j}$ is a large constant to be chosen later. Set $f = \sum_{i=\frac{d}{2}}^{d} f_i$, $g = \sum g_m$, $h_j = \sum h_{lj}$, $j = 1, \ldots, k$, where f_i, g_m, and h_{lj} are the characteristic function of the ith, mth, and lth component intervals of F, G, and H_j, $j = 1, \ldots, k$ respectively.

Let us consider the operator $H(f,g,h_1,\dots,h_k)$ of the theorem:

$$H(f,g,h_1,\dots,h_k)=\lim_{\epsilon\to 0}\left(\int_\epsilon^\infty f(x+t)g(x-t)\prod_{j=1}^k h_j(x-\theta_j t)\frac{dt}{t}\right.$$
$$\left.+\int_{-\infty}^{-\epsilon} f(x+t)g(x-t)\prod_{j=1}^k h_j(x-\theta_j t)\frac{dt}{t}\right).$$

Then the second term is zero since the intersection of $f(x+t)$ and $g(x-t)$ corresponding to x is on the positive axis of t. Thus

$$\begin{aligned}H(f,g,h_1,\dots,h_k)&=\lim_{\epsilon\to 0}\int_\epsilon^\infty f(x+t)g(x-t)\prod_{j=1}^k h_j(x-\theta_j t)\frac{dt}{t}\\&=\int_0^\infty f(x+t)g(x-t)\prod_{j=1}^k h_j(x-\theta_j t)\frac{dt}{t}.\end{aligned}$$

Since

$$\begin{aligned}\int_0^\infty f(x+t)g(x-t)\prod_{j=1}^k h_j(x-\theta_j t)\frac{dt}{t}&\geq\int_0^1 f(x+t)g(x-t)\prod_{j=1}^k h_j(x-\theta_j t)dt\\&=T(f,g,h_1,\dots,h_k)(x),\end{aligned}$$

it suffices to treat $T(f,g,h_1,\dots,h_k)$ instead of $H(f,g,h_1,\dots,h_k)$ to show the unboundedness of $H(f,g,h_1,\dots,h_k)$.

Each functions $f,g,h_1,\dots,h_k$ has $\|\cdot\|_p\sim(N\delta)^{1/p}\sim(N^{-1})^{1/p}\sim N^{-1/p}$. We see that $T(f_j,g_m,1,\dots,1)\sim\delta$ on the interval of length δ centered at $(id^{-1}-mc^{-1})/2$, and is supported on the concentric interval with length 2δ. Since $|id^{-1}-mc^{-1}|\geq(dc)^{-1}\sim N^{-2}$ for all $(i,m)\neq(0,0)$, the points $id^{-1}-mc^{-1}$ are distant more than N^{-2} for each other pair of indices i,m. Hence, $T(f_j,g_m,1,\dots,1)$ have pairwise disjoint supports for distinct pairs of indices $i=d/2,\dots,d,\ m=1,\dots,c/2$. Thus, $T(f,g,1,\dots,1)\sim\delta$ at a set $\mathcal{E}_N$ of measure $\geq N^2\delta\sim 1$. Here, the set $\mathcal{E}_N$ are subsets of some interval independent of N.

If $x+t=id^{-1}$ and $x-t=-mc^{-1}$, then for each $j=1,\dots,k$

$$\begin{aligned}(x-\theta_j t)&=\frac{1}{2}\left\{(1-\theta_j)\frac{a_j i}{a_j d}+(-1-\theta_j)\frac{b_j m}{b_j c}\right\}\\&=\frac{1}{2}(1-\theta_j)\frac{a_j i+b_j m}{a_j d}+mc^{-1}\cdot O(N^{-2})=(a_j i+b_j m)y+O(N^{-2}),\end{aligned}$$

thus setting $l=a_j i+b_j m$ there exists $C_{2,j}$ such that $|(x-\theta_j t)-ly|\leq C_{2,j}\delta$. Thus, we see that if $x+t\in\operatorname{supp}(f)$ and $x-t\in\operatorname{supp}(g)$, then $x+\theta_j t\in\operatorname{supp}(h_j)$. Hence $T(f,g,h_1,\dots,h_k)=T(f,g,1,\dots,1)$.

Therefore, we have

$$\begin{aligned}&T(f,g,h_1,\dots,h_k)(x)/(\|f\|_{p_1}\|g\|_{p_2}\prod_{j=1}^k\|h_j\|_{p_{j+2}})\\&\gtrsim\delta/(N\delta)^{\frac{1}{p_1}+\dots+\frac{1}{p_{k+2}}}\sim N^{\frac{1}{p_1}+\dots+\frac{1}{p_{k+2}}-2}=N^{\frac{1}{r}-2}\end{aligned}$$

for all $x \in \mathcal{E}_N$, for all N. If $r < 1/2$ then this exponent $\frac{1}{r} - 2$ is positive. The remained proof is straightforward; see Section 2 of [C].

We omit the detail of the trilinear case, which is the special case of the multilinear case. □

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