Math. Res. Lett. **19** (2012), *no.* 06, 1277–1281

MURPHY'S LAW FOR HILBERT FUNCTION STRATA IN THE HILBERT SCHEME OF POINTS

Daniel Erman

Abstract. An open question is whether the Hilbert scheme of points of a high-dimensional affine space satisfies Murphy's Law, as formulated by Vakil. In this short note, we instead consider the loci in the Hilbert scheme parameterizing punctual schemes with a given Hilbert function, and we show that these loci satisfy Murphy's Law. We also prove a related result for equivariant deformations of curve singularities with $\mathbb{G}_m$-action.

1. Introduction

It remains wide open whether the Hilbert scheme of points in $\mathbb{A}^n$ satisfies Murphy's Law, in the precise sense introduced in [Vak06, p. 2]. Vakil's version of Murphy's Law is based on the notion of smooth equivalence, which is the equivalence on local rings generated by the relations $R \sim S$, whenever there exists a smooth map of local rings $R \to S$. We say that R is a singularity of finite type over $\mathbb{Z}$ if R is the completion of a ring of finite type over $\mathbb{Z}$ at some prime ideal.

Question 1.1. *Up to smooth equivalence, does any singularity of finite type over $\mathbb{Z}$ occur on some Hilbert scheme of points in $\mathbb{A}^n$?*

In fact, beyond Iarrobino's foundational reducibility results [Iar72], very little is known about the singularities that occur on Hilbert schemes of points.

Inside the Hilbert scheme of points in $\mathbb{A}^n$, there are loci parameterizing the homogeneous ideals with a given Hilbert function. These strata, or variants thereof (including Gröbner strata, equivariant Hilbert schemes, etc.) have arisen in previous work on Hilbert schemes [CEVV09, Eva04, Led11]. We refer to these refined parameter spaces as punctual Hilbert function strata. These strata have a natural scheme structure (see Section 2), and our main result is a Murphy's Law for these strata.

Theorem 1.2. *Up to smooth equivalence, any singularity of finite type over $\mathbb{Z}$ occurs on some punctual Hilbert function strata.*

The idea behind this theorem is as follows. We fix a surface $X \subseteq \mathbb{P}^{n-1}$ with a pathological deformation space as constructed in [Vak06, Section 4]. We then consider the affine cone $CX \subseteq \mathbb{A}^n$ over X and define a zero-dimensional scheme Γ by taking a large infinitesimal neighborhood of the cone point of CX. In the language of ideals, we set $I_\Gamma := I_X + \mathfrak{m}^m$, where I_X is the ideal of CX, $\mathfrak{m}$ is the ideal of the cone point, and m is much larger than the Castelnuovo–Mumford regularity of X. Finally, we use a syzygetic argument to relate the $\mathbb{G}_m$-equivariant deformations of Γ with embedded deformations of $X \subseteq \mathbb{P}^{n-1}$, obtaining the main result.

Received by the editors May 24, 2012.

Beyond illustrating the unbounded pathologies of these strata, this theorem also suggests — but does not imply — an affirmative answer to Question 1.1.

Our techniques also prove a closely related result for the equivariant deformation spaces of curve singularities with a $\mathbb{G}_m$-action.

Theorem 1.3. *Up to smooth equivalence, any singularity of finite type over $\mathbb{Z}$ occurs on the $\mathbb{G}_m$-equivariant deformation space of some curve singularity with a $\mathbb{G}_m$-action.*

As above this theorem suggests — but does not imply — that deformation spaces of curve singularities satisfy Murphy's Law.

The paper is organized as follows. In Section 2, we review the deformation functors that will be used throughout. In Section 3, we prove Theorem 1.2, and in Section 4 we prove Theorem 1.3.

2. Review of deformation functors

Throughout, $\Bbbk$ denotes an arbitrary field. We mostly follow the notation of [Ser06, Section 2.4]. For an algebraic scheme Y, we use Def_Y to denote the functor from Artin rings to sets which sends an Artinian ring A to the set of deformations of Y over A. Similarly, for a closed subscheme $Y \subseteq Z$ we use $\mathrm{Def}_{Y|Z}$ to denote the functor of embedded deformations.

If we consider $\mathbb{A}^n_{\Bbbk}$ together with the $\mathbb{G}_m$-action of dilation, then this leads to a special case of the multigraded Hilbert scheme [HS04] where we give the polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$ the standard grading, and where we study families of graded ideals with a fixed Hilbert function h. The punctual Hilbert function strata correspond precisely to the multigraded Hilbert schemes in $\mathbb{A}^n$ with this standard grading and where the Hilbert function h has only finitely many nonzero entries. In particular, the punctual Hilbert function strata considered in this paper are actually projective schemes by [HS04, Corollary 1.2]. We refer the reader to [HS04] for proofs of these key facts about multigraded Hilbert schemes.

If $X \subseteq \mathbb{A}^n$ is $\mathbb{G}_m$-invariant, then we use $\mathrm{Def}^{\mathbb{G}_m}_{X|\mathbb{A}^n}$ to stand for the functor of $\mathbb{G}_m$-invariant embedded deformations of X, and we use $\mathcal{H}_X$ for the local ring at the point of the multigraded Hilbert scheme defined by $X \subseteq \mathbb{A}^n$.

With this notation, Theorem 1.2 amounts to the claim that, for any singularity R of finite type of $\mathbb{Z}$, there exists an n and a graded ideal $I_\Gamma \subseteq \Bbbk[x_1, \ldots, x_n]$ such that $\Bbbk[x_1, \ldots, x_n]/I_\Gamma$ is zero-dimensional and such that $\mathcal{H}_\Gamma$ is in the same smooth equivalence class as R.

3. Punctual Hilbert function strata

We begin with a result comparing the embedded deformation theory of $Y \subseteq \mathbb{P}^{n-1}$ with the deformation functors that will be essential to the proof of our main result. Throughout this section, we set $S := \Bbbk[x_1, \ldots, x_n]$ and $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle \subseteq S$.

Proposition 3.1. *Let $Y \subseteq \mathbb{P}^{n-1}$ be a projective scheme defined by the ideal $I_Y \subseteq S$. Let $\Gamma \subseteq \mathbb{A}^n$ be the 0-scheme defined by $I_\Gamma = I_Y + \mathfrak{m}^m$ for any $m \geq \mathrm{reg}(I_Y) + 2$. The functors $\mathrm{Def}^{\mathbb{G}_m}_{CY|\mathbb{A}^n}$ and $\mathrm{Def}^{\mathbb{G}_m}_{\Gamma|\mathbb{A}^n}$ are isomorphic.*

Proof. Consider a minimal free resolution of I_Y:

$$\cdots \longrightarrow F_3 \xrightarrow{\sigma_3} F_2 \xrightarrow{\sigma_2} F_1 \xrightarrow{\sigma_1} I_Y \longrightarrow 0.$$

Since $m > \operatorname{reg}(I_X)$, the minimal free resolution of $I_\Gamma = I_Y + \mathfrak{m}^m$ has the form

$$\cdots \longrightarrow F_3 \oplus S(-m-2)^{t_3} \xrightarrow{\left(\begin{smallmatrix}\sigma_3 & \tau_3\\ 0 & \tau_3'\end{smallmatrix}\right)} F_2 \oplus S(-m-1)^{t_2} \xrightarrow{\left(\begin{smallmatrix}\sigma_2 & \tau_2\\ 0 & \tau_2'\end{smallmatrix}\right)} F_1 \oplus S(-m)^{t_1} \xrightarrow{(f\ g)} I_\Gamma \longrightarrow 0,$$

for some positive integers t_i. The Hilbert function of I_Γ is given by:

$$h_d(I_\Gamma) = \begin{cases} h_d(I_Y), & \text{if } d < m,\\ 0, & \text{if } d \geq m.\end{cases}$$

We now construct a functorial map: $\mathrm{Def}^{\mathbb{G}_m}_{Y|\mathbb{A}^n} \to \mathrm{Def}^{\mathbb{G}_m}_{\Gamma|\mathbb{A}^n}$. For an Artin ring A, every element of $\mathrm{Def}^{\mathbb{G}_m}_{Y|\mathbb{A}^n}(A)$ is represented by an ideal $I'_Y \subseteq A[x_1,\dots,x_n]$, and our functorial map is given by sending the deformed ideal I'_Y to $I'_Y + \mathfrak{m}^m$. (By a minor abuse of notation, we also use $\mathfrak{m}$ to refer to $\mathfrak{m}A$). To check that this map is well defined, we must show that if I'_Y is a $\mathbb{G}_m$-invariant deformation of I_Y over A then $I'_Y + \mathfrak{m}^m$ is a $\mathbb{G}_m$-invariant deformation of I_Γ over A. By the definition of the multigraded Hilbert scheme [HS04, p. 1], this amounts to checking that for every $d \in \mathbb{N}$, the quotient

$$(A \otimes S_d)/(I'_Y + \mathfrak{m}^m)_d$$

is locally free of rank $h_d(I_\Gamma)$. This follows immediately from the observation that

$$(A \otimes S_d)/(I'_Y + \mathfrak{m}^m)_d \cong \begin{cases} (A \otimes S_d)/(I'_Y)_d, & \text{if } d < m,\\ 0, & \text{if } d \geq m.\end{cases}$$

To complete the proof of the proposition, it is now sufficient to prove that this map of functors induces an isomorphism on first order deformations and an injection on obstruction spaces (cf. [Pin74, Proof of Theorem 5.1]). The induced map on first order deformations is given by the composition of the natural maps:

$$\mathrm{Hom}_S(I_Y, S/I_Y)_0 \to \mathrm{Hom}_S(I_Y, S/I_\Gamma)_0 \to \mathrm{Hom}_S(I_\Gamma, S/I_\Gamma)_0. \tag{3.1}$$

An equivariant deformation of Y, i.e., an element of $\mathrm{Hom}_S(I_Y, S/I_Y)_0$, is equivalent to a degree 0 map $\widetilde{\alpha}\colon F_1 \to S/I_Y$ such that $\widetilde{\alpha} \circ \sigma_2 = 0$. The first map in (3.1) is sends $\widetilde{\alpha} \mapsto q \circ \widetilde{\alpha}$ where $q\colon S/I_Y \to S/I_\Gamma$ is the quotient map. Since each generator of F_2 has degree $\leq \operatorname{reg}(I_Y) + 1$, the image of $\widetilde{\alpha} \circ \sigma_2$ lands in $(S/I_Y)_{\leq \operatorname{reg}(I_\Gamma)+1}$. Since $m \geq \operatorname{reg}(I_\Gamma) + 2$, it then follows by degree reasons that the map $\widetilde{\alpha} \to q \circ \widetilde{\alpha}$ is both surjective and injective for degree zero maps.

An equivariant deformation of Γ, i.e., an element of $\mathrm{Hom}_S(I_\Gamma, S/I_\Gamma)_0$, is given by a pair of degree 0 morphisms (α, α') where $\alpha\colon F_1 \to S/I_\Gamma, \alpha\colon S(-m)^{t_1} \to S/I_\Gamma$ and where:

$$\alpha \circ \sigma_2 = 0 \quad \text{and} \quad \alpha \circ \tau_2 + \alpha' \circ \tau_2' = 0. \tag{3.2}$$

However, since $\operatorname{im}(\alpha') \subseteq (S/I_\Gamma)_m = 0$, it follows that α' is actually the zero map. Further, $\alpha \circ \tau_2$ lands in $(S/I_\Gamma)_m = 0$, so the second condition in (3.2) is trivially satisfied for any degree 0 map. Thus, the composition in (3.1), which sends $\widetilde{\alpha} \mapsto q \circ \widetilde{\alpha} \mapsto (q \circ \widetilde{\alpha}, 0)$, is a bijection.

The induced map of obstruction spaces is given by the composition:

$$\mathrm{Ext}^1_S(I_Y, S/I_Y)_0 \to \mathrm{Ext}^1_S(I_Y, S/I_\Gamma)_0 \to \mathrm{Ext}^1_S(I_\Gamma, S/I_\Gamma)_0. \tag{3.3}$$

We must check injectivity. A cycle for $\mathrm{Ext}^1_S(I_Y, S/I_Y)_0$ may be represented as a map: $\widetilde{\beta}\colon F_2 \to S/I_Y$, such that $\widetilde{\beta} \circ \sigma_3 = 0$. The map of obstruction spaces sends $\widetilde{\beta}$ to the cycle $(\beta, 0)\colon F_2 \oplus S(-m-1)^{t_1} \to S/I_\Gamma$ where $\beta := q \circ \widetilde{\beta}$. Since β has degree 0, the cycle $(\beta, 0)$ is a boundary if and only if there exists some $(\alpha, 0)\colon F_1 \oplus S(-m) \to S/I_\Gamma$ such that:

$$\alpha \circ \sigma_2 = \beta \quad \text{and} \quad \alpha \circ \tau_2 = 0.$$

Again by degree considerations, the second condition is automatically satisfied. Let $\widetilde{\alpha}$ any map $\widetilde{\alpha}\colon F_1 \to S/I_Y$ such that $q \circ \widetilde{\alpha} = \alpha$. Since the image of $\alpha \circ \sigma_2$ lands in $(S/I_\Gamma)_{\leq \mathrm{reg}(I_Y)+1}$, we conclude that

$$\alpha \circ \sigma_2 = \beta \iff \widetilde{\alpha} \circ \sigma_2 = \widetilde{\beta}.$$

This shows injectivity of the composition in (3.3), completing the proof. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. As in [Vak06, Theorems 4.4 and 4.5], we fix a smooth surface X whose abstract deformation space has the desired singularity type R and such that ω_X is very ample and $H^1(X, \mathcal{O}_X) = 0$. We may further choose a very ample $\mathcal{O}_X(1) := \omega_X^{\otimes e}$ for some large $e \gg 0$ such that each of the following holds (see [Vak06, Section 6.7]):

(i) $\oplus_{\nu \in \mathbb{Z}} H^1(X, \mathcal{O}_X(\nu)) = 0$;
(ii) $\oplus_{\nu \neq 0} H^1(X, T_X(\nu)) = 0$;
(iii) The embedding of $X \subseteq \mathbb{P}^n$ induced by $\mathcal{O}_X(1)$ is projectively normal.

By the argument in [Vak06, (4.6)], it follows that the embedded deformation theory of $X \subseteq \mathbb{P}^{n-1}$ is smoothly equivalent to the abstract deformation theory of X. Further, [Pie85, Comparison Theorem] then implies that the embedded deformation theory of $X \subseteq \mathbb{P}^{n-1}$ is equivalent to the $\mathbb{G}_m$-equivariant embedded deformation theory of the cone of $CX \subseteq \mathbb{A}^n$. Hence $\mathcal{H}_{CX}$ is smoothly equivalent to R.

We now let I_X be the ideal defining $X \subseteq \mathbb{P}^{n-1}$ and we set $I_\Gamma := I_X + \mathfrak{m}^m$ for any $m \geq \mathrm{reg}(I_X) + 2$. By Proposition 3.1, we conclude that $\mathcal{H}_{CX} \cong \mathcal{H}_\Gamma$, and hence $\mathcal{H}_\Gamma$ is smoothly equivalent to R. □

4. Curve singularities with a $\mathbb{G}_m$-action

If C is an affine curve $C \subseteq \mathbb{A}^n$ with a $\mathbb{G}_m$-action and an isolated singularity, then we use $\mathrm{Def}_C^{\mathbb{G}_m}$ to denote the functor of equivariant deformations of C. For any singularity type R, we will construct an equivariant curve $C \subseteq \mathbb{A}^n$ such that $\mathcal{H}_C$ lies in the same smooth equivalence class as R, and such that the map of functors $\mathrm{Def}^{\mathbb{G}_m}_{C|\mathbb{A}^n} \to \mathrm{Def}^{\mathbb{G}_m}_C$ is smooth. By definition, $\mathcal{H}_C$ represents the functor $\mathrm{Def}^{\mathbb{G}_m}_{C|\mathbb{A}^n}$, and hence this will prove the theorem.

Proof of Theorem 1.3. Fix a singularity type R. As in the proof of Theorem 1.2, we choose a codimension $n-3$ ideal $I_X = (f_1, \ldots, f_s) \subseteq S$ such that $\mathcal{H}_X$ has the same singularity type as R. Set $m \geq \mathrm{reg}(I_X) + 2$. If $\Bbbk$ is an infinite field, then we may define a regular sequence g_1, g_2 on S/I_X, where g_i is a generic homogeneous form of degreee m. If $\Bbbk$ is finite, then the same holds for some $m \gg \mathrm{reg}(I_X)$. Let $I_C := I_X + \langle g_1, g_2 \rangle$.

There is an induced map $\mathcal{H}_X \to \mathcal{H}_C$, by essentially the same argument as in Theorem 1.2. The condition of being a regular sequence on S/I_X is open, and hence there is an open set in the vector space $S_m \oplus S_m$ parameterizing the regular sequences. Thus the choice of g_1, g_2 is a smooth choice, and so $\mathcal{H}_C$ is in the same smooth equivalence class as $\mathcal{H}_X$.

Since C is affine, we have a formally smooth map of functors $\mathrm{Def}_{C|\mathbb{A}^n} \to \mathrm{Def}_C$ by [Art76, p. 4]. Further, by always choosing homogeneous representatives, one can extend Artin's argument to show that the map $\mathrm{Def}^{\mathbb{G}_m}_{C|\mathbb{A}^n} \to \mathrm{Def}^{\mathbb{G}_m}_C$ is formally smooth. Since the tangent spaces of both functors are finite-dimensional, the map of equivariant functors is actually smooth. As $\mathcal{H}_C$ is in the same smooth equivalence class as R, this completes the proof. $\square$

Acknowledgments

I would like to thank Ravi Vakil and Mauricio Velasco for conversations which inspired this project. I also thank Dustin Cartwright, David Eisenbud, Robin Hartshorne, Brian Osserman, Fred van der Wyck, and Bianca Viray for useful conversations. We thank the referee for a close reading and suggestions that improved this paper. The author was partially supported by an NSF fellowship and by a Simons Foundation fellowship.

References

[Art76] M. Artin, *Deformations of singularities*, Tata Institute of Fundamental Research Bombay, 1976

[CEVV09] D.A. Cartwright, D.Erman, M.Velasco and B.Viray, *Hilbert schemes of 8 points*, Algebra Number Theory **3**(7) (2009), 763–795.

[Eva04] L. Evain, *Irreducible components of the equivariant punctual Hilbert schemes*, Adv. Math. **185**(2), (2004), 328–346.

[HS04] M. Haiman and B. Sturmfels, *Multigraded Hilbert schemes*, J. Algebr. Geom. **13**(4) (2004), 725–769.

[Iar72] A. Iarrobino, *Reducibility of the families of 0-dimensional schemes on a variety*, Invent. Math. **15** (1972), 72–77.

[Led11] M. Lederer, *Gröbner strata in the Hilbert scheme of points*, J. Commut. Algebra **3**(3) (2011), 349–404.

[Pie85] R. Piene and M. Schlessigner, *On the Hilbert scheme compactification of the space of twisted cubics*, Amer. J. Math. **107**(4) (1985), 761–774.

[Pin74] H.C. Pinkham, Deformations of algebraic varieties with $\mathbb{G}_m$ action, Société Mathématique de France. Astérisque, No. 20. 1974.

[Ser06] E. Sernesi, Deformations of algebraic schemes, Vol. 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, 2006.

[Vak06] R. Vakil, *Murphy's law in algebraic geometry: badly-behaved deformation spaces*, Invent. Math. **164**(3) (2006), 569–590.

Department of Mathematics, University of Wisconsin, Madison, WI 53706-1325, USA
E-mail address: derman@math.wisc.edu
URL: http://www.math.wisc.edu/∼derman/