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LOWER BOUNDS ON THE HAUSDORFF MEASURE OF NODAL SETS II

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Abstract. We give a very short argument showing how the main identity (0.2) from our earlier paper [12] immediately leads to the best lower bound currently known [2] for the Hausdorff measure of nodal sets in dimensions $n \geq 3$.

Let (M, g) be a compact smooth Riemannian manifold of dimension n and let e_λ be real-valued eigenfunction of the associated Laplacian, i.e.,

$$-\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x)$$

with frequency $\lambda > 0$. Recent papers have been concerned with lower bounds for the $(n-1)$-dimensional Hausdorff measure, $|Z_\lambda|$, of the nodal set of e_λ,

$$Z_\lambda = \{x \in M : e_\lambda(x) = 0\}$$

in dimensions $n \geq 3$. When $n = 2$ the sharp lower bound by the frequency, $\lambda \lesssim |Z_\lambda|$, was obtained by Brüning in [1] and independently by Yau. For all dimensions, in the analytic case, the sharp upper and lower bounds $|Z_\lambda| \approx \lambda$ were obtained by Donnelly and Fefferman [4, 5].

Until recently, the best known lower bound when $n \geq 3$ seems to have been $e^{-c\lambda} \lesssim |Z_\lambda|$ (see [6]). Using a variation (0.2) of an identity of Dong [3], the authors showed in [12] that this can be improved to be $\lambda^{\frac{7}{4}-\frac{3n}{4}} \lesssim |Z_\lambda|$. Independently Colding and Minicozzi [2] obtained the more favorable lower bound

$$\lambda^{1-\frac{n-1}{2}} \lesssim |Z_\lambda| \tag{0.1}$$

by a different method. Subsequently, the first author and Hezari [7] were also able to obtain the lower bound (0.1) by an argument which was in the spirit of [12]. The purpose of this sequel to [12] is to show that the lower bound (0.1) can also be derived by a very small modification (indeed a simplification) of the original argument of [12].

The lower bounds of [7, 12] are based on the identity

$$\lambda^2 \int_M |e_\lambda|\, dV = 2 \int_{Z_\lambda} |\nabla_g e_\lambda|_g\, dS, \tag{0.2}$$

from [12] and the (sharp) lower bound for L^1-norms

$$\lambda^{-\frac{n-1}{4}} \lesssim \int_M |e_\lambda|\, dV, \tag{0.3}$$

which was also established in [12]. Here, dV is the volume element of (M, g).

The lower bound (0.1) is a very simple consequence of the identity (0.2) and the following lemma (which was implicit in [12]).

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Lemma 1. *If* $\lambda > 0$ *then*

$$\|\nabla_g e_\lambda\|_{L^\infty(M)} \lesssim \lambda^{1+\frac{n-1}{2}} \|e_\lambda\|_{L^1(M)}. \tag{0.4}$$

Indeed if we use (0.2) and then apply Lemma 1, we obtain

$$\begin{aligned} \lambda^2 \int_M |e_\lambda| \, dV = 2 \int_{Z_\lambda} |\nabla_g e_\lambda|_g \, dS &\le 2|Z_\lambda| \, \|\nabla_g e_\lambda\|_{L^\infty(M)} \\ &\lesssim 2|Z_\lambda| \, \lambda^{1+\frac{n-1}{2}} \|e_\lambda\|_{L^1(M)}, \end{aligned} \tag{0.5}$$

which of course implies (0.1).

Lemma 1 improves the upper bound on the integral given in Lemma 1 of [12], and its proof is almost the same as the proof of (0.3) in Proposition 2 of [12]:

Proof. For $\rho \in C_0^\infty(\mathbb{R})$, we define the λ-dependent family of operators

$$\chi_\lambda f = \int_{-\infty}^{\infty} \rho(t) e^{-it\lambda} e^{it\sqrt{-\Delta_g}} f \, dt = \hat{\rho}(\lambda - \sqrt{-\Delta_g}) f = \sum_{j=0}^{\infty} \hat{\rho}(\lambda - \lambda_j) E_j f, \tag{0.6}$$

on $L^2(M, dV)$ with $E_j f$ denoting the projection of f onto the jth eigenspace of $\sqrt{-\Delta_g}$. Here $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ are its eigenvalues, and if $\{e_j\}_{j=0}^\infty$ is the associated orthonormal basis of eigenfunctions (i.e. $\sqrt{-\Delta_g} e_j = \lambda_j e_j$), then

$$E_j f = \Big(\int_M f \, \overline{e_j} \, dV \Big) e_j.$$

We denote the kernel of χ_λ by $K_\lambda(x,y)$, i.e.,

$$\chi_\lambda f(x) = \int_M K_\lambda(x,y) f(y) dV(y), \quad (f \in C(M)).$$

If the Fourier transform of ρ satisfies $\hat{\rho}(0) = 1$, then $\chi_\lambda e_\lambda = e_\lambda$, or equivalently

$$\int_M K_\lambda(x,y) e_\lambda(y) dV(y) = e_\lambda(x).$$

Thus, K_λ is a reproducing kernel for e_λ if $\hat{\rho}(0) = 1$.

As in Section 5.1 in [10], we choose ρ so that the reproducing kernel $K_\lambda(x,y)$ is uniformly bounded by $\lambda^{\frac{n-1}{2}}$ on the diagonal as $\lambda \to +\infty$. This is essential for the proof of (0.4). If we assume that $\rho(t) = 0$ for $|t| \notin [\varepsilon/2, \varepsilon]$, with $\varepsilon > 0$ being a fixed number which is smaller than the injectivity radius of (M,g), then it is proved in Lemma 5.1.3 of [10] that

$$K_\lambda(x,y) = \lambda^{\frac{n-1}{2}} a_\lambda(x,y) e^{i\lambda r(x,y)}, \tag{0.7}$$

where $a_\lambda(x,y)$ is bounded with bounded derivatives in (x,y) and where $r(x,y)$ is the Riemannian distance between points. This WKB formula for $K_\lambda(x,y)$ is known as a parametrix and may be obtained from the Hörmander parametrix for $e^{it\sqrt{-\Delta}}$ in [8] or from the Hadamard parametrix for $\cos t\sqrt{-\Delta}$. We refer to [10,11] for the background.

It follows from (0.7) that

$$|\nabla_g K_\lambda(x,y)| \le C\lambda^{1+\frac{n-1}{2}}, \tag{0.8}$$

and therefore,

$$\begin{aligned}\sup_{x\in M}|\nabla_g\chi_\lambda f(x)| &= \sup_x\Big|\int f(y)\,\nabla_g K_\lambda(x,y)\,dV\Big|\\ &\le \big\|\nabla_g K_\lambda(x,y)\big\|_{L^\infty(M\times M)}\,\|f\|_{L^1}\\ &\le C\lambda^{1+\frac{n-1}{2}}\|f\|_{L^1}.\end{aligned}$$

To complete the proof of the Lemma, we set $f=e_\lambda$ and use that $\chi_\lambda e_\lambda = e_\lambda$. □

We note that $K_\lambda(x,y)$ has quite a different structure from the kernels of the spectral projection operators $E_{[\lambda,\lambda+1]}=\sum_{j:\lambda_j\in[\lambda,\lambda+1]}E_j$ and the estimate in Lemma 1 is quite different from the sup norm estimate in Lemma 4.2.4 of [10]. Indeed, in a λ^{-1} neighborhood of the diagonal, the spectral projections kernel $E_{[\lambda,\lambda+1]}(x,y)$ is of size λ^{n-1}. For instance, in the case of the standard sphere S^n, the kernel of the orthogonal projection E_k onto the space of spherical harmonics of degree $k\simeq\lambda$ is the constant $E_k(x,x)=\frac{\lambda^{n-1}}{Vol(S^n)}$ on the diagonal. We are able to choose the test function ρ above, so that the reproducing kernel $K_\lambda(x,y)$ is uniformly of size $\lambda^{\frac{n-1}{2}}$ (as in [9] and [10] Section 5.1) because we only need it to reproduce eigenfunctions e_λ of one eigenvalue and because it does not matter how K_λ acts on eigenfunctions of other eigenvalues. From the viewpoint of Lagrangian distributions, the Lagrangian manifold Λ_x associated to both $E_{[\lambda,\lambda+1]}(x,y)$ and $K_\lambda(x,y)$ for fixed x is the flowout $\Lambda_x=\bigcup_{t\in\text{supp }\rho}G^tS^*_xM\subset S^*M$ of the unit-cosphere S^*_xM under the geodesic flow G^t. The natural projection of Λ_x to M has a large singularity along S^*_xM which causes the λ^{n-1} blowup of $E_{[\lambda,\lambda+1]}(x,y)$ at $y=x$, but the projection is a covering map for the part of Λ_x where $t\in[\varepsilon,2\varepsilon]=\text{supp}\rho$. The parametrix (0.7) reflects the fact that the test function ρ cuts out all of Λ_x except where its projection to M is a covering map. For further discussion of the geometry underlying Lagrangian distributions we refer to [10, 11, 13].

Finally, we briefly compare the proof of (0.1) in this note with the estimates in [12]:

- Instead of Lemma 0.1, the estimate $\|\nabla_g e\|_{L^\infty(M)}\lesssim\lambda^{1+\frac{n-1}{2}}\|e_\lambda\|_{L^2}$ was used in [12]. The latter estimate is a consequence of the pointwise local Weyl law for $|\nabla e_\lambda(x)|^2$.
- In [12] the authors proved the lower bounds (0.3) by showing that

  $$\|e_\lambda\|_{L^\infty(M)}\lesssim\lambda^{\frac{n-1}{2}}\|e_\lambda\|_{L^1(M)},$$

  by essentially the same argument as in Lemma 1. In the proof given in this note, (0.3) is not used in the proof of (0.1) since the factor $\|e_\lambda\|_{L^1(M)}$ cancels out in the left and right sides.

References

[1] J. Brüning, *Über Knoten Eigenfunktionen des Laplace–Beltrami Operators*, Math. Z. **158** (1978), 15–21.

[2] T.H. Colding and W. P. Minicozzi II, *Lower bounds for nodal sets of eigenfunctions*, Comm. Math. Phys. **306** (2011), 777–784.

[3] R.T. Dong, *Nodal sets of eigenfunctions on Riemann surfaces*, J. Different. Geom. **36** (1992), 493–506.

[4] H. Donnelly and C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. **93** (1988), 161–183.
[5] H. Donnelly and C. Fefferman, *Nodal sets for eigenfunctions of the Laplacian on surfaces*, J. Amer. Math. Soc. **3** (1990), 333–353.
[6] Q. Han and F.H. Lin, *Nodal sets of solutions of Elliptic Differential Equations*, book in preparation (online at http://www.nd.edu/qhan/nodal.pdf).
[7] H. Hezari and C.D. Sogge, *A natural lower bound for the size of nodal sets*, Anal. PDE **5** (2012), no. 5, 1133–1137. `arXiv:1107.3440`.
[8] L. Hörmander, *The spectral function of an elliptic operator*, Acta Math., **121** (1968), 193–218.
[9] C.D. Sogge, *Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds*, J. Funct. Anal. **77** (1988), 123–138.
[10] C.D. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, 105, Cambridge University Press, Cambridge, 1993.
[11] C.D. Sogge, *Hangzhou lectures on eigenfunctions of the Laplacian*, Annals of Mathematics Studies, Princeton University Press, to appear.
[12] C.D. Sogge and S. Zelditch, *Lower bounds on the Hausdorff measure of nodal sets*, Math. Res. Lett. **18** (2011), 25–37.
[13] S. Zelditch, CBMS Lectures, in preparation.

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