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SINGULARITIES OF SOLUTIONS TO COMPRESSIBLE EULER EQUATIONS WITH VACUUM

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Abstract. Presented are two results on the formation of finite-time singularities of solutions to the compressible Euler equations in two and three space dimensions for isentropic, polytropic, ideal fluid flows. The initial velocity is assumed to be symmetric and the initial sound speed is required to vanish at the origin. They are smooth in Sobolev space H^3, but not required to have a compact support. It is shown that the H^3 norm of the velocity field and the sound speed will blow up in a finite time.

1. Introduction

Euler equation is one of the most fundamental equations in fluid dynamics. Many interesting fluid dynamic phenomena can be described by the Euler equation (see, for instance, [12]). Recently, singularity formation in fluid mechanics has attracted the attention of a number of researchers; see, for instance, [2, 3, 5–8, 11] and two recent review articles [1,4]. For compressible Euler equations of the motion of polytropic ideal fluid flows, Sideris (see [13]) proved, under various settings, several very interesting results on the formation of finite-time singularities to solutions whose initial velocity field has a compact support and initial density is strictly positive and is equal to a positive constant outside the support of the initial velocity field. It is very interesting to investigate the long-time behavior of solutions to compressible Euler equations with initial data containing vacuum states, as has been pointed out in [9].

In this short paper, we prove that solutions to compressible Euler equations will develop finite-time singularities for radially symmetric initial data whose initial velocity field has no compact support and initial density contains vacuum states. In particular, it is shown that the H^3 norm of the velocity field and the sound speed will blow up in a finite time.

To state our theorems, let us begin with the compressible Euler equations for isentropic, polytropic, ideal fluid flows:

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ (\rho u)_t + \nabla\cdot(\rho u\otimes u) + \nabla p = 0, \end{cases} \tag{1.1}$$

where $\rho(t,\cdot):\mathbb{R}^n\to\mathbb{R}$ is the scalar mass density, $u(t,\cdot):\mathbb{R}^n\to\mathbb{R}^n$ is the velocity field, $p(t,x)$ is the pressure, which is given by the equation of state

$$p = A\rho^\gamma. \tag{1.2}$$

Here, $A>0$ is an entropy constant, γ is the adiabatic index. For polytropic gases, one has $1<\gamma\le\frac{5}{3}$.

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The compressible Euler equations (1.1) are imposed on the following initial data:

$$\rho(0,x)=\rho_0(r),\quad u(0,x)=\frac{x}{r}v_0(r). \tag{1.3}$$

Here and in what follows, we will use

$$r=|x|,\quad x\in\mathbb{R}^n,$$

for notational convenience. Denote the sound speed $c(\rho)$ by

$$c(\rho)=\sqrt{\frac{\partial p(\rho)}{\partial\rho}}=\sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}}. \tag{1.4}$$

Our first result is on the formation of singularities of solutions to the compressible Euler equations in three dimensions:

Theorem 1.1. *Assume that* $\gamma>1$, $\rho_0\geq 0$ *and* $(c_0,u_0)\in H^3(\mathbb{R}^3)$ *with* $c_0=c(\rho_0)$. *Moreover, assume that* ρ_0 *and* v_0 *satisfy*

$$\rho_0(0)=0, \tag{1.5}$$

$$\int_{\mathbb{R}^3}\rho_0(r)dx>0, \tag{1.6}$$

and

$$-\int_{\mathbb{R}^3}\frac{(1+r)\rho_0v_0}{r^2e^r}dx\geq\sqrt{\frac{A}{(\gamma+1)(4\pi)^{\gamma-1}}}\Big(\int_{\mathbb{R}^3}\frac{\rho_0}{re^r}dx\Big)^{\frac{\gamma+1}{2}}. \tag{1.7}$$

Then the solution (ρ,u) *to the compressible Euler equations* (1.1) *with the initial data* (1.3) *will develop finite-time singularities.*

Our second result is on the formation of singularities of solutions to the compressible Euler equations in two dimensions:

Theorem 1.2. *Assume that* $\gamma>1$, $\rho_0\geq 0$ *and* $(c_0,u_0)\in H^3(\mathbb{R}^2)$ *with* $c_0=c(\rho_0)$. *Let* $K_0(r)$ *be the modified Bessel function*

$$K_0(r)=\int_0^\infty e^{-r\cosh t}\,dt. \tag{1.8}$$

If the initial data ρ_0 *and* v_0 *satisfy* (1.5),

$$\int_{\mathbb{R}^2}\rho_0(r)dx>0, \tag{1.9}$$

and

$$\int_{\mathbb{R}^2}\rho_0(r)v_0(r)K_0'(r)dx\geq\sqrt{\frac{A}{\gamma+1}}\frac{\big(\int_{\mathbb{R}^2}\rho_0(r)K_0(r)dx\big)^{\frac{\gamma+1}{2}}}{\big(\int_{\mathbb{R}^2}K_0(r)dx\big)^{\frac{\gamma-1}{2}}}, \tag{1.10}$$

then the solution (ρ,u) *to the compressible Euler equations* (1.1) *with the initial data* (1.3) *will develop finite-time singularities.*

Remark 1.3. For a polytropic ideal gas, the adiabatic index $\gamma \in (1, \frac{5}{3}]$ and hence $\frac{2}{\gamma-1} \geq 3$. Owing to the expression of the sound speed in (1.4), it is easy to get

$$\rho_0 = \big(A\gamma\big)^{-\frac{1}{\gamma-1}} c_0^{\frac{2}{\gamma-1}}.$$

Consequently, one also has $\rho_0 \in H^3(\mathbb{R}^n)$ under the condition that $c_0 \in H^3(\mathbb{R}^n)$, which in turn implies that $\rho(t,\cdot) \in H^2(\mathbb{R}^n)$ as long as the solution is smooth. For $\gamma > \frac{5}{3}$, local well-posedness theory in Theorem 2.1 implies that $\rho \in C([0,T) \times \mathbb{R}^n)$.

Remark 1.4. For the non-isentropic, polytropic, ideal gases, the entropy S is transported by the flows. It is easy to verify that our proofs of the blow up parts in Theorems 1.1 and 1.2 are still true. However, the local well-posedness of the coupled system with initial vacuum is much more complicated (see, for instance, [9]). We do not pursue this issue in this short paper.

There are several ingredients in the proofs of the above theorems. The first one is to write the compressible Euler equations (1.1) as a quasi-linear wave-type equation in terms of ρ with inhomogeneous terms involving u:

$$\rho_{tt} - \Delta p = \nabla \cdot [\nabla \cdot (\rho u \otimes u)].$$

This is in fact what Sideris did in [13]. However, we will treat this equation in a very different manner from that in [13] due to the facts that $\rho(r) \to 0$ as $r \to \infty$ and the initial velocity field u_0 has no compact support. We will choose the modified Bessel function $K_0(r)$ in two-dimensional (2D) case and $\frac{1}{re^r}$ in 3D case as test functions for the above quasi-linear wave type equation, respectively, to explore the nonlinear structure of the pressure p as a function of ρ, which eventually corresponds to the formation of finite-time singularities. The second one is making use of the symmetric structure of the solutions, which results in a good sign for the inhomogeneous term $\nabla \cdot [\nabla \cdot (\rho u \otimes u)]$ when taking the inner product of the above wave-type equation with test functions. Moreover, the symmetric structure of the solutions will also be used to eliminate boundary terms on the artificial boundary $r = 0$.

The remaining part of this paper is simply organized as follows: in Section 2, we will prove Theorem 1.1. Then in Section 3, we present the proof of Theorem 1.2.

2. Singularities of compressible Euler equations in 3D

In this section, we will prove Theorem 1.1. Before that, let us recall the local well-posedness of compressible Euler equations (for example, see [10]). To be precise, let us formulate it as a theorem for our use.

Theorem 2.1. *Assume that $\gamma > 1$, $\rho_0 \geq 0$ and $(c_0, u_0) \in H^3(\mathbb{R}^n)$ with $c_0 = c(\rho_0)$ and $n = 2,\ 3$. Then there exists a unique solution (ρ, u) to the compressible Euler equations* (1.1) *with initial data* (1.3) *on some time interval* $[0,T)$*, which satisfies*

$$c(\rho),\ u \in C([0,T), H^3(\mathbb{R}^n)) \cap C^1([0,T), H^2(\mathbb{R}^n)) \cap C^2([0,T), H^1(\mathbb{R}^n)),$$

and

$$\rho \in C([0,T) \times \mathbb{R}^n). \tag{2.1}$$

If $1 < \gamma \leq \frac{5}{3}$*, then*

$$\rho \in C([0,T), H^2(\mathbb{R}^n)) \cap C^1([0,T), H^1(\mathbb{R}^n)). \tag{2.2}$$

If ρ_0 and u_0 are radially symmetric and (1.5) *is satisfied, then*

$$\rho(t,0) \equiv 0,\ \ u(t,0) \equiv 0, \quad u(t,x) = \frac{x}{r}v(t,r). \tag{2.3}$$

The proof of the local well-posedness in the above theorem is based on the fact that the compressible Euler equations can be written as a symmetric hyperbolic system in terms of (c,u):

$$\begin{cases} c_t + u\cdot\nabla c + \dfrac{2c}{\gamma-1}\nabla\cdot u = 0, \\ u_t + u\cdot\nabla u + \dfrac{\gamma c}{\gamma-1}\nabla c = 0. \end{cases}$$

See [10] for more details. The fact (2.2) is due to Remark 1.3 and the equation of mass conservation in the original compressible Euler equations (1.1). The fact $u(t,0)\equiv 0$ is in fact a universal identity for smooth symmetric vector. To see $\rho(t,0)\equiv 0$, we use $u(t,0)\equiv 0$ and the equation of mass conservation to get

$$\rho(t,0) = \rho_0(0)e^{-\int_0^t \nabla\cdot u(s,0)ds}.$$

It is ready to present the proof of Theorem 1.1.

Proof. We prove Theorem 1.1 by contradiction. Suppose that the solution $(c,u)\in H^3(\mathbb{R}^3)$ for all time $t\geq 0$ and $T=\infty$ in Theorem 2.1. We will derive that the density blows up in the ball centered at the origin with an arbitrary small radius $r_0>0$ in a finite time, which contradicts with (2.1).

Applying the time derivative to the first equation of the compressible Euler system (1.1), we have (in the sense of distribution)

$$\rho_{tt} = -\nabla\cdot(\rho u)_t = \Delta p + \nabla\cdot[\nabla\cdot(\rho u\otimes u)]. \tag{2.4}$$

Taking the L^2 inner product of the above equation with the test function $\frac{1}{re^r}$, one has

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^3}\frac{\rho}{re^r}dx = \int_{\mathbb{R}^3}\Delta p\frac{1}{re^r}dx + \int_{\mathbb{R}^3}\nabla\cdot[\nabla\cdot(\rho u\otimes u)]\frac{1}{re^r}dx.$$

Let us first compute that

$$\begin{aligned} &\int_{\mathbb{R}^3}\Delta p\frac{1}{re^r}dx \\ &= \int_{\mathbb{R}^3}p\Delta\frac{1}{re^r}dx - \lim_{\varepsilon\to 0}\int_{|x|=\varepsilon}\left(\frac{\partial p}{\partial r}\frac{e^{-r}}{r} - \frac{d\left(\frac{1}{re^r}\right)}{dr}p\right)ds \\ &= \int_{\mathbb{R}^3}p\frac{1}{re^r}dx + \lim_{\varepsilon\to 0}\int_{|x|=\varepsilon}\left(\frac{1}{re^r}\right)'p\,ds. \end{aligned}$$

Using the equation of state (1.2) and the fact that $\rho(t,0)\equiv 0$, we have

$$\begin{aligned} A\int_{\mathbb{R}^3}\Delta\rho^\gamma\frac{1}{re^r}dx &= A\int_{\mathbb{R}^3}\rho^\gamma\frac{1}{re^r}dx \\ &\geq \frac{A}{(4\pi)^{\gamma-1}}\left(\int_{\mathbb{R}^3}\frac{\rho}{re^r}dx\right)^\gamma. \end{aligned}$$

On the other hand, noting (2.3), one has

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{re^r} \nabla \cdot [\nabla \cdot (\rho u \otimes u)] dx \\
&= \int_{\mathbb{R}^3} \partial_i \partial_j \left(\frac{1}{re^r} \right) (\rho u_i u_j) dx \\
&= \int_{\mathbb{R}^3} \rho v^2 \frac{x_j}{r} \partial_r \left[\frac{x_j}{r} \left(\frac{1}{re^r} \right)' \right] dx \\
&= \int_{\mathbb{R}^3} \rho v^2 \left(\frac{1}{re^r} \right)'' dx \\
&= \int_{\mathbb{R}^3} \rho v^2 \left[\frac{1}{r} + \frac{2}{r^2} + \frac{2}{r^3} \right] e^{-r} \, dx > 0.
\end{aligned}$$

Here and in what follows, we use Einstein's convention for summation over repeated indices. Consequently,

$$\frac{d^2}{dt^2} F(t) \geq \frac{A}{(4\pi)^{\gamma-1}} \big(F(t)\big)^{\gamma}, \tag{2.5}$$

where

$$F(t) = \int_{\mathbb{R}^3} \frac{\rho(t,r)}{re^r} dx.$$

Using the equation of conservation of mass and integration by parts, and noting (1.7), one has

$$F'(0) = \frac{d}{dt} \int_{\mathbb{R}^3} \frac{\rho}{re^r} dx \Big|_{t=0} = - \int_{\mathbb{R}^3} \frac{(1+r)\rho_0 v_0}{r^2 e^r} dx > 0. \tag{2.6}$$

The combination of (2.5) and (2.6) gives that

$$F'(t) = F'(0) + \int_0^t \frac{d^2}{ds^2} F(s) ds \geq F'(0) > 0.$$

Consequently, one can multiply the both sides of (2.5) by $\frac{d}{dt}F(t)$ to get

$$\begin{aligned}
\big(F'(t)\big)^2 \geq{}& \frac{A}{(\gamma+1)(4\pi)^{\gamma-1}} \big(F(t)\big)^{\gamma+1} \\
&+ \left(\frac{d}{dt} \int_{\mathbb{R}^3} \frac{\rho}{re^r} dx \right)^2 \Big|_{t=0} - \frac{A}{(\gamma+1)(4\pi)^{\gamma-1}} \left(\int_{\mathbb{R}^3} \frac{\rho_0}{re^r} dx \right)^{\gamma+1}.
\end{aligned}$$

Noting (1.7) and (2.6), and by denoting

$$C_0 = \sqrt{\frac{A}{(\gamma+1)(4\pi)^{\gamma-1}}},$$

one has

$$F'(t) \geq C_0 \big(F(t)\big)^{\frac{\gamma+1}{2}},$$

which implies that

$$\int_{\mathbb{R}^3} \frac{\rho}{re^r} dx \geq \left(F(0)^{-\frac{\gamma-1}{2}} - \frac{\gamma-1}{2} C_0 t \right)^{-\frac{2}{\gamma-1}}.$$

Noting (1.6), one has

$$\int_{\mathbb{R}^3} \frac{\rho_0}{re^r} dx > 0,$$

By using the mass conservation, one has

$$F(t) = \int_{\mathbb{R}^3} \frac{\rho}{re^r} dx \le \int_{B_{r_0}} \frac{\rho}{re^r} dx + \frac{1}{r_0} \int_{\mathbb{R}^3} \rho_0 dx,$$

for any given $r_0 > 0$, here B_{r_0} is the 3D ball centered at the origin with radius r_0. Consequently, one concludes that $\int_{r\le r_0} \rho(t,r) r dr$ cannot be bounded as $t \to \frac{2F(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_0}$, which in turn implies that $\rho(t,r)$ will blow up for $r \le r_0$ as $t \to \frac{2F(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_0}$. By the assumption $c \in L^\infty([0,\infty), H^3(\mathbb{R}^3)$ and Sobolev imbedding theorem, one has $c \in L^\infty([0,\infty)\times\mathbb{R}^3)$ and hence $\rho \in L^\infty([0,\infty)\times\mathbb{R}^3)$. We arrive at a contradiction. So, the H^3 norm of (c,u) cannot be bounded before the time $\frac{2F(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_0}$. The proof of Theorem 1.1 is completed. □

3. Singularities of compressible Euler equations in 2D

In this section, we prove Theorem 1.2. First of all, let us recall that the modified Bessel function $K_0(r)$ and its derivative $K_0'(r)$ decay sufficiently fast as $r \to \infty$. In fact, for any given $k > 1$, one has

$$\begin{aligned}
\sup_{0<r<\infty} r^k K_0(r) &= \sup_{0<r<\infty} r^k \int_0^\infty e^{-r\cosh t} dt \\
&= \sup_{0<r<\infty} r^k \int_0^1 e^{-\frac{re^t}{2}} e^{-\frac{re^{-t}}{2}} dt + \sup_{0<r<\infty} r^k \int_1^\infty e^{-\frac{re^t}{2}} e^{-\frac{re^{-t}}{2}} dt \\
&\le \sup_{0<r<\infty} r^k e^{-\frac{r}{2e}} \int_0^1 e^{-\frac{re^t}{2}} dt \\
&\quad + \sup_{0<r<\infty} \int_1^\infty r^k e^{-\frac{re^t}{2}} \left(\frac{r}{2}e^t\right)^k \left(\frac{2}{r}e^{-t}\right)^k dt \\
&\le \sup_{0<r<\infty} r^k e^{-\frac{r}{2e}} + \sup_{0<r<\infty} 2^k \int_1^\infty e^{-kt} dt \sup_{0<s<\infty} e^{-s} s^k \\
&< \infty
\end{aligned}$$

and

$$\begin{aligned}
\sup_{0<r<\infty} r^k |K_0'(r)| &= \sup_{0<r<\infty} r^k \int_0^\infty e^{-r\cosh t} \cosh t dt \\
&= \sup_{0<r<\infty} 2r^k \int_0^1 e^{-\frac{re^t}{2}} e^{-\frac{re^{-t}}{2}} dt + \sup_{0<r<\infty} r^k \int_1^\infty e^{-\frac{re^t}{2}} e^{-\frac{re^{-t}}{2}} e^t dt \\
&\le \sup_{0<r<\infty} 2r^k e^{-\frac{r}{2e}} + \sup_{0<r<\infty} 2^k \int_1^\infty e^{-(k-1)t} dt \sup_{0<s<\infty} e^{-s} s^k \\
&< \infty.
\end{aligned}$$

On the other hand, for $r > 0$, we also have

$$\begin{aligned} K_0(r) &\leq \int_0^1 e^{-r\cosh t}dt + \int_1^\infty e^{-\frac{re^t}{2}}e^{-\frac{re^{-t}}{2}}dt \\ &\leq 1 + \int_1^\infty e^{-\frac{re^t}{2}}dt \leq 1 + \frac{2}{r}, \end{aligned}$$

and

$$\begin{aligned} |K_0'(r)| &\leq \int_0^1 e^{-r\cosh t}e^t dt + \int_1^\infty e^{-\frac{re^t}{2}}e^{-\frac{re^{-t}}{2}}e^t dt \\ &\leq 2 + \int_1^\infty e^{-\frac{re^t}{2}}e^t dt \\ &\leq 2 + \int_1^\infty 2\big(\frac{re^t}{2}\big)^{-2}e^t dt \leq 2 + \frac{1}{2r^2}. \end{aligned}$$

We in fact have proved the following lemma:

Lemma 3.1. *The modified Bessel function $K_0(r) = \int_0^\infty e^{-r\cosh t}dt$ satisfies*

$$\begin{cases} K_0(r) \leq \dfrac{3}{r}, |K_0'(r)| \leq \dfrac{1}{r^2}, & 0 < r < \dfrac{1}{2}, \\ K_0(r) \leq \dfrac{C_k}{r^k}, |K_0'(r)| \leq \dfrac{C_k}{r^k}, & r > 1, \end{cases}$$

for constants C_k depending only on $k > 1$.

Now it is ready to present the proof of Theorem 1.2.

Proof. We prove Theorem 1.2 by contradiction. Suppose that the solution $(c, u) \in H^3(\mathbb{R}^3)$ for all time $t \geq 0$ and $T = \infty$ in Theorem 2.1. We will derive that the density blows up in the ball centered at the origin with an arbitrary small radius $r_0 > 0$ in a finite time, which contradicts with (2.1).

Taking the L^2 inner product of the above equation with the test function $K_0(r)$, one has

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^2} \rho K_0(r)dx = \int_{\mathbb{R}^2} \Delta p K_0(r)dx + \int_{\mathbb{R}^2} \nabla \cdot [\nabla \cdot (\rho u \otimes u)]K_0(r)dx.$$

Using a similar argument as in Section 2 and noting the decay properties of the Bessel function $K_0(r)$ in Lemma 3.1, we have

$$\int_{\mathbb{R}^2} \Delta p K_0(r)dx = \int_{\mathbb{R}^2} p\Delta K_0(r)dx.$$

Noting that the modified Bessel function $K_0(r)$ satisfies

$$K_0'' + \frac{1}{r}K_0' = K_0, \tag{3.1}$$

one has

$$\begin{aligned} \int_{\mathbb{R}^2} \Delta p K_0(r)dx &= A\int_{\mathbb{R}^2} \rho^\gamma K_0(r)dx \\ &\geq \frac{A}{\left(\int_{\mathbb{R}^2} K_0(r)dx\right)^{\gamma-1}}\left(\int_{\mathbb{R}^2} \rho K_0(r)dx\right)^\gamma. \end{aligned}$$

On the other hand, noting (2.3) and using (3.1) and Lemma 3.1, one has

$$\begin{aligned}&\int_{\mathbb{R}^2} K_0(r)\nabla\cdot[\nabla\cdot(\rho u\otimes u)]dx\\&=\int_{\mathbb{R}^2}\rho v^2 K_0''(r)dx\\&=\int_{\mathbb{R}^2}\rho v^2\big(K_0(r)-\frac{1}{r}K_0'(r)\big)dx>0.\end{aligned}$$

Here, we also used the fact that $K_0'(r)<0$ by the expression of the modified Bessel function $K_0(r)$. Consequently, we have

$$\frac{d^2}{dt^2}G(t)\geq\frac{AG(t)^\gamma}{\big(\int_{\mathbb{R}^2}K_0(r)dx\big)^{\gamma-1}}. \tag{3.2}$$

where

$$G(t)=\int_{\mathbb{R}^2}\rho(t,r)K_0(r)dx.$$

Using the equation of conservation of mass and integration by parts, and noting (1.9), one has

$$G'(0)=\frac{d}{dt}\int_{\mathbb{R}^2}\rho K_0(r)dx\Big|_{t=0}=\int_{\mathbb{R}^2}\rho_0 v_0 K_0'(r)dx>0. \tag{3.3}$$

The combination of (3.2) and (3.3) gives that

$$G'(t)=G'(0)+\int_0^t\frac{d^2}{ds^2}G(s)ds\geq G'(0)>0.$$

Consequently, one can multiply the both sides of (3.2) by $\frac{d}{dt}G(t)$ to get

$$\begin{aligned}\big(G'(t)\big)^2\geq&\frac{A}{(\gamma+1)\big(\int_{\mathbb{R}^2}K_0(r)dx\big)^{\gamma-1}}\big(G(t)\big)^{\gamma+1}\\&+\Big(\frac{d}{dt}\int_{\mathbb{R}^2}\rho K_0(r)dx\Big)^2\Big|_{t=0}-\frac{A\big(\int_{\mathbb{R}^2}\rho_0K_0(r)dx\big)^{\gamma+1}}{(\gamma+1)\big(\int_{\mathbb{R}^2}K_0(r)dx\big)^{\gamma-1}}.\end{aligned}$$

Noting (1.10) and (3.3), and by denoting

$$C_1=\sqrt{\frac{A}{(\gamma+1)}}\Big(\int_{\mathbb{R}^2}K_0(r)dx\Big)^{-\frac{\gamma-1}{2}},$$

one has

$$G'(t)\geq C_1\big(G(t)\big)^{\frac{\gamma+1}{2}},$$

which implies that

$$\int_{\mathbb{R}^2}\rho K_0(r)dx\geq\Big(G(0)^{-\frac{\gamma-1}{2}}-\frac{\gamma-1}{2}C_1t\Big)^{-\frac{2}{\gamma-1}}.$$

Noting (1.9), one has

$$G(0)=\int_{\mathbb{R}^2}\rho_0K_0(r)dx>0.$$

By using the mass conservation, one has

$$G(t) = \int_{\mathbb{R}^2} \frac{\rho}{re^r} dx \leq \int_{B_{r_0}} \rho K_0(r) dx + \frac{\max_{r \geq r_0} K_0(r)}{r_0} \int_{\mathbb{R}^2} \rho_0 dx,$$

for any given $r_0 > 0$, here B_{r_0} is the 2D ball centered at the origin with radius r_0. Consequently, one concludes that $\rho(t, r)$ will blow up for $r \leq r_0$ as $t \to \frac{2G(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_1}$. By the assumption $c \in L^\infty([0,\infty), H^3(\mathbb{R}^2)$ and Sobolev imbedding theorem, one has $c \in L^\infty([0,\infty) \times \mathbb{R}^2)$ and hence $\rho \in L^\infty([0,\infty) \times \mathbb{R}^2)$. We arrive at a contradiction. So, the H^3 norm of (c, u) cannot be bounded before the time $\frac{2G(0)^{-\frac{\gamma-1}{2}}}{(\gamma-1)C_1}$. The proof of Theorem 1.2 is completed. □

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