Math. Res. Lett. **20** (2013), *no.* 01, 51–65

VARIANCE OF THE EXPONENTS OF ORBIFOLD LANDAU–GINZBURG MODELS

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Abstract. We prove a formula for the variance of the set of exponents of a non-degenerate weighted homogeneous polynomial with an action of a diagonal subgroup of $\mathrm{SL}_n(\mathbb{C})$.

Introduction

Let X be a smooth compact Kähler manifold of dimension n. The Hodge numbers $h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$, $p, q \in \mathbb{Z}$, are some of the most important numerical invariants of X. They satisfy

$$h^{p,q}(X) = h^{q,p}(X), \quad p, q \in \mathbb{Z},$$

and the Serre duality

$$h^{p,q}(X) = h^{n-p,n-q}(X), \quad p, q \in \mathbb{Z}.$$

The Euler number $\chi(X)$ can also be written in terms of the Hodge numbers as

$$\chi(X) = \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} h^{p,q}(X).$$

One can easily calculate the expectation value of the distribution $\{q \in \mathbb{Z} \,|\, h^{p,q}(X) \neq 0\}$, which is given by the formula

$$\sum_{p,q \in \mathbb{Z}} (-1)^{p+q} q \cdot h^{p,q}(X) = \frac{1}{2} n \cdot \chi(X).$$

Equivalently, this can be rewritten as

$$\sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \left(q - \frac{n}{2}\right) h^{p,q}(X) = 0.$$

This means nothing else but that the mean of the distribution $\{q \in \mathbb{Z} \,|\, h^{p,q}(X) \neq 0\}$ is $n/2$. It is then natural to ask what is the variance of this distribution. A formula for this variance was given by Libgober and Wood [9] and Borisov [2]:

Theorem 1 (Libgober–Wood, Borisov). *One has*

$$(0.1) \qquad \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \left(q - \frac{n}{2}\right)^2 h^{p,q}(X) = \frac{1}{12} n \cdot \chi(X) + \frac{1}{6} \int_X c_1(X) \cup c_{n-1}(X),$$

where $c_i(X)$ denotes the ith Chern class of X.

Received by the editors June 22, 2012.

2010 *Mathematics Subject Classification.* 32S25, 32S35, 14L30.

If the first Chern class, $c_1(X)$ is numerically zero, then the above formula becomes

$$\sum_{p,q\in\mathbb{Z}} (-1)^{p+q}\left(q-\frac{n}{2}\right)^2 h^{p,q}(X) = \frac{1}{12}n\cdot\chi(X). \tag{0.2}$$

Similar phenomena were discovered in singularity theory. Let us consider a polynomial $f(x_1,\ldots,x_n)$ with an isolated singularity at the origin. There, the analogue of the set $\{q\in\mathbb{Z}\,|\,h^{p,q}(X)\neq 0\}$ above will be the set of the *exponents* of $f(x_1,\ldots,x_n)$, which is a set of rational numbers and is also one of the most important numerical invariants defined by the mixed Hodge structure associated to $f(x_1,\ldots,x_n)$. Let us give two important examples.

First, suppose that $f(x_1,\ldots,x_n)$ is a non-degenerate weighted homogeneous polynomial, namely, a polynomial with an isolated singularity at the origin with the property that there are positive rational numbers w_i, $i=1,\ldots,n$, such that

$$f(\lambda^{w_1}x_1,\ldots,\lambda^{w_n}x_n)=\lambda f(x_1,\ldots,x_n),\quad \lambda\in\mathbb{C}\backslash\{0\}.$$

We have the following properties of the exponents of f:

Theorem 2 (**cf.** [10])**.** *Let $q_1\le q_2\le\cdots\le q_\mu$ be the exponents of f, where μ is the Milnor number of f defined by*

$$\mu:=\dim_{\mathbb{C}}\mathbb{C}[x_1,\ldots,x_n]\Big/\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right).$$

Then one has

$$\mu=(-1)^n\prod_{i=1}^n\left(1-\frac{1}{w_i}\right)$$

and

$$\sum_{i=1}^{\mu} y^{q_i-\frac{n}{2}} = (-1)^n\prod_{i=1}^n\frac{y^{\frac{1}{2}}-y^{w_i-\frac{1}{2}}}{1-y^{w_i}}.$$

In particular, one has a duality of exponents $q_i+q_{\mu-i+1}=n$, $i=1,\ldots,\mu$, and hence

$$\sum_{i=1}^{\mu} q_i=\frac{1}{2}n\cdot\mu.$$

The following formula was proven by Hertling [6] in the context of Frobenius manifolds and an elementary proof was given by Dimca [3].

Theorem 3 (**Hertling, Dimca**)**.** *Let $q_1\le q_2\le\cdots\le q_\mu$ be the exponents of f. One has*

$$\sum_{i=1}^{\mu}\left(q_i-\frac{n}{2}\right)^2=\frac{1}{12}\hat{c}\cdot\mu,\quad \hat{c}:=n-2\sum_{i=1}^n w_i.$$

Next, consider the polynomial $f(x_1,x_2,x_3):=x_1^{\alpha_1}+x_2^{\alpha_2}+x_3^{\alpha_3}-x_1x_2x_3$ such that $1/\alpha_1+1/\alpha_2+1/\alpha_3<1$. We have the following properties of the exponents of f:

Theorem 4 (**cf.** [1])**.** *The set of exponents* $\{q_i\}$ *of* f *is given by*

$$\Big\{1, \frac{1}{\alpha_1}+1, \frac{2}{\alpha_1}+1, \dots, \frac{\alpha_1-1}{\alpha_1}+1, \frac{1}{\alpha_2}+1, \frac{2}{\alpha_2}+1, \dots, \frac{\alpha_2-1}{\alpha_2}+1, \frac{1}{\alpha_3}+1, \frac{2}{\alpha_3}+1, \dots, \frac{\alpha_3-1}{\alpha_3}+1, 2\Big\}.$$

In particular, one has

$$\sum_{i=1}^{\mu}\left(q_i-\frac{3}{2}\right)^2=\frac{1}{12}\mu+\frac{1}{6}\chi, \quad \chi := 2+\sum_{i=1}^{3}\left(\frac{1}{\alpha_i}-1\right).$$

The purpose of this paper is to generalize these results to pairs (f, G), where $G \subset \mathrm{SL}_n(\mathbb{C})$ is a finite abelian subgroup leaving f invariant. If f is weighted homogeneous, such a pair is also called an *orbifold Landau–Ginzburg model* because f is the potential of such a model. Our main theorem in this paper is Theorem 19. The generalization of Theorem 4 is given as Theorem 21. The similarity between smooth compact Kähler manifolds and isolated hypersurface singularities with a group action is not an accident but a matter of course. Mirror symmetry predicts a correspondence between Landau–Ginzburg models and (non-commutative) Calabi–Yau orbifolds. For example, a mirror partner of a weighted homogeneous polynomial with a group action is a fractional Calabi–Yau manifold of dimension $\hat{c}$, which has lead us to the statement of Theorem 19.

1. Basic properties of E-functions

Let G be a finite abelian subgroup of $\mathrm{SL}_n(\mathbb{C})$ acting diagonally on $\mathbb{C}^n$. For $g \in G$, we denote by $\mathrm{Fix}\, g := \{x \in \mathbb{C}^n \mid g \cdot x = x\}$ the fixed locus of g and by $n_g := \dim \mathrm{Fix}\, g$ its dimension.

We first introduce the notion of the age of an element of a finite group as follows:

Definition ([8])**.** Let $g \in G$ be an element and r be the order of g. Then g has a unique expression of the following form:

$$g = \mathrm{diag}(\mathbf{e}[a_1/r], \dots, \mathbf{e}[a_n/r]) \quad \text{with } 0 \le a_i < r,$$

where $\mathbf{e}[-] = e^{2\pi\sqrt{-1}\cdot -}$. Such an element g is often simply denoted by

$$g = \frac{1}{r}(a_1, \dots, a_n).$$

The *age* of g is defined as

$$\mathrm{age}(g) := \frac{1}{r}\sum_{i=1}^{n} a_i.$$

Since we assume that $G \subset \mathrm{SL}_n(\mathbb{C})$, the number $\mathrm{age}(g)$ is a non-negative integer for all $g \in G$.

Definition. An element $g \in G$ of age 1 with $\mathrm{Fix}\, g = \{0\}$ is called a *junior element*. The number of junior elements is denoted by j_G.

Let $f = f(x_1, \dots, x_n)$ be a polynomial with an isolated singularity at the origin, which is invariant under the natural action of G. For $g \in G$, set $f^g := f|_{\mathrm{Fix}\, g}$.

Proposition 5. *The function f^g has an isolated singularity at the origin.*

Proof. Since G acts diagonally on $\mathbb{C}^n$, we may assume that $\mathrm{Fix}\, g = \{x_{n_g+1} = \cdots = x_n = 0\}$ by a suitable renumbering of indices. Since f is invariant under G, $g \cdot x_i \neq x_i$ for $i = n_g+1, \ldots, n$ and $\frac{\partial f}{\partial x_{n_g+1}}, \ldots, \frac{\partial f}{\partial x_n}$ form a regular sequence, we have

$$\left(\frac{\partial f}{\partial x_{n_g+1}}, \ldots, \frac{\partial f}{\partial x_n}\right) \subset \left(x_{n_g+1}, \ldots, x_n\right).$$

Therefore, we have

$$\begin{aligned} &\dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_{n_g}\} \Big/ \left(\frac{\partial f^g}{\partial x_1}, \ldots, \frac{\partial f^g}{\partial x_{n_g}}\right) \\ &= \dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\} \Big/ \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n_g}}, x_{n_g+1}, \ldots, x_n\right) \\ &\leq \dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\} \Big/ \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) < \infty. \end{aligned}$$

□

We shall associate to f the following bi-graded vector space:

Definition. Let $H^{n-1}(Y_\infty, \mathbb{C})$ be the vanishing cohomology of f on which Steenbrink constructed a canonical mixed Hodge structure in [10]. Denote by $F^\bullet$ the Hodge filtration on $H^{n-1}(Y_\infty, \mathbb{C})$.

Define the bi-graded vector space $\mathcal{H}_f := \bigoplus_{p,q \in \mathbb{Q}} \mathcal{H}_f^{p,q}$ as

(i) If $p + q \neq n$, then $\mathcal{H}_f^{p,q} := 0$.

(ii) If $p + q = n$ and $p \in \mathbb{Z}$, then

$$\mathcal{H}_f^{p,q} := \mathrm{Gr}^p_{F^\bullet} H^{n-1}(Y_\infty, \mathbb{C})_1.$$

(iii) If $p + q = n$ and $p \notin \mathbb{Z}$, then

$$\mathcal{H}_f^{p,q} := \mathrm{Gr}^{[p]}_{F^\bullet} H^{n-1}(Y_\infty, \mathbb{C})_{e^{2\pi\sqrt{-1}p}},$$

where $[p]$ is the largest integer less than p.

We shall use the fact that $\mathcal{H}_{f^g}$ admits a natural G-action by restricting the G-action on $\mathbb{C}^n$ to $\mathrm{Fix}\, g$ (which is well-defined since G acts diagonally on $\mathbb{C}^n$).

To the pair (f, G) we can associate the following bi-graded vector space:

Definition. Define the bi-graded $\mathbb{C}$-vector space $\mathcal{H}_{f,G}$ as

$$\mathcal{H}_{f,G} := \bigoplus_{g \in G} (\mathcal{H}_{f^g})^G(-\mathrm{age}(g), -\mathrm{age}(g)), \tag{1.1}$$

where $(\mathcal{H}_{f^g})^G$ denotes the G-invariant subspace of $\mathcal{H}_{f^g}$.

Since the bi-graded vector space $\mathcal{H}_{f,G}$ is the analog of $\bigoplus_{p,q \in \mathbb{Z}} H^q(X, \Omega_X^p)$ for a smooth compact Kähler manifold X, we introduce the following notion:

Definition. The *Hodge numbers* for the pair (f, G) are

$$h^{p,q}(f, G) := \dim_{\mathbb{C}} \mathcal{H}_{f,G}^{p,q}, \quad p, q \in \mathbb{Q}.$$

Definition. The rational number q with $\mathcal{H}^{p,q}_{f,G} \neq 0$ is called an *exponent* of the pair (f,G). The *set of exponents* of the pair (f,G) is the multi-set of exponents

$$\{q * h^{p,q}(f,G) \mid p,q \in \mathbb{Q},\ h^{p,q}(f,G) \neq 0\},$$

where by $u * v$ we denote v copies of the rational number u.

Note that $p+q \in \mathbb{Z}$ for the rational number q with $h^{p,q}(f,G) \neq 0$ since $G \subset \mathrm{SL}_n(\mathbb{C})$.

Definition. The E-function for the pair (f,G) is

$$E(f,G)(t,\bar{t}) := \sum_{p,q\in\mathbb{Q}} (-1)^{(p-n)+q} h^{p,q}(f,G) \cdot t^{p-\frac{n}{2}} \bar{t}^{q-\frac{n}{2}}. \tag{1.2}$$

Definition. The *Milnor number* for the pair (f,G) is

$$\mu_{(f,G)} := E(f,G)(1,1) = \sum_{p,q\in\mathbb{Q}} (-1)^{(p-n)+q} h^{p,q}(f,G).$$

Theorem 6. *Assume that f is a non-degenerate weighted homogeneous polynomial. Write $g \in G$ in the form $(\lambda_1(g),\dots,\lambda_n(g))$ where $\lambda_i(g) = \mathbf{e}[a_i w_i]$. The E-function for the pair (f,G) is given by the following formula:*

$$E(f,G)(t,\bar{t}) = \sum_{g\in G} E_g(f,G)(t,\bar{t}), \tag{1.3}$$

$$E_g(f,G)(t,\bar{t}) := (-1)^n \left(\prod_{a_i w_i \notin \mathbb{Z}} (t\bar{t})^{w_i a_i - [w_i a_i] - \frac{1}{2}} \right) \cdot \frac{1}{|G|} \sum_{h\in G} \prod_{a_i w_i \in \mathbb{Z}} \frac{\left(\frac{\bar{t}}{t}\right)^{\frac{1}{2}} - \lambda_i(h) \left(\frac{\bar{t}}{t}\right)^{w_i - \frac{1}{2}}}{1 - \lambda_i(h) \left(\frac{\bar{t}}{t}\right)^{w_i}}.$$

Here $[a]$ for $a \in \mathbb{Q}$ denotes the largest integer less than or equal to a.

Proof. Theorem 2 enables us to obtain $E_g(f,G)(t,\bar{t})$. In particular, the term

$$\frac{1}{|G|} \sum_{h\in G} (-1)^{n_g} \prod_{a_i w_i \in \mathbb{Z}} \frac{\left(\frac{\bar{t}}{t}\right)^{\frac{1}{2}} - \lambda_i(h) \left(\frac{\bar{t}}{t}\right)^{w_i - \frac{1}{2}}}{1 - \lambda_i(h) \left(\frac{\bar{t}}{t}\right)^{w_i}}$$

calculates the G-invariant part of $E(f^g,\{1\})(t,\bar{t})$ and the term

$$(-1)^{n-n_g} \prod_{w_i a_i \notin \mathbb{Z}} (t\bar{t})^{w_i a_i - [w_i a_i] - \frac{1}{2}}$$

gives the contribution from the age shift $(-\mathrm{age}(g), -\mathrm{age}(g))$. □

We have the following properties of the Hodge numbers $h^{p,q}(f,G)$.

Corollary 7. *Assume that f is a non-degenerate weighted homogeneous polynomial. We have*

$$h^{p,q}(f,G) = h^{q,p}(f,G), \quad p,q \in \mathbb{Q}.$$

In other words, we have

$$E(f,G)(t,\bar{t}) = E(f,G)(\bar{t},t).$$

Proof. This is shown by an elementary direct calculation. □

Corollary 8. *Assume that f is a non-degenerate weighted homogeneous polynomial. The Hodge numbers satisfy the "Serre duality"*

$$h^{p,q}(f,G) = h^{n-p,n-q}(f,G), \quad p,q \in \mathbb{Q}.$$

In other words, we have

$$E(f,G)(t,\bar{t}) = E(f,G)(t^{-1},\bar{t}^{-1}).$$

Proof. By using the formula

$$w_i(-a_i) - [w_i(-a_i)] - \frac{1}{2} = -w_i a_i + [w_i a_i] + \frac{1}{2},$$

an easy calculation yields the formula. □

Corollary 9. *Assume that f is a non-degenerate weighted homogeneous polynomial. The mean of the set of exponents of (f,G) is $n/2$. Namely, we have*

$$\sum_{p,q\in\mathbb{Q}} (-1)^{(p-n)+q} \left(q - \frac{n}{2}\right) h^{p,q}(f,G) = 0.$$

Proof. This is obvious from the previous corollary. □

Definition. Define the *variance of the set of exponents* of (f,G) by

$$\mathrm{Var}_{(f,G)} := \sum_{p,q\in\mathbb{Q}} (-1)^{(p-n)+q} \left(q - \frac{n}{2}\right)^2 h^{p,q}(f,G).$$

In order to state our formula for the variance, we introduce the following notion of dimension for a polynomial f with an isolated singularity at the origin.

Definition. The non-negative rational number $\hat{c}$ defined as the difference of the maximal exponent of the pair $(f,\{1\})$ and the minimal exponent of the pair $(f,\{1\})$ is called the *dimension* of f.

Proposition 10. *Assume that f is a non-degenerate weighted homogeneous polynomial. The dimension $\hat{c}$ of f is given by*

$$\hat{c} := n - 2\sum_{i=1}^{n} w_i.$$

Proof. It easily follows from Theorem 2 that the maximal exponent and the minimal exponent are given by $n - \sum_{i=1}^{n} w_i$ and $\sum_{i=1}^{n} w_i$, respectively. □

It is natural from the mirror symmetry point of view to expect that the variance of the set of exponents of (f,G) should be given by

$$\mathrm{Var}_{(f,G)} = \frac{1}{12}\hat{c}\cdot \mu_{(f,G)}. \tag{1.4}$$

This will be proved in the next section.

2. Variance of the exponents

Definition. The *χ_y-genus* for the pair (f,G) is

$$\chi(f,G)(y) := E(f,G)(1,y).$$

We have

$$\chi(f,G)(y) = (-1)^n \sum_{g\in G}\left(y^{\mathrm{age}(g)-\frac{n-n_g}{2}} \cdot \frac{1}{|G|}\sum_{h\in G}\prod_{\lambda_i(g)=1} \frac{y^{\frac{1}{2}}-\lambda_i(h)y^{w_i-\frac{1}{2}}}{1-\lambda_i(h)y^{w_i}}\right).$$

One has

$$\mu_{(f,G)} = \lim_{y\to 1}\chi(f,G)(y),$$
$$\mathrm{Var}_{(f,G)} = \lim_{y\to 1}\frac{d}{dy}\left(y\frac{d}{dy}\chi(f,G)(y)\right).$$

Proposition 11. *Let*

$$p_i(y) := \frac{y^{\frac{1}{2}}-\lambda_i(h)y^{w_i-\frac{1}{2}}}{1-\lambda_i(h)y^{w_i}}.$$

(i) *For $\lambda_i(h)=1$ one has*

$$\lim_{y\to 1} p_i(y) = 1-\frac{1}{w_i},\quad \lim_{y\to 1}\frac{\frac{d}{dy}p_i(y)}{p_i(y)} = 0,\quad \lim_{y\to 1}\frac{d}{dy}\left(y\frac{\frac{d}{dy}p_i(y)}{p_i(y)}\right) = \frac{1-2w_i}{12}.$$

(ii) *For $\lambda_i(h)\neq 1$ one has*

$$\lim_{y\to 1} p_i(y) = 1,\quad \lim_{y\to 1}\frac{\frac{d}{dy}p_i(y)}{p_i(y)} = \frac{1}{2}\frac{1+\lambda_i(h)}{1-\lambda_i(h)},$$
$$\lim_{y\to 1}\frac{d}{dy}\left(y\frac{\frac{d}{dy}p_i(y)}{p_i(y)}\right) = -\frac{(1-2w_i)\lambda_i(h)}{(1-\lambda_i(h))^2}.$$

Proof. For (i) see the proof of [3, Proposition 5.2]. Statement (ii) follows from a similar elementary but tedious computation. □

Let $I_0 := \{1,\dots,n\}$ and let $H\subset G$ be a subgroup of G. For a subset $I\subset I_0$ ($I=\emptyset$ is admitted) let H^I be the maximal subgroup of H fixing the coordinates x_i, $i\in I$.

Lemma 12. *Let $H\subset G$ be a subgroup of G and $i\in I_0$. Then*

$$\sum_{h\in H\setminus H^{\{i\}}}\frac{1+\lambda_i(h)}{1-\lambda_i(h)} = 0$$

Proof. One has

$$\sum_{h\in H\setminus H^{\{i\}}}\frac{1+\lambda_i(h)}{1-\lambda_i(h)} = \sum_{h\in H\setminus H^{\{i\}}}\frac{1}{1-\lambda_i(h)} + \sum_{h\in H\setminus H^{\{i\}}}\frac{1}{\lambda_i(h^{-1})-1} = 0.$$

□

Proposition 13. *Let $r \in \mathbb{Z}$, $r \geq 2$, and $\zeta_r = \mathbf{e}[1/r]$ be a primitive rth root of unity. Then one has*
$$-\sum_{k=1}^{r-1} \frac{\zeta_r^k}{(1-\zeta_r^k)^2} = \frac{r^2-1}{12}.$$

Proof. One has
$$-\sum_{k=1}^{r-1} \frac{\zeta_r^k}{(1-\zeta_r^k)^2} = \lim_{t\to 1} q'(t) \text{ where } q(t) := -\sum_{k=1}^{r-1} \frac{1}{1-\zeta_r^k t}.$$
One can easily see that
$$q(t) = \frac{-r\left(\sum_{k=0}^{r-2} t^k\right) + \sum_{k=0}^{r-2}(k+1)t^k}{\sum_{k=0}^{r-1} t^k}.$$
This implies
$$\lim_{t\to 1} q'(t) = \frac{1}{r^2}\left[\sum_{k=1}^{r-2} k(k-r+1)r - \left(\sum_{\ell=1}^{r-1}(\ell - r)\right)\left(\sum_{k=1}^{r-1} k\right)\right] = \frac{r^2-1}{12}.$$
□

Corollary 14. *Let $H \subset G$ be a subgroup of G and $i \in I_0$. Then*
$$-\sum_{h\in H\setminus H^{\{i\}}} \frac{\lambda_i(h)}{(1-\lambda_i(h))^2} = \frac{|H\cap H^{\{i\}}|(|H/H\cap H^{\{i\}}|^2-1)}{12}.$$

Proof. The image of the factor group $H/H\cap H^{\{i\}}$ under the induced character $\lambda_i : H/H\cap H^{\{i\}} \to \mathbb{C}^*$ is a finite abelian subgroup of the unit circle S^1 and hence cyclic. Therefore, the formula follows from Proposition 13. □

Let
$$((x)) := \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \in \mathbb{R}, x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Proposition 15. *Let $r \in \mathbb{Z}$, $r \geq 2$, $\zeta_r = \mathbf{e}[1/r]$ be a primitive rth root of unity, and a, b be integers satisfying $0 < a, b < r$. Then one has*
$$\frac{1}{4r} \sum_{\substack{k=1,\\ r\not| ak,bk}}^{r-1} \frac{1+\zeta_r^{ak}}{1-\zeta_r^{ak}} \frac{1+\zeta_r^{bk}}{1-\zeta_r^{bk}} = -\sum_{k=1}^{r-1}\left(\left(\frac{ak}{r}\right)\right)\left(\left(\frac{bk}{r}\right)\right).$$

Remark 16. The right-hand side of the formula of Proposition 15 is a generalized Dedekind sum and Proposition 15 is a slight generalization of [7, 5.2 Theorem 1], since
$$\frac{1+\mathbf{e}[x]}{1-\mathbf{e}[x]} = \sqrt{-1}\cot \pi x$$
for any real number x. The difference is that [7, 5.2 Theorem 1] is only formulated for integers a, b prime to r.

Proof of Proposition 15. We follow the proof of [7, 5.2 Theorem 1]. For simplicity, we assume $b = 1$. By the formula [7, 5.2 (2)], which goes back to Eisenstein [5], we have

$$\left(\left(\frac{q}{r}\right)\right) = -\frac{1}{2r}\sum_{\ell=1}^{r-1} \zeta_r^{\ell q}\frac{\zeta_r^{\ell}+1}{\zeta_r^{\ell}-1}$$

for any integers q and r. (Note that there is a minor misprint in [7, 5.2 (2)].) Applying this formula, we get

$$\begin{aligned}
&\sum_{\ell=1}^{r-1}\left(\left(\frac{a\ell}{r}\right)\right)\left(\left(\frac{\ell}{r}\right)\right)\\
&\quad= \sum_{\ell=1}^{r}\left(\left(\frac{a\ell}{r}\right)\right)\left(\left(\frac{\ell}{r}\right)\right) = \frac{1}{4r^2}\sum_{\ell=1}^{r}\sum_{m=1}^{r-1}\sum_{k=1}^{r-1}\zeta_r^{(m+ak)\ell}\frac{\zeta_r^m+1}{\zeta_r^m-1}\frac{\zeta_r^k+1}{\zeta_r^k-1}\\
&\quad= \frac{1}{4r}\sum_{\substack{k=1,\\ r\not| ak}}^{r-1}\frac{\zeta_r^{-ak}+1}{\zeta_r^{-ak}-1}\frac{\zeta_r^k+1}{\zeta_r^k-1} = -\frac{1}{4r}\sum_{\substack{k=1,\\ r\not| ak}}^{r-1}\frac{1+\zeta_r^{ak}}{1-\zeta_r^{ak}}\frac{1+\zeta_r^k}{1-\zeta_r^k},
\end{aligned}$$

since

$$\sum_{\ell=1}^{r}\zeta_r^{(m+ak)\ell} = \begin{cases} 0 & \text{if } m+ak \not\equiv 0 \bmod r,\\ r & \text{if } m+ak \equiv 0 \bmod r.\end{cases}$$

□

Corollary 17. *Let $K \subset J \subset I_0$. Then*

$$\frac{1}{4}\sum_{h\in G^K}\left(\sum_{\substack{j\in J\setminus K,\\ \lambda_j(h)\neq 1}}\frac{1+\lambda_j(h)}{1-\lambda_j(h)}\right)^2 = -|G^K|\sum_{h\in G^K}\left(\sum_{j\in J\setminus K}((a_jw_j))\right)^2,$$

where $\lambda_j(h) = \mathbf{e}[a_jw_j]$ for all $h \in G^K$ and $j \in J\setminus K$.

Proof. This follows from Proposition 15 by the same arguments as in the proof of Corollary 14. □

Proposition 18. *For a non-degenerate weighted homogeneous polynomial f, one has*

$$\mu_{(f,G)} = \frac{(-1)^n}{|G|}\left\{\sum_{I\subset I_0}\prod_{i\in I}\left(1-\frac{1}{w_i}\right)\left[\sum_{I\subset J\subset I_0}(-1)^{|J|-|I|}|G^J|^2\right]\right\}. \tag{2.1}$$

Proof. Let $J \subset I_0$. Let G_J be the set of elements $g \in G$ with $\lambda_j(g) = 1$ for $j \in J$ and $\lambda_j(g) \neq 1$ for $j \notin J$, i.e., the set of elements of G which fix the coordinates x_j, $j \in J$, and only these coordinates. Then

$$|G_J| = \sum_{\substack{K,\\ J\subset K\subset I_0}}(-1)^{|K|-|J|}|G^K|.$$

Let $I \subset J$. Let $G_{I,J}$ be the set of elements g of G with $\lambda_i(g) = 1$ for $i \in I$ and $\Lambda_j(g) \neq 1$ for $j \in J\setminus I$ (and $\lambda_k(g)$ arbitrary for $k \in I_0\setminus J$). Then

$$|G_{I,J}| = \sum_{\substack{K,\\ I\subset K\subset J}}(-1)^{|K|-|I|}|G^K|.$$

By Proposition 11 one has

$$\lim_{y\to 1}\chi(f,G)(y) = \frac{(-1)^n}{|G|}\sum_{\substack{J,\\ J\subset I_0}}|G_J|\left(\sum_{\substack{I,\\ I\subset J}}\prod_{i\in I}\left(1-\frac{1}{w_i}\right)|G_{I,J}|\right)$$
$$= \frac{(-1)^n}{|G|}\sum_{\substack{I,\\ I\subset I_0}}\prod_{i\in I}\left(1-\frac{1}{w_i}\right)\left(\sum_{\substack{J,\\ I\subset J\subset I_0}}|G_J||G_{I,J}|\right).$$

Now let $I\subset I_0$ be fixed. Then

$$\sum_{\substack{J,\\ I\subset J\subset I_0}}|G_J||G_{I,J}| = \sum_{\substack{J,\\ I\subset J\subset I_0}}\left(\sum_{\substack{K,\\ J\subset K\subset I_0}}(-1)^{|K|-|J|}|G^K|\right)\left(\sum_{\substack{L,\\ I\subset L\subset J}}(-1)^{|L|-|I|}|G^L|\right)$$
$$= \sum_{\substack{L,\\ I\subset L\subset I_0}}\sum_{\substack{K,\\ L\subset K\subset I_0}}\left(\sum_{\substack{J,\\ L\subset J\subset K}}(-1)^{|K|+|L|-|I|-|J|}\right)|G^K||G^L|$$
$$= \sum_{\substack{K,\\ I\subset K\subset I_0}}(-1)^{|K|-|I|}|G^K|^2,$$

since for fixed $L\subset I_0$ and $K\subset I_0$ with $L\subset K$

(2.2)
$$\sum_{\substack{J,\\ L\subset J\subset K}}(-1)^{|K|+|L|-|I|-|J|} = (-1)^{|K|-|I|}(1-1)^{|K|-|L|} = \begin{cases}(-1)^{|K|-|I|} & \text{for } L=K,\\ 0 & \text{otherwise.}\end{cases}$$

□

Now, we are ready to state the main result of our paper.

Theorem 19. *For a non-degenerate weighted homogeneous polynomial f, one has*

$$\mathrm{Var}_{(f,G)} = \sum_{p,q\in\mathbb{Q}}(-1)^{(p-n)+q}\left(q-\frac{n}{2}\right)^2 h^{p,q}(f,G) = \frac{1}{12}\hat{c}\cdot\mu_{(f,G)}.$$

Proof. We use the notation introduced in the proof of Proposition 18. By Proposition 11 and Lemma 12 we have

$$\lim_{y\to 1}\frac{d}{dy}\left(y\frac{d}{dy}\chi(f,G)(y)\right) = A+B+C,$$

where

$$A := \frac{(-1)^n}{|G|}\sum_{\substack{J,\\ J\subset I_0}}\sum_{g\in G_J}\left(\mathrm{age}(g)-\frac{n-n_g}{2}\right)^2\left[\sum_{\substack{I,\\ I\subset J}}\prod_{i\in I}\left(1-\frac{1}{w_i}\right)|G_{I,J}|\right],$$

$$B := \frac{(-1)^n}{|G|} \sum_{\substack{J,\\ J\subset I_0}} |G_J| \left[\sum_{\substack{I,\\ I\subset J}} \prod_{i\in I} \left(1 - \frac{1}{w_i}\right) \sum_{h\in G_{I,J}} \frac{1}{4} \left(\sum_{j\in J\setminus I} \frac{1+\lambda_j(h)}{1-\lambda_j(h)} \right)^2 \right],$$

$$C := \frac{(-1)^n}{|G|} \sum_{\substack{J,\\ J\subset I_0}} |G_J|$$

$$\times \left[\sum_{\substack{I,\\ I\subset J}} \prod_{i\in I} \left(1 - \frac{1}{w_i}\right) \left(|G_{I,J}| \left(\sum_{i\in I} \frac{1-2w_i}{12} \right) - \sum_{h\in G_{I,J}} \sum_{\substack{j\in J,\\ j\notin I}} \frac{(1-2w_j)\lambda_j(h)}{(1-\lambda_j(h))^2} \right) \right].$$

(a) We first show that $A+B=0$. We first take the sums in A and B in a different order:

$$A = \frac{(-1)^n}{|G|} \sum_{\substack{I,\\ I\subset I_0}} \prod_{i\in I} \left(1 - \frac{1}{w_i}\right) A_I, \; A_I := \sum_{\substack{J,\\ I\subset J\subset I_0}} \sum_{g\in G_J} \left(\operatorname{age}(g) - \frac{n-n_g}{2} \right)^2 |G_{I,J}|,$$

$$B = \frac{(-1)^n}{|G|} \sum_{\substack{I,\\ I\subset I_0}} \prod_{i\in I} \left(1 - \frac{1}{w_i}\right) B_I,$$

$$B_I := \sum_{\substack{J,\\ I\subset J\subset I_0}} |G_J| \left(\sum_{h\in G_{I,J}} \frac{1}{4} \left(\sum_{j\in J\setminus I} \frac{1+\lambda_j(h)}{1-\lambda_j(h)} \right)^2 \right).$$

Now let $I\subset I_0$ be fixed. Let $\lambda_i(g) = \mathbf{e}[a_i w_i]$. Then, we have on the one hand:

$$\begin{aligned} A_I &= \sum_{\substack{J,\\ I\subset J\subset I_0}} |G_{I,J}| \sum_{g\in G_J} \left(\sum_{j\in I_0\setminus J} ((a_j w_j)) \right)^2 \\ &= \sum_{\substack{J,\\ I\subset J\subset I_0}} |G_{I,J}| \sum_{\substack{K,\\ J\subset K\subset I_0}} (-1)^{|K|-|J|} \sum_{g\in G^K} \left(\sum_{j\in I_0\setminus K} ((a_j w_j)) \right)^2. \end{aligned}$$

On the other hand, we have by Corollary 17

$$\begin{aligned} B_I &= \sum_{\substack{J,\\ I\subset J\subset I_0}} |G_J| \sum_{h\in G_{I,J}} \frac{1}{4} \left(\sum_{j\in J\setminus I} \frac{1+\lambda_j(h)}{1-\lambda_j(h)} \right)^2 \\ &= \sum_{\substack{J,\\ I\subset J\subset I_0}} |G_J| \sum_{\substack{K,\\ I\subset K\subset J}} (-1)^{|K|-|I|} \sum_{h\in G^K} \frac{1}{4} \left(\sum_{j\in J\setminus K} \frac{1+\lambda_j(h)}{1-\lambda_j(h)} \right)^2 \\ &= - \sum_{\substack{J,\\ I\subset J\subset I_0}} |G_J| \sum_{\substack{K,\\ I\subset K\subset J}} (-1)^{|K|-|I|} |G^K| \sum_{h\in G^K} \left(\sum_{j\in J\setminus K} ((a_j w_j)) \right)^2. \end{aligned}$$

For $I \subset K \subset J \subset I_0$ let

$$s(K,J) := \sum_{g\in G^K} \left(\sum_{j\in J\setminus K} ((a_j w_j)) \right)^2 .$$

Then

$$\begin{aligned}
A_I &= \sum_{\substack{K,\\ I\subset K\subset I_0}} \sum_{\substack{J,\\ I\subset J\subset K}} (-1)^{|K|-|J|} |G_{I,J}| s(K,I_0) \\
&= \sum_{\substack{K,\\ I\subset K\subset I_0}} \sum_{\substack{J,\\ I\subset J\subset K}} (-1)^{|K|-|J|} \left(\sum_{\substack{L,\\ I\subset L\subset J}} (-1)^{|L|-|I|} |G^L| \right) s(K,I_0) \\
&= \sum_{\substack{L,\\ I\subset L\subset I_0}} \sum_{\substack{K,\\ L\subset K\subset I_0}} \left(\sum_{\substack{J,\\ L\subset J\subset K}} (-1)^{|K|+|L|-|I|-|J|} \right) |G^L| s(K,I_0) \\
&= \sum_{\substack{K,\\ I\subset K\subset I_0}} (-1)^{|K|-|I|} |G^K| s(K,I_0)
\end{aligned}$$

by Formula (2.2). On the other hand, we have

$$\begin{aligned}
B_I &= - \sum_{\substack{K,\\ I\subset K\subset I_0}} \sum_{\substack{J,\\ K\subset J\subset I_0}} (-1)^{|K|-|I|} |G_J||G^K| s(K,J) \\
&= - \sum_{\substack{K,\\ I\subset K\subset I_0}} \sum_{\substack{J,\\ K\subset J\subset I_0}} (-1)^{|K|-|I|} \left(\sum_{\substack{L,\\ J\subset L\subset I_0}} (-1)^{|L|-|J|} |G^L| \right) |G^K| s(K,J) \\
&= - \sum_{\substack{L,\\ I\subset L\subset I_0}} \sum_{\substack{K,\\ I\subset K\subset L}} \left(\sum_{\substack{J,\\ L\subset J\subset I_0}} (-1)^{|K|+|L|-|I|-|J|} \right) |G^L||G^K| s(K,J) \\
&= - \sum_{\substack{K,\\ I\subset K\subset I_0}} (-1)^{|K|-|I|} |G^K| s(K,I_0) = -A_I,
\end{aligned}$$

again by Formula (2.2) and since $|G^{I_0}| = 1$. This shows that $A + B = 0$.

(b) We now consider the term C. Let $J \subset I_0$, $I \subset J$ and $j \in J$, $j \notin I$. Then it follows from Corollary 14 that

$$-\sum_{h\in G_{I,J}} \frac{\lambda_j(h)}{(1-\lambda_j(h))^2} = \frac{1}{12} m^J_{I,j},$$

where

$$m^J_{I,j} := \sum_{\substack{K, j\notin K,\\ I\subset K\subset J}} (-1)^{|K|-|I|} |G^{K\cup\{i\}}| \left(\left| G^K / G^{K\cup\{i\}} \right|^2 - 1 \right).$$

By (a) we have

$$\begin{aligned}
&\lim_{y\to 1}\frac{d}{dy}\left(y\frac{d}{dy}\chi(f,G)(y)\right)\\
&\quad = C = \frac{(-1)^n}{|G|}\sum_{\substack{J,\\ J\subset I_0}}|G_J|\\
&\quad\times\left[\sum_{\substack{I,\\ I\subset J}}\prod_{i\in I}\left(1-\frac{1}{w_i}\right)\left(|G_{I,J}|\left(\sum_{i\in I}\frac{1-2w_i}{12}\right)+\sum_{\substack{j\in J,\\ j\notin I}}m^J_{I,j}\left(\frac{1-2w_j}{12}\right)\right)\right]\\
&\quad = \frac{(-1)^n}{|G|}\sum_{\substack{I,\\ I\subset I_0}}\prod_{i\in I}\left(1-\frac{1}{w_i}\right)\\
&\quad\times\left[\sum_{\substack{J,\\ I\subset J\subset I_0}}|G_J|\left(|G_{I,J}|\left(\sum_{i\in I}\frac{1-2w_i}{12}\right)+\sum_{\substack{j\in J,\\ j\notin I}}m^J_{I,j}\left(\frac{1-2w_j}{12}\right)\right)\right].
\end{aligned}$$

Now let $I\subset I_0$ and $j\notin I$ be fixed. Then

$$\begin{aligned}
\sum_{\substack{J, j\in J,\\ I\subset J\subset I_0}}|G_J]m^J_{I,j} &= \sum_{\substack{J, j\in J,\\ I\subset J\subset I_0}}\left(\sum_{\substack{K,\\ J\subset K\subset I_0}}(-1)^{|K|-|J|}|G^K|\right)\\
&\quad\times\left(\sum_{\substack{L, j\notin L,\\ I\subset L\subset J}}(-1)^{|L|-|I|}|G^{L\cup\{j\}}|\left(\left|G^L/G^{L\cup\{j\}}\right|^2-1\right)\right)\\
&= \sum_{\substack{L, j\notin L,\\ I\subset L\subset I_0}}\sum_{\substack{K, j\in K,\\ L\subset K\subset I_0}}\left(\sum_{\substack{J, j\in J,\\ L\subset J\subset K}}(-1)^{|K|+|L|-|I|-|J|}\right)\\
&\quad\times|G^K||G^{L\cup\{j\}}|\left(\left|G^L/G^{L\cup\{j\}}\right|^2-1\right).
\end{aligned}$$

Since $j\notin L$ but $j\in J$, the case $J=L$ and hence also $K=L$ is excluded in the sum

$$\sum_{\substack{J, j\in J,\\ L\subset J\subset K}}(-1)^{|K|+|L|-|I|-|J|}.$$

Therefore

$$\sum_{\substack{J, j\in J,\\ L\subset J\subset K}}(-1)^{|K|+|L|-|I|-|J|}=\begin{cases}(-1)^{|L|-|I|} & \text{for } K=L\cup\{j\},\\ 0 & \text{otherwise.}\end{cases}$$

Hence, we obtain

$$\begin{aligned}\sum_{\substack{J, j\in J,\\ I\subset J\subset I_0}} |G_J]m_{I,j}^J &= \sum_{\substack{L, j\notin L,\\ I\subset L\subset I_0}} (-1)^{|L|-|I|}|G^{L\cup\{j\}}|^2 \left(\left|G^L/G^{L\cup\{j\}}\right|^2 - 1\right).\\ &= \sum_{\substack{L, j\notin L,\\ I\subset L\subset I_0}} (-1)^{|L|-|I|}\left(|G^L|^2 - |G^{L\cup\{j\}}|^2\right)\\ &= \sum_{\substack{K,\\ I\subset K\subset I_0}} (-1)^{|K|-|I|}|G^K|^2.\end{aligned}$$

Therefore, the statement follows from Proposition 18. □

3. Variance of the exponents for cusp singularities with group actions

Let $f(x_1,x_2,x_3) := x_1^{\alpha_1}+x_2^{\alpha_2}+x_3^{\alpha_3}-x_1x_2x_3$ and G be a finite subgroup of $SL_n(\mathbb{C})$ acting diagonally on $\mathbb{C}^n$ under which f is invariant. Let $K_i \subset G$ be the maximal subgroup fixing the coordinate x_i, $i=1,2,3$. Define numbers $\gamma_1,\ldots,\gamma_s$ by

$$(\gamma_1,\ldots,\gamma_s) = \left(\frac{\alpha_i}{|G/K_i|} * |K_i|, i=1,2,3\right),$$

where we omit numbers which are equal to one on the right-hand side. Define a number $\chi_{(f,G)}$ by

$$\chi_{(f,G)} := 2-2j_G + \sum_{i=1}^{s}\left(\frac{1}{\gamma_i}-1\right).$$

Lemma 20. *Let the pair (f,G) be as above.*

(i) *The Milnor number of the pair (f,G) is given by*

$$\mu_{(f,G)} = 2-2j_G+\sum_{i=1}^{s}(\gamma_i-1). \tag{3.1}$$

(ii) *The set of exponents for the pair (f,G) is given by*

$$\begin{aligned}&\{1,2\}\coprod\left\{\frac{1}{\gamma_1}+1,\frac{2}{\gamma_1}+1,\ldots,\frac{\gamma_1-1}{\gamma_1}+1\right\}\\ &\coprod\left\{\frac{1}{\gamma_2}+1,\frac{2}{\gamma_2}+1,\ldots,\frac{\gamma_2-1}{\gamma_2}+1\right\}\coprod\cdots\\ &\cdots\coprod\left\{\frac{1}{\gamma_s}+1,\frac{2}{\gamma_s}+1,\ldots,\frac{\gamma_s-1}{\gamma_s}+1\right\}\end{aligned} \tag{3.2}$$

Proof. See Corollary 5.13 and the proof of Theorem 5.12 of [4]. □

We have the following formula for the variance. Note that we have $\hat{c}=1$ by Theorem 4.

Theorem 21. *Let the pair (f,G) be as above. The variance of the set of exponents of (f,G) is given by*

$$\mathrm{Var}_{(f,G)} = \frac{1}{12}\mu_{(f,G)}+\frac{1}{6}\chi_{(f,G)} = \frac{1}{12}\hat{c}\cdot\mu_{(f,G)}+\frac{1}{6}\chi_{(f,G)}. \tag{3.3}$$

Proof. Some elementary calculation yields the statement. □

Note that the pair (f, G) can be considered as a mirror partner of the orbifold curve (Deligne–Mumford stack) $\mathcal{C}$ which is a smooth projective curve of genus j_G with s isotropic points of orders $\gamma_1, \dots, \gamma_s$ (cf. Theorem 7.1 of [4]). The above formula for the variance is compatible with this observation. In particular, the dimension of $\mathcal{C}$ is 1, $\mu_{(f,G)}$ is the orbifold Euler number $\chi(\mathcal{C})$ of $\mathcal{C}$ and $\chi_{(f,G)}$ is the orbifold Euler characteristic of $\mathcal{C}$, which is the degree of the first Chern class $c_1(\mathcal{C})$ of $\mathcal{C}$. Applying this to the formula in Theorem 1, we recover the equation (3.3).

Acknowledgments

This work was supported by the DFG-program SPP1388 "Representation Theory" (Eb 102/6-1). The second named author is also supported by JSPS KAKENHI Grant Number 24684005. We are very grateful to the anonymous referee for carefully reading our paper, drawing our attention to a serious gap in the proof of the main result in the first version, and for most valuable comments which led to a major revision of the article.

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