

SINGULARITIES OF RICHARDSON VARIETIES

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ABSTRACT. This paper gives a short proof that essentially all questions concerning singularities of Richardson varieties reduce to corresponding questions about Schubert varieties. Consequently, one quickly deduces new and previously known results.

1. Introduction and the main result

Let G denote a simple reductive linear algebraic group over an algebraically closed field \mathbb{k} . Fix a choice of Borel and opposite Borel subgroups B and B_- , and let $T = B \cap B_-$ be the associated maximal torus. A **generalized flag variety** is a variety G/P , where $P \supseteq B$ is a parabolic subgroup. A B -orbit closure $X_w := \overline{BwP/P} \subseteq G/P$ is known as a **Schubert variety**. Here w is an element of the Weyl group $W \cong N(T)/T$ of G . If we let W_P be the parabolic subgroup of W associated to P , the choice of a different coset representative in wW_P gives the same Schubert variety.

There has been significant interest in invariants of singularities of Schubert varieties; two surveys include [BilLak01, Bri05]. More generally, one wishes to similarly understand singularities of other special subvarieties X of various G/P . Two examples of such subvarieties X are K -orbit closures (where K is a symmetric subgroup of G) and Peterson varieties. Combinatorics important to the study of Schubert varieties shows up in these cases; for example, permutation pattern avoidance appears in both the study of K -orbits [McgTra09] and Peterson varieties [InsYon12].

This paper concerns **Richardson varieties**, which are the varieties $X_w^v := X_w \cap X^v$, where $X^v = \overline{B_-vP/P}$ is an **opposite Schubert variety**. When $v \leq w$ in Bruhat order, X_w^v is nonempty, reduced and irreducible [Ric92]. Richardson varieties play a prominent role in Schubert calculus, as their classes in the cohomology ring of G/P satisfy $[X_w^v] = [X_w] \cdot [X_{w_o v}]$. Hence one can study the product of Schubert classes and more specifically the **generalized Littlewood–Richardson coefficients** $C_{v,u}^w$ defined by $[X_u] \cdot [X_v] = \sum_w C_{v,u}^w [X_w]$ by studying Richardson varieties. Several recent papers have studied the singularities of Richardson varieties using various techniques, including standard monomial theory [KreLak04], intersection theory [BilCos12, Bal12], and Frobenius splitting [KnuLamSpe09]. Indeed, [BilCos12, KnuLamSpe09] study a generalization known as the *projected Richardson varieties*.

Our main result states that, given any point $x \in X_w^v$, the local properties of X_w^v at x are completely determined by the Schubert cell X_σ° and opposite Schubert cell X_τ° to which x belongs. Furthermore, for any invariant or property whose behavior under

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product of varieties is understood, its behavior on X_w^v at x can be calculated from its behavior on X_w at points of X_σ^τ and its behavior on X^v at points of X_σ° . More precisely:

Theorem 1.1. *Let \mathcal{P} be a local invariant of varieties that is preserved under products with affine space. Suppose there exists a function $f_{\mathcal{P}}$ such that, for any varieties X and Y ,*

$$(1.1) \quad \mathcal{P}(x \times y, X \times Y) = f_{\mathcal{P}}(\mathcal{P}(x, X), \mathcal{P}(y, Y)), \quad \text{for any } x \in X, y \in Y.$$

Let $xP \in X_w^v$, and suppose $xP \in (B\sigma P/P) \cap (B_\tau P/P)$ for some $\sigma, \tau \in W$. Then

$$\mathcal{P}(xP, X_w^v) = f_{\mathcal{P}}(\mathcal{P}(\sigma P, X_w), \mathcal{P}(\tau P, X^v)) = f_{\mathcal{P}}(\mathcal{P}(xP, X_w), \mathcal{P}(xP, X^v)).$$

Our proof gives a local isomorphism, up to a product with affine space, between a neighborhood of a point $xP \in X_w^v$ and a product of local charts on X_w and X^v . This isomorphism is T -equivariant. Furthermore, the reader can see from the proof that our theorem applies not only to (T -equivariant) intrinsic invariants, but also to local invariants that are relative to the embedding of X_w^v in G/P (for example local T -equivariant cohomology classes).

We emphasize that in Theorem 1.1, xP need not be a T -fixed point. For Schubert varieties, the elements of B provide local isomorphisms between any point $xP \in X_w$ and a T -fixed point; thus one can assume $xP \in (G/P)^T$ if one is interested in local properties. However, in general the largest subgroup of G that fixes X_w^v is T , which is not large enough to put every point in the orbit of a T -fixed point.

Call an invariant \mathcal{P} **factorizable** with respect to $f_{\mathcal{P}}$ if it satisfies (1.1). Almost all interesting local invariants are factorizable. We now mention three important previously studied special cases of our theorem.

Example 1. \mathcal{P} = “is smooth” is factorizable with respect to $f_{\mathcal{P}}$ = “logical and”. \square

Let $\text{Singlocus}(X)$ denote the singular locus of a variety X . The following was first stated in [BilCos12, Corollary 2.9]. Their proof invokes Kleiman’s transversality theorem; it is also immediate from Theorem 1.1 combined with Example 1:

Corollary 1.2. $\text{Singlocus}(X_w^v) = (\text{Singlocus}(X_w) \cap X^v) \cup (X_w \cap \text{Singlocus}(X^v))$.

Example 2. \mathcal{P} = “is normal and Cohen–Macaulay with rational singularities” is also factorizable with respect to $f_{\mathcal{P}}$ = “logical and”. \square

It is well known that every Schubert variety is normal and Cohen–Macaulay with rational singularities; see [Bri05] for proofs and historical remarks. The following was originally proved in [Bri02] in characteristic zero and [KnuLamSpe09, Appendix A] in general. Their proofs use a generalization of the Bott–Samelson resolution and, in the latter case, the Frobenius splitting of G/P .

Corollary 1.3. *The Richardson variety X_w^v is Cohen–Macaulay and normal with rational singularities.*

Example 3. Consider the \mathbb{Z} -graded Hilbert series of $\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,Z}$, the associated graded ring of the local ring $\mathcal{O}_{p,Z}$, which is denoted $\text{Hilb}(\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,Z}, q)$. The **H -polynomial** $H_{p,Z}(q)$ is defined by

$$\text{Hilb}(\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,Z}, q) = \frac{H_{p,Z}(q)}{(1-q)^{\dim Z}},$$

and $H_{p,Z}(1)$ is the **Hilbert–Samuel multiplicity** $\text{mult}_{p,Z}$. Let \mathcal{P} be either $H_{p,Z}(q)$ or $\text{mult}_{p,Z}$, and $f_{\mathcal{P}}$ = “(usual) multiplication”. Since “taking associated graded commutes with tensor product,” \mathcal{P} is factorizable with respect to $f_{\mathcal{P}}$. \square

Corollary 1.4. *Let $xP \in X_w^v$. Then $H_{xP, X_w^v}(q) = H_{xP, X_w}(q) \cdot H_{xP, X^v}(q)$.*

In [LiYon11], the H -polynomial was studied for Schubert varieties, where it was conjectured $H_{vP, X_w}(q) \in \mathbb{N}[q]$. Corollary 1.4 extends the conjecture to Richardson varieties.

The consequence that

$$(1.2) \quad \text{mult}_{xP, X_w^v} = \text{mult}_{xP, X_w} \cdot \text{mult}_{xP, X^v}$$

also appears to be new. Previously, (1.2) was proved for *minuscule* G/P in [KreLak04, Remark 7.6.6] when $xP \in (G/P)^T$ (which is the case $\sigma = \tau$).

This result was generalized by M. Balan [Bal12] who proved (1.2) for all points xP , again assuming that P is minuscule. (A different generalization in the minuscule setting appears in [BilCos12, Remark 2.15].)

In Section 2, we prove Theorem 1.1. Our proof is elementary, at least if one accepts standard algebraic groups language. It uses a variation of [KazLus79, Lemma A.4], that relies on a variation on standard results about unipotent groups found, for example, in [Hum75, Section 28.1]. We also include a variant of the proof that avoids the language of algebraic groups (valid only in type A , but otherwise essentially the same as the general proof) in Section 3. In that section, we also discuss some further consequences of Theorem 1.1.

2. Proof of Theorem 1.1

The projection map $\rho : G/B \rightarrow G/P$ is a fibration with fibers locally isomorphic to affine space. Thus, since our local invariant is constant under product with affine space, its value at a point $xP \in \overline{BwP/P} \cap \overline{B_vP/P}$ is the same as that for any point in $\rho^{-1}(xP) \subseteq \overline{Bw^PB/B} \cap \overline{Bv_PB/B}$, where w^P is the maximal length coset representative of $w \in W/W_P$ and v_P is the minimal length coset representative of $v \in W/W_P$. Therefore, it follows that to check correctness of Theorem 1.1, we can assume $P = B$.

Let $X_u^\circ = BuB/B$ and $X_u^\circ = B_-uB/B$ be the Schubert and opposite Schubert cells for u . The proof depends on the following version of [KazLus79, Lemma A.4].

Lemma 2.1. *Given any $u \in W$, there exists an isomorphism*

$$\eta = (\eta_1, \eta_2) : uX_\circ^{\text{id}} \rightarrow X_u^\circ \times X_u^\circ$$

such that, for any $x \in uX_\circ^{\text{id}}$, $\eta_1(x)$ and x are in the same Schubert cell, and $\eta_2(x)$ and x are in the same opposite Schubert cell.

The result [KazLus79, Lemma A.4] states that there is such a T -equivariant isomorphism but does not state the additional properties about it asserted in Lemma 2.1.

Our proof of Lemma 2.1 depends on the following lemma about algebraic groups.

Lemma 2.2. *Let U be a unipotent group on which T acts without nontrivial fixed points, while $\mathfrak{u} = \text{Lie}(U)$ is the direct sum of one-dimensional eigenspaces for T*

corresponding to characters α with distinct connected kernels T_α . Furthermore, let U_1 and U_2 be T -stable subgroups of U such that $U_1 \cap U_2 = \{\text{id}\}$, $U_1 U_2 = U$, and the corresponding Lie algebras satisfy $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$. Then there exists an isomorphism of varieties

$$\sigma : U \rightarrow U_1 \times U_2$$

such that the factors σ_1 and σ_2 satisfy $\sigma_1(u) \in U_2 u$ and $\sigma_2(u) \in U_1 u$.

Note that σ will not be a group homomorphism unless U_1 and U_2 commute.

Proof. It is a standard fact (see, for example, [Hum75, Section 28.1]) that the product morphisms $\pi : U_1 \times U_2 \rightarrow U$ and $\kappa : U_1 \times U_2 \rightarrow U$ given by $\pi(u_1, u_2) = u_1 u_2$ and $\kappa(u_1, u_2) = u_2 u_1$ are bijective morphisms of varieties. The hypothesis on the Lie algebras implies they are actually isomorphisms (of varieties) since the differential is everywhere injective; see the local isomorphism criterion, for example, in [Har92, Theorem 14.9]. However, the inverses of these two maps each only satisfy one of our two requirements that $\sigma_1(u) \in U_2 u$ and $\sigma_2(u) \in U_1 u$.

Let π_2^{-1} and κ_1^{-1} be respectively the inverses of π and κ followed by projection onto respectively the second and first factors. We now have a map $\sigma = \kappa_1^{-1} \times \pi_2^{-1} : U \rightarrow U_1 \times U_2$. It remains to prove that $\kappa_1^{-1} \times \pi_2^{-1}$ is an isomorphism (of schemes), as once that is done, σ is our desired map, as follows. Since $\sigma_1 = \kappa_1^{-1}$, so $\sigma_1(u) \in U_2 u$, and since $\sigma_2 = \pi_2^{-1}$, so $\sigma_2(u) \in U_1 u$.

We prove $\kappa_1^{-1} \times \pi_2^{-1}$ is an isomorphism by induction on the dimension of U . Since U is unipotent, it is nilpotent and thus has a nontrivial center U' [Hum75, Lemma 17.4(c)]. This center is clearly T -stable. Therefore $U'_1 = U_1 \cap U'$ and $U'_2 = U_2 \cap U'$ are T -stable subgroups of U' .

As U' , U'_1 and U'_2 are unipotent and T -stable, they are spanned by products of root subgroups U_α [Hum75, Proposition 28.1]. These root subgroups are a subset of those spanning U , U_1 and U_2 respectively. Hence, it follows $U'_1 U'_2 = U'$. Therefore, π and κ restrict to isomorphisms $\tilde{\pi}, \tilde{\kappa} : U'_1 \times U'_2 \rightarrow U'$. Since U'_1 and U'_2 commute, $\tilde{\kappa}_1^{-1} \times \tilde{\pi}_2^{-1}$ is actually the same as both $\tilde{\kappa}^{-1}$ and $\tilde{\pi}^{-1}$ on U' and hence invertible.

Furthermore, because U' is central, π and κ induce maps $\bar{\pi}, \bar{\kappa} : U_1/U'_1 \times U_2/U'_2 \rightarrow U/U'$.

(It is straightforward to check the maps are well defined.) By induction, the map

$$\bar{\kappa}_1^{-1} \times \bar{\pi}_2^{-1} : U/U' \rightarrow U_1/U'_1 \times U_2/U'_2$$

is an isomorphism. (Technically, the induction applies to the natural embeddings, which are isomorphisms onto their images, of U_1/U'_1 and U_2/U'_2 in U/U' .)

Summarizing, we have maps in a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & U' & \longrightarrow & U & \longrightarrow & U/U' \longrightarrow 0 \\ & & \downarrow \tilde{\kappa}_1^{-1} \times \tilde{\pi}_2^{-1} & & \downarrow \sigma = \kappa_1^{-1} \times \pi_2^{-1} & & \downarrow \bar{\kappa}_1^{-1} \times \bar{\pi}_2^{-1} \\ 0 & \longrightarrow & U'_1 \times U'_2 & \longrightarrow & U_1 \times U_2 & \longrightarrow & U_1/U'_1 \times U_2/U'_2 \longrightarrow 0 \end{array}$$

It is easy to check from the definitions that σ not only restricts to a group homomorphism on U' but moreover that $\sigma(ab) = \sigma(a)\sigma(b)$ whenever a or b is in U' . Using this, a straightforward diagram chase similar to the proof of the five-lemma implies σ is a bijection.

By our hypotheses, the equality of tangent spaces $T_p(U) = \mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2 = T_{(a,b)}(U_1 \times U_2)$ holds when p, a, b are the identity. Hence, $d\sigma$ is an injection at the identity.

However, injectivity holds everywhere else since σ is T -equivariant, the identity is the only T -fixed point, and failure for the differential to be injective is a closed condition. Therefore, σ is an isomorphism of schemes, by another use of the local criterion for isomorphism. \square

Proof of Lemma 2.1: Let $U_+ \subset B$ and $U_- \subset B_-$ respectively denote the unipotent subgroups of the Borel and opposite Borel subgroups of G . Given $u \in W$, we use Lemma 2.2 in the case where $U = uU_-u^{-1}$, $U_1 = U_- \cap uU_-u^{-1}$, and $U_2 = U_+ \cap uU_-u^{-1}$. Lemma 2.2 gives us an isomorphism

$$\sigma : U \rightarrow U_1 \times U_2.$$

such that $\sigma_1(u)$ and u differ by left multiplication by an element of U_2 and $\sigma_2(u)$ and u differ by left multiplication by an element of U_1 .

Now note that we have isomorphisms $m_0 : U \rightarrow uX_\circ^{\text{id}}$, $m_1 : U_1 \rightarrow X_\circ^u$, and $m_2 : U_2 \rightarrow X_u^\circ$ each defined by $m_i(g) = guB$. Let $\eta = (m_1 \times m_2) \circ \sigma \circ m_0^{-1}$. Since by Lemma 2.2, $\sigma_1(g) \in U_2g$ for any $g \in U$ and $U_2 \subseteq U_+ \subset B$, $\eta_1(guB) \in BguB$ and hence this point is in the same Schubert cell as guB . Similarly since $U_1 \subset B_-$, the points $\eta_2(guB)$ and guB are in the same opposite Schubert cell. \square

Proof of Theorem 1.1: Let $xB \in G/B$. Since the charts $\{uX_\circ^{\text{id}}\}_{u \in W}$ cover G/B , we can fix $u \in W$ such that $xB \in uX_\circ^{\text{id}}$. Also let η be the map from Lemma 2.1.

We have the Bruhat and opposite Bruhat decompositions

$$(2.1) \quad X_w = \coprod_{\sigma \leq w} X_\sigma^\circ \quad \text{and} \quad X^v = \coprod_{\tau \geq v} X_\tau^\tau.$$

Therefore by (2.1), $xB \in X_w^v$ if and only if $xB \in X_\sigma^\circ$ for some fixed $\sigma \leq w$ and $xB \in X_\tau^\tau$ for some fixed $\tau \geq v$. By Lemma 2.1 we know $\eta_1(xB)$ and xB are in the same Schubert cell, and we also know $\eta_2(xB)$ and xB are in the same opposite Schubert cell. Therefore, $xB \in X_w^v$ if and only if $\eta_1(xB) \in X_w$ and $\eta_2(xB) \in X^v$.

Therefore, η restricts to an isomorphism

$$\eta|_{X_w^v} : uX_\circ^{\text{id}} \cap X_w^v \rightarrow (X_\circ^u \cap X_w) \times (X_u^\circ \cap X^v).$$

This isomorphism is scheme-theoretic since both sides are reduced. (The right-hand side is reduced since it is a product of two integral schemes over their base field and thus integral.)

Now suppose $xB \in X_w^v$. Since \mathcal{P} is a local invariant,

$$\mathcal{P}(xB, X_w^v) = \mathcal{P}(xB, uX_\circ^{\text{id}} \cap X_w^v).$$

Since η is an isomorphism,

$$\mathcal{P}(xB, uX_\circ^{\text{id}} \cap X_w^v) = \mathcal{P}(\eta(xB), (X_\circ^u \cap X_w) \times (X_u^\circ \cap X^v)).$$

By the hypothesis that \mathcal{P} factorizes with respect to $f_{\mathcal{P}}$,

$$\mathcal{P}(\eta(xB), (X_\circ^u \cap X_w) \times (X_u^\circ \cap X^v)) = f_{\mathcal{P}}(\mathcal{P}(\eta_1(xB), X_\circ^u \cap X_w), \mathcal{P}(\eta_2(xB), X_u^\circ \cap X^v)).$$

If $xB \in X_\sigma^\circ \cap X_\tau^\tau$, then by Lemma 2.1 we know $\eta_1(xB) \in X_\sigma^\circ$ and $\eta_2(xB) \in X_\tau^\tau$.

Since in a Schubert variety every point is locally isomorphic (by the action of B) to the T -fixed point in its Schubert cell, we have:

$$\mathcal{P}(\eta_1(xB), X_\circ^u \cap X_w) = \mathcal{P}(\eta_1(xB), X_w) = \mathcal{P}(\sigma B, X_w).$$

The first equality uses [KazLus79, Lemma A.4] (of which we have just proved a stronger version), which states that $(X_u^\circ \cap X_w) \times \mathbb{k}^{\ell(u)} \cong uX_\circ^{\text{id}} \cap X_w$. We are also using the assumption that \mathcal{P} is invariant under Cartesian product with affine space.

The same arguments apply to the opposite Schubert varieties (using the action of B_-); hence:

$$\mathcal{P}(\eta_2(xB), X_u^\circ \cap X^v) = \mathcal{P}(\eta_2(xB), X^v) = \mathcal{P}(\tau B, X^v).$$

Combining the above we obtain

$$\mathcal{P}(xB, X_w^v) = f_{\mathcal{P}}(\mathcal{P}(\sigma B, X_w), \mathcal{P}(\tau B, X^v)),$$

as desired. \square

3. Further consequences and comments

3.1. Other singularity invariants. For brevity, we refer the reader to [WooYon08] for discussion of properties of Schubert varieties of interest to us. The purpose of this section is to explain the extension of this discussion to Richardson varieties.

There are two kinds of invariants \mathcal{P} we are interested in. The first is of the “yes/no” kind. As in Corollary 1.2, they are factorizable by $f_{\mathcal{P}}$ = “logical **and**”. If $\text{non}\mathcal{P}\text{locus}(X)$ is the set of points in X where \mathcal{P} takes on the value “no”, then we have

$$(3.1) \quad \text{non}\mathcal{P}\text{locus}(X_w^v) = (\text{non}\mathcal{P}\text{locus}(X_w) \cap X^v) \cup (X_w \cap \text{non}\mathcal{P}\text{locus}(X^v)).$$

In [WooYon06] it was determined which Schubert varieties are Gorenstein. The property \mathcal{P} = “Gorenstein” is factorizable with respect to $f_{\mathcal{P}}$ = “logical **and**” since being Gorenstein is a homological property (and the total complex of the double complex formed by tensoring two free resolutions is exact in this case). Thus, we record:

Corollary 3.1. *(3.1) holds for \mathcal{P} = “is Gorenstein”.*

In [WooYon06, WooYon08] a conjectural description was given of the non-Gorenstein locus of any Schubert variety. This conjecture was stated using interval pattern avoidance in the case of GL_n/B ; furthermore, the conjecture was made explicit and proved for minuscule G/P by N. Perrin [Per09]. In such cases, one can combine these combinatorial descriptions with Corollary 3.1 to obtain descriptions of $\text{non}\mathcal{P}\text{locus}(X_w^v)$. Recently, Darayon [Dar12] determined which Richardson varieties in Grassmannians are *arithmetically* Gorenstein under the standard embedding. (Theorem 1.1 does not apply to the cone singularity.)

Unfortunately, even Corollary 3.1 combined with the results in [WooYon06] does not provide a characterization for when $X_w^v \subseteq GL_n/B$ is Gorenstein. One seems to need a (compatible) characterization for the non-Gorenstein locus of a Schubert variety in general.

A stronger property than Gorensteinness is that of \mathcal{P} = “is a local complete intersection (lci)”. This \mathcal{P} factorizes with respect to $f_{\mathcal{P}}$ = “logical **and**”, and thus (3.1) again applies. Recently, a characterization of which $X_w \subseteq GL_n/B$ are local complete intersections has been determined by H. Úlfarsson and the second author [UlfWoo11]. To further determine when X_w^v is a local complete intersection, one also seems to need a characterization of the lci locus of a Schubert variety.

The second type of invariant we are interested in takes values in \mathbb{Z} or some (Laurent) polynomial ring. Suppose \mathcal{P} is factorizable with respect to $f_{\mathcal{P}}$ = “usual multiplication of numbers or (Laurent) polynomials”. If we let $\mathcal{P}_{x,X}$ denote the value taken at $x \in X$, then just as in Corollary 1.4, we have

$$(3.2) \quad \mathcal{P}_{xP, X_w^v} = \mathcal{P}_{xP, X_w} \cdot \mathcal{P}_{xP, X^v}.$$

for any $xP \in X_w^v$.

More refined than knowing if a point $x \in X$ is Gorenstein is to know the *Cohen–Macaulay type* of that point. Let $\mathcal{P}_{x,X} = CM_{x,X} \in \mathbb{N}$ denote this statistic. (When $CM_{x,X} = 1$ then x is a Gorenstein point, assuming X is Cohen–Macaulay, which is always true in our situation.) Again, since \mathcal{P} is a homological property, it factorizes with respect to $f_{\mathcal{P}}$ = “(usual multiplication)”. Thus, we obtain another example of (3.2):

Corollary 3.2. *(3.2) holds for \mathcal{P} = “Cohen–Macaulay type”.*

Let $P_{v,w}(q) \in \mathbb{N}[q]$ denote a Kazhdan–Lusztig polynomial. Owing to its interpretation as a Poincaré polynomial for local intersection cohomology at vB in $X_w \subseteq G/B$ (together with a Kunneth-type formula for local intersection cohomology), one can apply (3.2), i.e.,

$$Q_{x,w,v}(q) = P_{\sigma,w}(q) \cdot P_{w_0\tau, w_0v}(q),$$

where $Q_{x,w,v}(q)$ is the Poincaré polynomial for local intersection cohomology of X_w^v at xB and where $xB \in X_{\sigma}^{\circ}$ and $xB \in X_{\tau}^{\circ}$ (as in the Introduction section).

Work of Li and the second author [LiYon11] suggests an analogy between the polynomials $P_{v,w}(q)$ and $H_{vP, X_w}(q)$. In particular, for Grassmannians and covexillary Schubert varieties, it is shown that $P_{v,w}(q) \preceq H_{vP, X_w}(q)$, where \preceq means “coefficient-wise \leq ”. This therefore lifts to Richardson varieties.

Wyser [Wys12] has recently proved that certain K -orbits (for example, certain $K = GL_p \times GL_q$ orbits in GL_n/B) are isomorphic to Richardson varieties. Therefore, our results apply to those K -orbits. Further discussion may appear elsewhere.

3.2. Another proof of Lemma 2.1, when in Type A. Assume x is the generic matrix in uX_{\circ}^{id} : the rows are a generic matrix in U_- , but are u -permuted. Let z_{ij} be the entry in row i and column j . Note $u(i) = j$ means there is a 1 in column i and row j of the matrix. See Example 4.

Define $\eta_1(x)$ to be obtained by “sweeping up using the 1’s in each column”: take the column j_1 whose “1” (coming from u) is in row n (so $j_1 = u^{-1}(n)$) and do upward row operations, so that all entries in column j_1 strictly above row n are 0. Then find the column j_2 whose “1” is in row $n - 1$ and do operations that place 0 above that “1” etc. Declare the result after sweeping in all n columns to be $\eta_1(x)$. (The resulting matrix can be obtained by any sequence of row operations that put 0’s above each “1”; however the stated order is most efficient, as no column has to be swept more than once.) Clearly $\eta_1(x)$ is in BuB/B and is in the same Schubert cell as x since the former comes from the latter by left multiplication of a matrix from B .

Similarly, define $\eta_2(x) \in B_-uB/B$ by downward sweeping operations, starting with the column whose “1” (from u) appears in row 1, etc. Likewise, $\eta_2(x)$ and x are in the same opposite Schubert cell.

Let

$$D^{\text{up}} = \{(i, j) \in [n] \times [n] : i > \pi(j) \text{ and } j < \pi^{-1}(i)\},$$

$$D^{\text{down}} = \{(i, j) \in [n] \times [n] : i < \pi(j) \text{ and } j < \pi^{-1}(i)\}.$$

These are the positions in $\eta_1(x)$ and $\eta_2(x)$ respectively not (*a priori*) equal to 0 or 1.

Claim 3.3. (i) In $\eta_1(x)$, the entry in position $(i, j) \in D^{\text{up}}$ is an expression of the form $z_{ij} + f$ where f is a polynomial in the generic entries $z_{a,b}$ where either $b = j$ and $a > i$, or $b > j$.

(ii) Similarly, in $\eta_2(x)$, the entry in position $(i, j) \in D^{\text{down}}$ is an expression of the form $z_{ij} + g$ where g is a polynomial in the generic entries $z_{a,b}$ where either $b = j$ and $a < i$, or $b > j$.

Proof. The argument for (i) is the upside down version of (ii). Let us argue the latter case (where one does downward sweeping). The only 1's we use to sweep that affect some position $(i, j) \in D^{\text{down}}$ are those strictly northeast of (i, j) . Say this 1 is in position (i', j') (with $i' < i$ and $j' > j$). Then the row operation involving this 1 converts the entry $y_{i,j}$ in position (i, j) to $y_{ij} - y_{i',j}y_{i,j'}$, and the result follows by induction on $j \leq n$. \square

We now recover all generic entries z_{ij} in x from the entries of $\eta_1(x)$ and $\eta_2(x)$. Inductively, we argue that, if we know all such entries in columns $k, k+1, \dots, n$ of x , we can deduce the entries in column $k-1$ of x .

For the base case of the argument, note that column n of x consists of only 0's and 1's, so there is nothing to do. In the induction step, the argument depends on the relative positions of the 1's in columns $k-1$ and k :

Case I – The 1 in column $k-1$ is higher (lower index) than the one in column k : Suppose $u(k-1) = b$ and $u(k) = c$ where $b < c$. First consider positions $(q, k-1)$ for $q < b$. We use $\eta_2(x)$. Either position $(1, k-1)$ is 0 in x (because there is a 1 to the left of column $k-1$ in the first row), or $z_{1,k-1}$ appears in that position of $\eta_2(x)$ and hence is determined. Then by Claim 3.3(ii) and induction, working top to bottom, we determine each $z_{q,k-1}$ for $q < b$.

Now consider indices $(r, k-1)$ for $r > b$. This time note position $(n, k-1)$ is either 0 in x or $z_{n,k-1}$ appears there in both x and $\eta_1(x)$ and hence is determined. Then the remaining generic variables of x in column $k-1$ are obtained from $\eta_1(x)$ by working bottom to top, applying induction and Claim 3.3(i).

Case II – The 1 in column $k-1$ is lower (higher index) than the one in column k : The argument is the upside down version of Case I, reversing the roles of $\eta_1(x)$ and $\eta_2(x)$ in the obvious way. \square

Our second proof of Theorem 1.1 (in type A) now concludes as in Section 2.

Example 4. Let $u = 31542 \in S_5$. A generic matrix in uX_{\circ}^{id} has a standard form

$$x = \begin{pmatrix} z_{11} & 1 & 0 & 0 & 0 \\ z_{21} & z_{22} & z_{23} & z_{24} & 1 \\ 1 & 0 & 0 & 0 & 0 \\ z_{41} & z_{42} & z_{43} & 1 & 0 \\ z_{51} & z_{52} & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$\eta_1(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & z_{22} - z_{24}(z_{42} - z_{52}z_{43}) - z_{23}z_{52} & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ z_{41} - z_{51}z_{43} & z_{42} - z_{52}z_{43} & 0 & 1 & 0 \\ z_{51} & z_{52} & 1 & 0 & 0 \end{pmatrix}$$

and

$$\eta_2(x) = \begin{pmatrix} z_{11} & 1 & 0 & 0 & 0 \\ z_{21} - z_{11}z_{22} & 0 & z_{23} & z_{24} & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_{43} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We show how to recover the entries of x as polynomials of the entries in $\eta_1(x)$ and $\eta_2(x)$. The variables z_{51} , z_{52} , z_{43} , z_{23} , z_{24} , and z_{11} appear in $\eta_1(x)$ or $\eta_2(x)$. (These are the variables that have no 0's between them and the 1 to their right in $\eta_1(x)$ or $\eta_2(x)$.) Given these, we can recover z_{41} , z_{42} , and z_{22} from the entries of η_1 and η_2 , since $z_{41} = (z_{41} - z_{51}z_{43}) + z_{51}z_{43}$, $z_{42} = (z_{42} - z_{52}z_{43}) + z_{52}z_{43}$, and $z_{22} = (z_{22} - z_{24}(z_{42} - z_{52}z_{43}) - z_{23}z_{52}) + z_{23}z_{52} + z_{24}(z_{42} - z_{52}z_{43})$. Given these, we can recover z_{21} , since $z_{21} = (z_{21} - z_{11}z_{22}) + z_{11}z_{22}$. \square

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