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MULTIVARIABLE LUBIN–TATE (φ, Γ)-MODULES AND FILTERED φ-MODULES

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Abstract. We define some rings of power series in several variables, that are attached to a Lubin–Tate formal module. We then give some examples of (φ, Γ)-modules over those rings. They are the global sections of some reflexive sheaves on the p-adic open unit polydisk, that are constructed from a filtered φ-module using a modification process. We prove that we obtain every crystalline (φ, Γ)-module over those rings in this way.

Contents

Introduction

Let F be the unramified extension of $\mathbf{Q}_p$ of degree h and let $q = p^h$ so that the residue field of $\mathcal{O}_F$ is $\mathbf{F}_q$. We fix an embedding $F \subset \overline{\mathbf{Q}}_p$ so that if $\sigma : F \to F$ denotes the absolute Frobenius map, which lifts $x \mapsto x^p$ on $\mathbf{F}_q$, then the h embeddings of F into $\overline{\mathbf{Q}}_p$ are given by $\mathrm{Id}, \sigma, \ldots, \sigma^{h-1}$. The symbol φ_q denotes a σ^h-semilinear Frobenius map. If K is a subfield of $\overline{\mathbf{Q}}_p$, then let $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$.

The goal of this article is to present a first attempt at constructing some "multivariable Lubin–Tate (φ, Γ)-modules," that is some (φ_q, Γ_F)-modules over rings of power series in h variables, on which $\Gamma_F = \mathcal{O}_F^\times$ acts by a formula arising from a Lubin–Tate formal $\mathcal{O}_F$-module. A construction of such (φ_q, Γ_F)-modules, but "in one variable," was carried out by Kisin and Ren in [18]: they prove that in certain cases, the (φ_q, Γ_F)-modules arising from Fontaine's standard construction of [12] are overconvergent.

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In order to do so, Kisin and Ren adapt the construction of (φ,Γ)-modules attached to filtered (φ,N)-modules given in [4] to their setting, which allows them to attach a (φ_q,Γ_F)-module in one variable to a filtered φ_q-module. They then point out in the introduction of [18] that "it seems likely that in order to obtain a classification valid for any crystalline G_K-representation one needs to consider higher dimensional subrings of $W(\mathrm{Fr}\, R)$, constructed using the periods of all the conjugates of [the Lubin–Tate group]."

The motivation for these computations is the hope that we can construct some representations of the Borel subgroup of $\mathrm{GL}_2(F)$, for example using the recipe given by Colmez in [11], that would shed some light on the p-adic local Langlands correspondence for $\mathrm{GL}_2(F)$ (see [9]). Theorems A, B and C below are a very first step in this direction, but remain insufficient. In particular, the "p-adic Fourier theory" of Schneider and Teitelbaum (see [20]) will very likely play an important role in the sequel.

We now describe our results in more detail. Let LT_h be the Lubin–Tate formal $\mathcal{O}_F$-module for which multiplication by p is given by $[p](T) = pT + T^q$. We denote by $[a](T)$ the element of $\mathcal{O}_F[\![T]\!]$ that gives the action of $a \in \mathcal{O}_F$ on LT_h. We consider two rings $\mathcal{R}^+(Y)$ and $\mathcal{R}(Y)$ of power series in the h variables $Y_0,\dots,Y_{h-1}$, with coefficients in F. The ring $\mathcal{R}^+(Y)$ is the ring of power series that converge on the open unit polydisk, and $\mathcal{R}(Y)$ is the Robba ring that corresponds to it, by adapting Schneider's construction given in the appendix of [23]. The action of the group $\mathcal{O}_F^\times$ on those rings is given by the formula $a(Y_j) = [\sigma^j(a)](Y_j)$, and the Frobenius map φ_q is given by $\varphi_q(Y_j) = [p](Y_j)$.

The construction of p-adic periods for Lubin–Tate groups gives rise to a map $\mathcal{R}^+(Y) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$, where $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ is the Fréchet completion of $\widetilde{\mathbf{B}}^+ = W(\widetilde{\mathbf{E}}^+)[1/p]$, and we prove (Corollary 3.7) that this map is in fact injective (remark: if $\widetilde{\mathcal{R}}^+(Y)$ denotes the completion of the perfection of $\mathcal{R}^+(Y)$, then the map above extends to a map $\widetilde{\mathcal{R}}^+(Y) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ but note that, by the theory of the field of norms of [14, 22], this latter map is not injective anymore if $h \geqslant 2$. This has prevented us from studying étale φ_q-modules using Kedlaya's methods, so such considerations are absent from this paper).

Let D be a finite dimensional F-vector space, endowed with an F-linear Frobenius map $\varphi_q : D \to D$, and an action of G_F on D that factors through Γ_F and commutes with φ_q. For each $0 \leqslant j \leqslant h-1$, let $\mathrm{Fil}_j^\bullet$ be a filtration on $F \otimes_F^{\sigma^j} D \simeq D$ that is stable under Γ_F.

For example, if V is an F-linear crystalline representation of G_F of dimension d, then $\mathrm{D}_{\mathrm{cris}}(V)$ is a free $F \otimes_{\mathbf{Q}_p} F$-module of rank d, and we have

$$\mathrm{D}_{\mathrm{cris}}(V) = D \oplus \varphi(D) \oplus \cdots \oplus \varphi^{h-1}(D),$$

according to the decomposition of $F \otimes_{\mathbf{Q}_p} F$ as $\prod_{\sigma^i : F \to F} F$. Each $\varphi^j(D)$ has the filtration induced from $\mathrm{D}_{\mathrm{cris}}(V)$, and we set $\mathrm{Fil}_j^k D = \varphi^{-j}(\mathrm{Fil}^k \mathrm{D}_{\mathrm{cris}}(V) \cap \varphi^j(D))$.

The composite of the map $\mathcal{R}^+(Y) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ with the map $\varphi^{-k} : \widetilde{\mathbf{B}}^+_{\mathrm{rig}} \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ gives rise to a map $\iota_k : \mathcal{R}^+(Y) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$. Let $\log_{\mathrm{LT}}(T)$ be the logarithm of LT_h, and let $\lambda_j = \log_{\mathrm{LT}}(Y_j)/Y_j$ and $\lambda = \lambda_0 \times \cdots \times \lambda_{h-1}$ (note that the image of $\prod_{j=0}^{h-1} \log_{\mathrm{LT}}(Y_j)$ in

$\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ is some $\mathbf{Q}_p$-multiple of $t = \log(1+X)$, so that λ is an analogue of t/X). Define

$$\mathrm{M}^+(D) = \{y \in \mathcal{R}^+(Y)[1/\lambda] \otimes_F D,\ \iota_k(y) \in \mathrm{Fil}^0_{-k}(\mathbf{B}_{\mathrm{dR}} \otimes_F^{\sigma^{-k}} D) \text{ for all } k \geqslant h\}.$$

The ring $\mathcal{R}^+(Y)$ is a Fréchet–Stein algebra in the sense of [21], and we therefore have the notion of coadmissible $\mathcal{R}^+(Y)$-modules, which are the global sections of coherent sheaves on the open unit polydisk.

Theorem A. *The module* $\mathrm{M}^+(D)$ *is a reflexive coadmissible* $\mathcal{R}^+(Y)$*-module, for all* $0 \leqslant j \leqslant h-1$, $\mathrm{M}^+(D)[\lambda_j/\lambda]$ *is a free* $\mathcal{R}^+(Y)[\lambda_j/\lambda]$*-module of rank* d*, and we have* $\mathrm{M}^+(D) = \cap_{j=0}^{h-1} \mathrm{M}^+(D)[\lambda_j/\lambda]$.

The definition of $\mathrm{M}^+(D)$ is analogous to the one given in [4,18] and similar articles. When $h=1$, the proof of theorem A relies on the fact that $\mathrm{M}^+(D)$ can be seen as a vector bundle on the open unit disk. Our proof of theorem A relies on the one dimensional case, and on the interpretation of $\mathrm{M}^+(D)$ as the global sections of a coherent sheaf on the open unit polydisk.

Remark. If $h \leqslant 2$, then $\mathcal{R}^+(Y)$ is of dimension $\leqslant 2$ and one can then prove that $\mathrm{M}^+(D)$, being reflexive, is actually free of rank d (see Remark 5.7). If $h \geqslant 3$, I do not know whether $\mathrm{M}^+(D)$ is free of rank d in general.

Let $\mathrm{M}(D) = \mathcal{R}(Y) \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$, so that $\mathrm{M}(D)$ is a (φ_q, Γ_F)-module over the multivariable Robba ring $\mathcal{R}(Y)$ (see Definition 6.4).

Theorem B. *If* V *is an* F*-linear crystalline representation of* G_F*, and if* D *arises from* $\mathrm{D}_{\mathrm{cris}}(V)$ *as above, then there is a natural map* $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathcal{R}(Y)} \mathrm{M}(D) \to \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_F V$*, and this map is an isomorphism.*

If M is a (φ_q, Γ_F)-module over $\mathcal{R}(Y)$, then we set $\mathrm{D}_{\mathrm{cris}}(\mathrm{M}) = (\mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} \mathrm{M})^{\Gamma_F}$, and we say that M is crystalline if (1) $\mathrm{M}[\lambda_j/\lambda]$ is a free $\mathcal{R}(Y)[\lambda_j/\lambda]$-module of some rank d for all j, (2) $\mathrm{M} = \cap_{j=0}^{h-1} \mathrm{M}[\lambda_j/\lambda]$, and (3) $\dim \mathrm{D}_{\mathrm{cris}}(\mathrm{M}) = d$. For example, if D is a filtered φ_q-module with h filtrations $\mathrm{Fil}^\bullet_j$ as above, on which the action of Γ_F is trivial, then $\mathrm{M}(D)$ is a crystalline (φ_q, Γ_F)-module.

Theorem C. *The functors* $\mathrm{M} \mapsto \mathrm{D}_{\mathrm{cris}}(\mathrm{M})$ *and* $D \mapsto \mathrm{M}(D)$*, between the category of crystalline* (φ_q, Γ_F)*-modules over* $\mathcal{R}(Y)$ *and the category of* φ_q*-modules with* h *filtrations, are mutually inverse.*

Note that if $h=1$, then the (φ, Γ)-modules that we construct are the classical cyclotomic ones, and theorems A, B and C are well-known.

We now give a short description of the contents of this paper: in Section 1, we give some reminders about the p-adic periods of Lubin–Tate formal $\mathcal{O}_F$-modules. In Section 2, we define the various rings of power series that we use, and establish some of their properties. In Section 3, we embed those rings in the usual rings of p-adic periods. In Section 4, we briefly survey Kisin and Ren's construction and explain why (φ_q, Γ_F)-modules over rings of power series in several variables are needed. In Section 5, we attach such objects to filtered φ_q-modules and prove Theorem A. In Section 6, we define (φ_q, Γ_F)-modules and prove Theorem B. In Section 7, we study crystalline (φ_q, Γ_F)-modules and prove Theorem C.

1. Periods of Lubin–Tate formal groups

Let LT_h be the Lubin–Tate formal $\mathcal{O}_F$-module for which multiplication by p is given by $[p](T) = pT + T^q$. We denote by $[a](T)$ the element of $\mathcal{O}_F[\![T]\!]$ that gives the action of $a \in \mathcal{O}_F$ on LT_h and by $S(T,U) = T \oplus U$ the element of $\mathcal{O}_F[\![T,U]\!]$ that gives addition.

Let $\pi_0 = 0$ and for each $n \geqslant 1$, let $\pi_n \in \overline{\mathbf{Q}}_p$ be such that $[p](\pi_n) = \pi_{n-1}$, with $\pi_1 \neq 0$. We have $\mathrm{val}_p(\pi_n) = 1/q^{n-1}(q-1)$ if $n \geqslant 1$. Let $F_n = F(\pi_n)$ and let $F_\infty = \cup_{n\geqslant 1} F_n$. Recall that $\mathrm{Gal}(F_\infty/F) \simeq \mathcal{O}_F^\times$ and that the maximal abelian extension of F is $F_\infty \cdot F^{\mathrm{unr}}$. Denote by H_F the group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_\infty)$, by Γ_F the group $\mathrm{Gal}(F_\infty/F)$ and by χ_{LT} the isomorphism $\chi_{\mathrm{LT}} : \Gamma_F \to \mathcal{O}_F^\times$. In the sequel, we sometimes directly identify Γ_F with $\mathcal{O}_F^\times$, that is we drop "χ_{LT}" from the notation to make the formulas less cumbersome.

Let $\widetilde{\mathbf{E}}^+ = \varprojlim_{x\mapsto x^q} \mathcal{O}_{\mathbf{C}_p}/p$ and $\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+)$ denote Fontaine's rings of periods (see [13]). Note that we take the limit with respect to the maps $x \mapsto x^q$, which does not change the rings. Let $\varphi_q : \widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{A}}^+$ be given by $\varphi_q = \varphi^h$. Recall that in Section 9.2 of [10], Colmez has constructed a map $\{\cdot\} : \widetilde{\mathbf{E}}^+ \to \widetilde{\mathbf{A}}^+$ having the following property: if $x \in \widetilde{\mathbf{E}}^+$, then $\{x\}$ is the unique element of $\widetilde{\mathbf{A}}^+$ that lifts x and satisfies $\varphi_q(\{x\}) = [p](\{x\})$. Let $\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbf{C}_p}$ denote Fontaine's map (see [13]). If $x = (x_0, x_1, \ldots)$, then $\theta(\{x\}) = \lim_{n\to\infty}[p^n](\widehat{x}_n)$, where $\widehat{x}_n \in \mathcal{O}_{\mathbf{C}_p}$ is any lift of x_n.

Let $u = \{(\overline{\pi}_0, \overline{\pi}_1, \ldots)\} \in \widetilde{\mathbf{A}}^+$, so that $g(u) = [\chi_{\mathrm{LT}}(g)](u)$ if $g \in G_F$.

Let $\log_{\mathrm{LT}}(T) \in F[\![T]\!]$ denote the Lubin–Tate logarithm map, which converges on the open unit disk and satisfies $\log_{\mathrm{LT}}([a](T)) = a \cdot \log_{\mathrm{LT}}(T)$ if $a \in \mathcal{O}_F$. Recall (see Section 9.3 of [10]) that $\log_{\mathrm{LT}}(u)$ converges in $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ to an element t_F which satisfies $g(t_F) = \chi_{\mathrm{LT}}(g) \cdot t_F$.

Let $Q_k(T)$ be the minimal polynomial of π_k over F. We have $Q_0(T) = T$, $Q_1(T) = p + T^{q-1}$ and $Q_{k+1}(T) = Q_k([p](T))$ if $k \geqslant 1$. Note that $\log_{\mathrm{LT}}(T) = T \cdot \prod_{k\geqslant 1} Q_k(T)/p$. Indeed, $\log_{\mathrm{LT}}(T) = \lim_{k\to\infty} p^{-k} \cdot [p^k](T)$ (Section 9.3 of [10]) and $[p^k](T) = Q_0(T) \cdots Q_k(T)$. Let $\exp_{\mathrm{LT}}(T)$ denote the inverse of $\log_{\mathrm{LT}}(T)$. We have $\exp_{\mathrm{LT}}(T) = \sum_{k=1}^\infty e_k T^k$ with $v_p(e_k) \geqslant -k/(q-1)$. For example, $\log_{\mathbf{G}_{\mathrm{m}}}(T) = \log(1+T)$ and $\exp_{\mathbf{G}_{\mathrm{m}}}(T) = \exp(T) - 1$.

Remark 1.1. Our special choice of $[p](T) = pT + T^q$ is the simplest. Since $[p](T)$ belongs to $\mathbf{Z}_p[T]$, the series $Q_k(T)$, $\log_{\mathrm{LT}}(T)$ and $\exp_{\mathrm{LT}}(T)$ all have coefficients in $\mathbf{Q}_p$. It also implies that $[\sigma(a)](T) = \sigma([a](T))$, since $[a](T) = \exp_{\mathrm{LT}}(a \cdot \log_{\mathrm{LT}}(T))$.

Lemma 1.2. *If $z \in \mathfrak{m}_{\mathbf{C}_p}$, then*

$$\frac{[1+a](z) - z}{a} = \log_{\mathrm{LT}}(z) \cdot \frac{dS}{dU}(z,0) + \mathrm{O}(a),$$

as $a \to 0$ in $\mathcal{O}_F$.

Proof. We are looking at the limit of $(S(z,[a](z)) - z)/a$ as $a \to 0$. If a is small enough, then $[a](z) = \exp_{\mathrm{LT}}(a \cdot \log_{\mathrm{LT}}(z)) = a \cdot \log_{\mathrm{LT}}(z) + \mathrm{O}(a^2)$, which implies the lemma. □

2. Rings of multivariable power series

We consider power series in the h variables $Y_0, \ldots, Y_{h-1}$. If $Y^m = Y_0^{m_0} \ldots Y_{h-1}^{m_{h-1}}$ is a monomial, then its weight is $w(m) = m_0 + pm_1 + \cdots + p^{h-1}m_{h-1}$. If I is a subinterval of

$[0;+\infty]$ and if $J = \{j_1, \dots, j_k\}$ is a subset of $\{0, \dots, h-1\}$, then (adapting Appendix A of [23] to our situation) we define $\mathcal{R}^I(\{Y_j\}_{j\in J})$ to be the ring of power series

$$f(Y_{j_1}, \dots, Y_{j_k}) = \sum_{m_1,\dots,m_k \in \mathbf{Z}} a_{m_1\dots m_k} Y_{j_1}^{m_1} \dots Y_{j_k}^{m_k},$$

such that $\mathrm{val}_p(a_m) + w(m)/r \to +\infty$ for all $r \in I$. In other words, $f(Y)$ is required to converge on the polyannulus $\{(Y_0, \dots, Y_{h-1})$ such that $|Y_0| = p^{-1/r}, \dots, |Y_{h-1}| = p^{-p^{h-1}/r}\}$ for all $r \in I$. We then define $W(f(Y), r) = \inf_{m\in\mathbf{Z}}(\mathrm{val}_p(a_m) + w(m)/r)$ and, if I is closed, $W(f(Y), I) = \inf_{r\in I} W(f(Y), r)$.

We let $\mathcal{R}^+(\{Y_j\}_{j\in J}) = \mathcal{R}^{[0;+\infty[}(\{Y_j\}_{j\in J})$ be the ring of holomorphic functions on the open unit polydisk corresponding to J. The Robba ring $\mathcal{R}(\{Y_j\}_{j\in J})$ is defined as $\mathcal{R}(\{Y_j\}_{j\in J}) = \cup_{r\geqslant 0}\mathcal{R}^{[r;+\infty[}(\{Y_j\}_{j\in J})$. In order to lighten the notation, we write $\mathcal{R}^I(Y)$, $\mathcal{R}^+(Y)$ and $\mathcal{R}(Y)$ instead of $\mathcal{R}^I(Y_0, \dots, Y_{h-1})$, $\mathcal{R}^+(Y_0, \dots, Y_{h-1})$ and $\mathcal{R}(Y_0, \dots, Y_{h-1})$.

The rings $\mathcal{R}^I(\{Y_j\}_{j\in J})$ are endowed with an F-linear action of Γ_F, given by the formula $a(Y_j) = [\sigma^j(a)](Y_j)$. There is also an F-linear Frobenius map:

$$\varphi_q : \mathcal{R}^I(\{Y_j\}_{j\in J}) \to \mathcal{R}^{I'}(\{Y_j\}_{j\in J}),$$

given by $Y_j \mapsto [p](Y_j)$, for appropriate I and I'.

On the ring $\mathcal{R}^I(Y)$, we can define in addition an absolute σ-semilinear Frobenius map φ by $Y_j \mapsto Y_{j+1}$ for $0 \leqslant j \leqslant h-2$ and $Y_{h-1} \mapsto [p](Y_0)$. This map φ has the property that $\varphi^h = \varphi_q$, and it also commutes with Γ_F.

Let $t_i = \log_{\mathrm{LT}}(Y_i)$. Since $a(Y_i) = [\sigma^i(a)](Y_i)$ if $a \in \Gamma_F$, we have $a(t_i) = \sigma^i(a) \cdot t_i$ so that $g(t_0 \dots t_{h-1}) = \mathrm{N}_{F/\mathbf{Q}_p}(\chi_{\mathrm{LT}}(g)) \cdot t_0 \dots t_{h-1} = \chi_{\mathrm{cyc}}(g) \cdot t_0 \dots t_{h-1}$ if $g \in G_F$ as well as $\varphi(t_0 \dots t_{h-1}) = p \cdot t_0 \dots t_{h-1}$. The element $t_0 \dots t_{h-1}$ therefore behaves like a $\mathbf{Q}_p$-multiple of the "usual" t of p-adic Hodge theory (see Proposition 3.4 for a more precise statement).

The following two propositions are variations on the "Weierstrass division theorem."

Proposition 2.1. *Let $I = [0; s]$ or $[0; s[$ and let $P(T) \in \mathcal{O}_F[T]$ be a monic polynomial of degree d whose non-leading coefficients are all divisible by p. If $f \in \mathcal{R}^I(\{Y_j\}_{j\in J})$, then there exists $g \in \mathcal{R}^I(\{Y_j\}_{j\in J})$ and $f_0, \dots, f_{d-1} \in \mathcal{R}^I(\{Y_j\}_{j\in J\setminus\{i\}})$ such that*

$$f = f_0 + f_1 Y_i + \dots + f_{d-1} Y_i^{d-1} + g \cdot P(Y_i).$$

Proof. If $I = [0; s]$ is closed, then this is a straightforward consequence of the Weierstrass division theorem. Since g and the f_i's are uniquely determined, the result extends to the case when $I = [0; s[$. □

Proposition 2.2. *Let $I = [s; s]$ and let $P(T) \in \mathcal{O}_F[T]$ be a monic polynomial of degree d, all of whose roots are of valuation $-1/s$. If $f \in \mathcal{R}^I(\{Y_j\}_{j\in J})$, then there exists $g \in \mathcal{R}^I(\{Y_j\}_{j\in J})$ and $f_0, \dots, f_{d-1} \in \mathcal{R}^I(\{Y_j\}_{j\in J\setminus\{i\}})$ such that*

$$f = f_0 + f_1 Y_i + \dots + f_{d-1} Y_i^{d-1} + g \cdot P(Y_i).$$

Proof. The polynomial $Q(T) = P(1/T)T^d/P(0)$ is monic and all its roots are of valuation $1/s$. Write $f = f^+ + f^-$ where f^+ contains positive powers of Y_i and f^- contains negative powers of Y_i. One may Weierstrass divide f^+ by $P(Y_i)$ and f^- by $Q(1/Y_i)$, which implies the proposition. □

Lemma 2.3. *If I is a closed interval, then the action of Γ_F on $\mathcal{R}^I(Y)$ is locally $\mathbf{Q}_p$-analytic, and we have*

$$[1+a](f(Y)) = f(Y) + \sum_{j=0}^{h-1} \sigma^j(a) \cdot \log_{\mathrm{LT}}(Y_j) \cdot \frac{dS}{dU}(Y_j,0) \cdot \frac{df}{dY_j}(Y) + \mathrm{O}(a^2).$$

Proof. The above formula follows from the fact that $[1+a](Y_j) = Y_j \oplus [a](Y_j) = Y_j \oplus (\sigma^j(a)\cdot \log_{\mathrm{LT}}(Y_j) + \mathrm{O}(a^2))$. □

Proposition 2.4. *Let $\rho = (\rho_1,\ldots,\rho_{h-1})$ and let $\mathcal{R}^\rho_{F_k}(T_1,\ldots,T_{h-1})$ denote the ring of Laurent series converging for $|T_i| = \rho_i$, with coefficients in F_k. If the $z_i \in \mathfrak{m}_{\widehat{F}_\infty}$ are such that $\log_{\mathrm{LT}}(z_i) \neq 0$, $|z_i| = \rho_i$ and $g(z_i) = [\sigma^i(g)](z_i)$ for $g \in \mathcal{O}_F^\times$, then the map $\mathcal{R}^\rho_{F_k}(T_1,\ldots,T_{h-1}) \to \mathbf{C}_p$ given by evaluating at $(z_1,\ldots,z_{h-1})$ is injective.*

Proof. Suppose that $f(z_1,\ldots,z_{h-1}) = 0$ for some $f \in \mathcal{R}^\rho_{F_k}(T_1,\ldots,T_{h-1})$. If $g \in \Gamma_{F_k}$, then $f(g(z_1),\ldots,g(z_{h-1})) = 0$. If $g = 1+a$ with a small, then Lemma 1.2 provides us with $h-1$ elements $y_1,\ldots,y_{h-1}$ of $\widehat{F}_\infty$ such that $g(z_i) = z_i + \sigma^i(a)\cdot y_i + \mathrm{O}(a^2)$. Since $y_i = \log_{\mathrm{LT}}(z_i)\cdot dS/dU(z_i,0)$ and dS/dU is a unit and $\log_{\mathrm{LT}}(z_i) \neq 0$, the elements $y_1,\ldots,y_{h-1}$ are all non-zero.

If $f \neq 0$ and m is the smallest index for which f has a non-zero partial derivative of order m at $(z_1,\ldots,z_{h-1})$ and if we expand $f(g(z_1),\ldots,g(z_{h-1}))$ around $(z_1,\ldots,z_{h-1})$ (which generalizes Lemma 2.3), then we get

$$\sum_{j_1+\cdots+j_{h-1}=m} (\sigma^1(a)y_1)^{j_1}\cdots(\sigma^{h-1}(a)y_{h-1})^{j_{h-1}} \times \frac{d^m f}{dT_1^{j_1}\cdots dT_{h-1}^{j_{h-1}}}(z_1,\ldots,z_{h-1}) + \mathrm{O}(a^{m+1}).$$

Since $f(g(z_1),\ldots,g(z_{h-1})) = 0$, the above linear combination is a homogeneous polynomial, of degree m in $h-1$ variables and coefficients in $\widehat{F}_\infty$, that is identically zero on $(\sigma^1(a),\ldots,\sigma^{h-1}(a))$. The shortest non-zero polynomial that is identically zero on $(\sigma^1(a),\ldots,\sigma^{h-1}(a))$ can be taken to have coefficients in F and Artin's theorem on the algebraic independence of characters implies that it is equal to zero. Since all the y_i's are non-zero, all the partial derivatives of order m of f are zero, so that finally $f=0$. □

3. Embeddings in $\mathbf{B}_{\mathbf{dR}}$

We now explain how to embed the rings of power series of the previous section in the usual rings of p-adic periods. Let $\widetilde{\mathbf{B}}^I$ be the ring defined in Section 2.1 of [2]. This ring is complete with respect to the valuation $V(\cdot,I)$ (an equivalent valuation is denoted by $V_I(\cdot)$ in Section 2.1 of ibid.). Recall that if $x = \sum_{k\geqslant 0} p^k[x_k] \in \widetilde{\mathbf{A}}^+$, then $V(x,r) = \inf_k(\mathrm{val}_{\mathbf{E}}(x_k) + krp/(p-1))$. Set $r_F = p^{h-1}\cdot q/(q-1)\cdot(p-1)/p$ (for example, $r_{\mathbf{Q}_p} = 1$ and if $h>1$, then $r_F < p^{h-1}$).

Proposition 3.1. *If $r \geqslant r_F$ and $m \in \mathbf{Z}$, then $V(\varphi^j(u)^m, r) = m\cdot p^j\cdot q/(q-1)$ for $0 \leqslant j \leqslant h-1$.*

Proof. Recall that $u = \{\pi\}$ where $\pi = (\pi_0, \pi_1, \ldots)$ with $\mathrm{val}_p(\pi_n) = 1/q^{n-1}(q-1)$ for $n \geqslant 1$, so that $\mathrm{val}_{\mathbf{E}}(\pi) = q/(q-1)$. We have $\varphi^j(u) = [\pi^{p^j}] + \sum_{k\geqslant 1} p^k [u_{k,j}]$ where $\mathrm{val}_{\mathbf{E}}(u_{k,j}) > 0$, so that if $r \geqslant r_F$, then $\varphi^j(u)/[\pi^{p^j}]$ is a unit of $\widetilde{\mathbf{A}}^{\dagger,r}$ and the proposition follows. □

Note that a better estimate on the $\mathrm{val}_{\mathbf{E}}(u_{k,j})$ would allow us to take a smaller value for r_F. Let $s_n = p^{n-h}(q-1)$ and let $r_n = p^{n-1}(p-1)$ (so that $s_n \cdot q/(q-1) = r_n \cdot p/(p-1)$).

Proposition 3.2. *If $n \geqslant h$, and if $f(Y) \in \mathcal{R}^{[s_n;s_n]}(Y)$, then $f(u, \ldots, \varphi^{h-1}(u))$ converges in $\widetilde{\mathbf{B}}^{[r_n;r_n]}$.*

Proof. If $f(Y) = \sum_{m\in\mathbf{Z}^h} a_m Y^m \in \mathcal{R}^{[s_n;s_n]}(Y)$, then $\mathrm{val}_p(a_m) + w(m)/(p^{n-h}(q-1)) \to +\infty$. If $n \geqslant h$, then $r_n > r_F$ so that $V(\varphi^j(u)^{m_j}, r) = m_j \cdot p^j \cdot q/(q-1)$ for $0 \leqslant j \leqslant h-1$ by Proposition 3.1, and then

$$V(a_{m_0,\ldots,m_{h-1}} u^{m_0} \cdots \varphi^{h-1}(u)^{m_{h-1}}, r_n) \to +\infty.$$

The series $f(u, \ldots, \varphi^{h-1}(u))$ therefore converges in $\widetilde{\mathbf{B}}^{[r_n;r_n]}$. □

Corollary 3.3. *If $n \geqslant h$, and if $f(Y) \in \mathcal{R}^{[0;s_n]}(Y)$, then $f(u, \ldots, \varphi^{h-1}(u))$ converges in $\widetilde{\mathbf{B}}^{[0;r_n]}$. If $f(Y) \in \mathcal{R}^+(Y)$, then $f(u, \ldots, \varphi^{h-1}(u))$ converges in $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$.*

Proof. If $f \in \mathcal{R}^{[0;s_n]}(Y)$, then each term of the series $f(u, \ldots, \varphi^{h-1}(u))$ belongs to $\widetilde{\mathbf{B}}^+$ so that it converges in $\widetilde{\mathbf{B}}^{[0;r_n]}$ by the maximum modulus principle (corollary 2.20 of [2]). The second assertion follows by passing to the limit. □

The image of $\log_{\mathrm{LT}}(Y_0) \cdots \log_{\mathrm{LT}}(Y_{h-1})$ in $\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \subset \mathbf{B}^+_{\mathrm{dR}}$ is $a \cdot t$ with $a \in \mathbf{Q}_p$, as we have seen above. We henceforth denote by t the element of $\mathcal{R}^+(Y)$ whose image in $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ is t, that is $t = \log_{\mathrm{LT}}(Y_0) \cdots \log_{\mathrm{LT}}(Y_{h-1})/a$. In the following proposition, we determine the valuation of a (this is not used in the rest of this article).

Proposition 3.4. *In the ring $\mathbf{B}^+_{\mathrm{dR}}$, the product $\log_{\mathrm{LT}}(u) \cdots \log_{\mathrm{LT}}(\varphi^{h-1}(u))$ belongs to $p^{h-1} \cdot \mathbf{Z}_p^\times \cdot t$, where t is the usual t of p-adic Hodge theory.*

Proof. We have seen that $\log_{\mathrm{LT}}(u) \cdots \log_{\mathrm{LT}}(\varphi^{h-1}(u)) = a \cdot t$ with $a \in \mathbf{Q}_p$, and we now compute $\mathrm{val}_p(a)$. We have $\log_{\mathrm{LT}}(u) = u \cdot \prod_{k\geqslant 1} Q_k(u)/p$ and likewise, if $\pi = [\varepsilon] - 1$, then $t = \pi \cdot \prod_{k\geqslant 1} Q_k^{\mathrm{cyc}}(\pi)/p$. This implies that $\theta(t/\log_{\mathrm{LT}}(u)) = \theta(\pi/u)$. Since both $\pi/\varphi^{-1}(\pi)$ and $u/\varphi_q^{-1}(u)$ are generators of $\ker(\theta)$ in $\widetilde{\mathbf{A}}^+$, we have $\mathrm{val}_p(\theta(t/\log_{\mathrm{LT}}(u))) = 1/(p-1) - 1/(q-1)$. On the other hand, $\mathrm{val}_p(\theta \circ \varphi^j(u)) = \mathrm{val}_p(\lim_{n\to\infty} [p^n](\pi_n^{p^j})) = 1 + p^j/(q-1)$ if $1 \leqslant j \leqslant h-1$, so that $\mathrm{val}_p(\theta(\log_{\mathrm{LT}}(\varphi^j(u)))) = 1 + p^j/(q-1)$. This implies that $\mathrm{val}_p(a) = \mathrm{val}_p(\theta(a)) = h-1$, and hence the proposition. □

Definition 3.5. Let $\iota_n : \mathcal{R}^{[s_n;s_n]}(Y) \to \mathbf{B}^+_{\mathrm{dR}}$ be the compositum of the map defined above, with the map $\varphi^{-n} : \widetilde{\mathbf{B}}^{[r_n;r_n]} \to \widetilde{\mathbf{B}}^{[r_0;r_0]}$ and the map $\widetilde{\mathbf{B}}^{[r_0;r_0]} \subset \mathbf{B}^+_{\mathrm{dR}}$ defined in Section 2.2 of [2].

It follows from the definition as well as the formulas for φ and the action of Γ_F on $\mathcal{R}^I(Y)$ that $\iota_{n+1}(\varphi(f)) = \iota_n(f)$ when applicable, and that $g(\iota_n(f)) = \iota_n(g(f))$ if $g \in G_F$. Since $\iota_n(t) = p^{-n}t$, we can extend ι_n to $\iota_n : \mathcal{R}^{[s_n;s_n]}(Y)[1/t] \to \mathbf{B}_{\mathrm{dR}}$.

Theorem 3.6. *If $n \geqslant h$, if $f \in \mathcal{R}^{[s_n;s_n]}(Y)$, and if $n = hk + i$ with $0 \leqslant i \leqslant h-1$, then we have $\iota_n(f) \in \mathrm{Fil}^1 \mathbf{B}_{\mathrm{dR}}^+$ if and only if $f \in Q_k(Y_i) \cdot \mathcal{R}^{[s_n;s_n]}(Y)$.*

Proof. Recall that $u = \{(\pi_0, \pi_1, \ldots)\} \in \widetilde{\mathbf{A}}^+$. If $m \geqslant 1$ and $u_m = \theta(\varphi^{-m}(u)) \in \widehat{F}_\infty$, then $g(u_m) = [\sigma^{-m}(g)](u_m)$. Note that if $m = h\ell$, then $u_m = \theta(\varphi_q^{-\ell}(u)) = \pi_\ell$. The theorem is equivalent to the assertion that $f^{\sigma^{-n}}(u_n, \ldots, u_{n-h+1}) = 0$ in $\mathbf{C}_p$ if and only if $f \in Q_k(Y_i) \cdot \mathcal{R}^{[s_n;s_n]}(Y)$. We have $u_{n-i} = \pi_k$ so that if f belongs to $Q_k(Y_i) \cdot \mathcal{R}^{[s_n;s_n]}(Y)$, then $f^{\sigma^{-n}}(u_n, \ldots, u_{n-h+1}) = 0$.

Since $Q_k(T)$ is a monic polynomial of degree $d = q^{k-1}(q-1)$, whose non-leading coefficients are divisible by p, we can use Proposition 2.2 to write $f^{\sigma^{-n}} = f_0 + Y_i f_1 + \cdots + Y_i^{d-1} f_{d-1} + Q_k(Y_i) r$ with f_i a power series in the Y_j's with $j \neq i$. Proposition 2.4 applied to $f_0 + \pi_k f_1 + \cdots + \pi_k^{d-1} f_{d-1}$, with the T_j's a suitable permutation of the Y_j's, shows that $f_0 + \pi_k f_1 + \cdots + \pi_k^{d-1} f_{d-1} = 0$. Therefore, $f = Q_k(Y_i) r^{\sigma^n}$, which proves the theorem. □

Corollary 3.7. *If $n \geqslant h$, then the map $\iota_n : \mathcal{R}^{[s_n;s_n]}(Y) \to \mathbf{B}_{\mathrm{dR}}^+$ is injective. If $n \in \mathbf{Z}$, then the map $\iota_n : \mathcal{R}^+(Y) \to \mathbf{B}_{\mathrm{dR}}^+$ is injective.*

Proof. The first assertion follows from Theorem 3.6. The second follows from that, and from the fact that $\iota_{n+1}(\varphi(f)) = \iota_n(f)$ for the other n. □

Corollary 3.8. *If $I \subset [s_h; +\infty[$, and if $f(Y) \in \mathcal{R}^I(Y)[1/t]$, then $f(Y) \in \mathcal{R}^I(Y)$ if and only if $\iota_n(f) \in \mathbf{B}_{\mathrm{dR}}^+$ for all n such that $s_n \in I$.*

4. (φ_q, Γ_F)-modules in one variable

Before constructing (φ_q, Γ_F)-modules over $\mathcal{R}(Y)$, we review Kisin and Ren's construction of (φ_q, Γ_F)-modules in one variable and explain why we need rings in several variables.

Let Y_0 be the variable of Section 2, and let $\mathcal{E}(Y_0)$ be Fontaine's field of [12] with coefficients in F, that is $\mathcal{E}(Y_0) = \mathcal{O}_{\mathcal{E}}(Y_0)[1/p]$ where $\mathcal{O}_{\mathcal{E}}(Y_0)$ is the p-adic completion of $\mathcal{O}_F[\![Y_0]\!][1/Y_0]$. We let $\mathcal{E}^\dagger(Y_0)$ and $\mathcal{R}(Y_0)$ denote the corresponding overconvergent and Robba rings. If I is a subinterval of $[0; +\infty]$, then we denote as above by $\mathcal{R}^I(Y_0)$ the set of power series $f(Y_0) = \sum_{m \in \mathbf{Z}} a_m Y_0^m$ that belong to $\mathcal{R}^I(Y_0, \ldots, Y_{h-1})$ via the natural inclusion.

If K/F is a finite extension, then by the theory of the field of norms (see [14, 22]), there corresponds to it a finite extension $\mathcal{E}_K(Y_0)$ of $\mathcal{E}(Y_0)$, of degree $[K_\infty : F_\infty]$. A (φ_q, Γ_K)-module over $\mathcal{E}_K(Y_0)$ is a finite dimensional $\mathcal{E}_K(Y_0)$-vector space D, along with a semilinear φ_q and a compatible action of Γ_K. We say that D is étale if $\mathrm{D} = \mathcal{E}_K(Y_0) \otimes_{\mathcal{O}_{\mathcal{E}_K}(Y_0)} \mathrm{D}_0$ where D_0 is a (φ_q, Γ_K)-module over $\mathcal{O}_{\mathcal{E}_K}(Y_0)$. By specializing the constructions of [12], Kisin and Ren prove the following theorem in their paper (Theorem 1.6 of [18]).

Theorem 4.1. *The functors*

$$V \mapsto (\widehat{\mathcal{E}}(Y_0)^{\mathrm{unr}} \otimes_F V)^{H_K} \text{ and } \mathrm{D} \mapsto (\widehat{\mathcal{E}}(Y_0)^{\mathrm{unr}} \otimes_{\mathcal{E}_K(Y_0)} \mathrm{D})^{\varphi_q = 1}$$

give rise to mutually inverse equivalences of categories between the category of F-linear representations of G_K and the category of étale (φ_q, Γ_K)-modules over $\mathcal{E}_K(Y_0)$.

We say that an F-linear representation of G_K is F-analytic if it is Hodge-Tate with weights 0 (i.e., $\mathbf{C}_p$-admissible) at all embeddings $\tau \neq \mathrm{Id}$. Kisin and Ren then go on to show that if $K \subset F_\infty$, and if V is a crystalline representation of G_K, that is F-analytic, then the (φ_q, Γ_K)-module attached to V is overconvergent (see Section 3.3 of ibid.).

Assume from now on that $K \subset F_\infty$, so that $\mathcal{E}_K(Y_0) = \mathcal{E}(Y_0)$. If D is a (φ_q, Γ_K)-module over $\mathcal{R}(Y_0)$, and if $g \in \Gamma_K$ is close enough to 1, then by standard arguments (see Section 4.1 of [2] or Section 2.1 of [18]), the series $\log(g) = \log(1 + (g-1))$ gives rise to a differential operator $\nabla_g : \mathrm{D} \to \mathrm{D}$. The map $\mathrm{Lie}\,\Gamma_F \to \mathrm{End}(\mathrm{D})$ arising from $v \mapsto \nabla_{\exp(v)}$ is $\mathbf{Q}_p$-linear, and we say that D is F-analytic if this map is F-linear (see Section 2.1 of [18] and Section 1.3 of [15]). This is equivalent to the requirement that $\nabla_j = 0$ on D for $1 \leqslant j \leqslant h-1$, where ∇_j is the partial derivative in the direction σ^j.

Theorem 4.2. *If V is an overconvergent F-linear representation of G_K, and if $\mathrm{D}(V) = \mathcal{R}(Y_0) \otimes_{\mathcal{E}^\dagger(Y_0)} \mathrm{D}^\dagger(V)$, then $\mathrm{D}(V)$ is F-analytic if and only if V is F-analytic.*

Proof. Choose $1 \leqslant j \leqslant h-1$, and take $n \gg 0$ such that $n = -j \bmod h$. By Proposition 3.2, we have a map $\theta \circ \varphi^{-n} : \mathcal{R}^{[s_n;s_n]}(Y_0) \to \mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{C}_p$, giving rise to an isomorphism

$$\mathbf{C}_p \otimes^{\theta\circ\varphi^{-n}}_{\mathcal{R}^{[s_n;s_n]}(Y_0)} \mathrm{D}^{[s_n;s_n]}(V) \to \mathbf{C}_p \otimes_F^{\sigma^j} V.$$

We first prove that if $\mathrm{D}(V)$ is F-analytic, then V is $\mathbf{C}_p$-admissible at the embedding σ^j. Let $\widehat{F}_\infty^{(j)}$ denote the field of locally σ^j-analytic vectors of $\widehat{F}_\infty$ for the action of Γ_K. Note that $\theta \circ \varphi^{-n}(\mathcal{R}^{[s_n;s_n]}(Y_0)) \subset \widehat{F}_\infty^{(j)}$. Let $\mathrm{D}_{\mathrm{Sen}}^{(j)}(V)$ be the $\widehat{F}_\infty^{(j)}$-vector space

$$\mathrm{D}_{\mathrm{Sen}}^{(j)}(V) = \widehat{F}_\infty^{(j)} \otimes_{\theta\circ\varphi^{-n}(\mathcal{R}^{[s_n;s_n]}(Y_0))} \theta \circ \varphi^{-n}(\mathrm{D}^{[s_n;s_n]}(V)).$$

It is of dimension d, its image in $(\mathbf{C}_p \otimes_F^{\sigma^j} V)^{H_F}$ generates $\mathbf{C}_p \otimes_F^{\sigma^j} V$, and its elements are all locally σ^j-analytic vectors of $(\mathbf{C}_p \otimes_F^{\sigma^j} V)^{H_F}$ because $\mathrm{D}(V)$ is F-analytic and $\varphi^{-n} \circ \nabla_j = \nabla_0 \circ \varphi^{-n}$. If $y \in \mathrm{D}_{\mathrm{Sen}}^{(j)}(V)$, then $(g(y)-y)/(\sigma^j \circ \chi_{\mathrm{LT}}(g) - 1)$ has a limit as $g \to 1$, and we call $\nabla_j(y)$ this limit. We then have $g(y) = \exp(\log_p(\sigma^j \circ \chi_{\mathrm{LT}}(g)) \cdot \nabla_j)(y)$ if $g \in \Gamma_K$ is close to 1.

Recall that there exists $a_j \in \mathbf{C}_p$ such that $\log_p(\sigma^j \circ \chi_{\mathrm{LT}}(g)) = g(a_j) - a_j$. For example, one can take $a_j = \log_p(\theta \circ \iota_0(t_j))$. The element a_j then belongs to $\widehat{F}_\infty^{(j)}$ for obvious reasons and satisfies $\nabla_j(a_j) = 1$. Take $y \in \mathrm{D}_{\mathrm{Sen}}^{(j)}(V)$, and choose $a_{j,0} \in F_\infty$ such that $|a_j - a_{j,0}|_p$ is small enough. The series

$$C(y) = \sum_{k\geqslant 0} (-1)^k \frac{(a_j - a_{j,0})^k}{k!} \nabla_j^k(y)$$

then converges for the topology of $\mathrm{D}_{\mathrm{Sen}}^{(j)}(V)$ (the technical details concerning convergence in such spaces of locally analytic vectors can be found in [6]) and a short computation shows that $\nabla_j(C(y)) = 0$, so that $C(y) \in (\mathbf{C}_p \otimes_F^{\sigma^j} V)^{G_{F_n}}$ for some $n = n(y) \gg 0$. In addition, $n(y) = n(\nabla_j^k(y))$ for $k \geqslant 0$, the series for $C(\nabla_j^k(y))$ also converges for the topology of $\mathrm{D}_{\mathrm{Sen}}^{(j)}(V)$, and $y = \sum_{k\geqslant 0} (a_j - a_{j,0})^k/k! \cdot C(\nabla_j^k(y))$.

If $y_1, \ldots, y_d$ is a basis of $\mathrm{D}_{\mathrm{Sen}}^{(j)}(V)$, and if $n \geqslant \max n(y_i)$, then the above computations show that the elements y_i belong to $\widehat{F}_\infty^{(j)} \otimes_{F_n} (\mathbf{C}_p \otimes_F^{\sigma^j} V)^{G_{F_n}}$, so that

$(\mathbf{C}_p \otimes_F^{\sigma^j} V)^{G_{F_n}}$ generates $(\mathbf{C}_p \otimes_F^{\sigma^j} V)^{H_F}$. This implies that V is $\mathbf{C}_p$-admissible at the embedding σ^j. This is true for all $1 \leqslant j \leqslant h-1$, and therefore V is F-analytic.

We now prove that if V is $\mathbf{C}_p$-admissible at the embedding σ^j, then $\nabla_j = 0$ on $\mathrm{D}(V)$. Choose $n = hm - j$ with $m \gg 0$. Since $j \neq 0 \bmod h$, the map $\theta \circ \varphi^{-n} : \mathcal{R}^{[s_n;s_n]}(Y_0) \to \mathbf{C}_p$ is injective by Theorem 3.6. This implies that the map

$$\mathrm{D}^{[s_n;s_n]}(V) \to \mathbf{C}_p \otimes^{\theta\circ\varphi^{-n}}_{\mathcal{R}^{[s_n;s_n]}(Y_0)} \mathrm{D}^{[s_n;s_n]}(V)$$

is injective, and hence the map $\mathrm{D}^{[s_n;s_n]}(V) \to \mathbf{C}_p \otimes_F^{\sigma^j} V$ is also injective. Therefore, we have an injection $\mathrm{D}^{[s_n;s_n]}(V) \to ((\mathbf{C}_p \otimes_F^{\sigma^j} V)^{H_F})^{\mathrm{an}}$ where $((\mathbf{C}_p \otimes_F^{\sigma^j} V)^{H_F})^{\mathrm{an}}$ denotes the set of locally $\mathbf{Q}_p$-analytic vectors of $(\mathbf{C}_p \otimes_F^{\sigma^j} V)^{H_F}$. If V is $\mathbf{C}_p$-admissible at the embedding σ^j, then $((\mathbf{C}_p \otimes_F^{\sigma^j} V)^{H_F})^{\mathrm{an}} = (\widehat{F}_\infty^{\mathrm{an}})^d$. One of the main results of [6] is that $\nabla_0 = 0$ on $\widehat{F}_\infty^{\mathrm{an}}$ (it is shown in [6] that, in a suitable sense, $\widehat{F}_\infty^{\mathrm{an}}$ is generated by F_∞ and the elements $a_1, \dots, a_{h-1}$). This implies that $\nabla_j = 0$ on $\mathrm{D}^{[s_n;s_n]}(V)$, since $\varphi^{-n} \circ \nabla_j = \nabla_0 \circ \varphi^{-n}$. □

Note that an analogous argument for the proof of the implication "$\mathrm{D}(V)$ is F-analytic implies V is F-analytic" was given by Bingyong Xie for those V that are trivial on H_F.

Corollary 4.3. *If V is an absolutely irreducible F-linear overconvergent representation of G_K, then there exists a character δ of Γ_K such that $V \otimes \delta$ is F-analytic.*

Proof. We give a sketch of the proof. Choose some $g \in \Gamma_K$ such that $\log_p(\chi_{\mathrm{LT}}(g)) \neq 0$, and let $\nabla = \log(g)/\log_p(\chi_{\mathrm{LT}}(g))$. Choose $r > 0$ large enough and $s \geqslant qr$. If $a \in \mathcal{O}_F$, and if $\mathrm{val}_p(a) \geqslant n$ for $n = n(r,s)$ large enough, then the series $\exp(a \cdot \nabla)$ converges to an operator on $\mathrm{D}^{[r;s]}(V)$. This way, we can define a twisted action of Γ_{K_n} on $\mathrm{D}^{[r;s]}(V)$, by the formula $h \star x = \exp(\log_p(\chi_{\mathrm{LT}}(h)) \cdot \nabla)(x)$. This action is now F-analytic by construction.

Since $s \geqslant qr$, the modules $\mathrm{D}^{[q^m r;q^m s]}(V)$ for $m \geqslant 0$ are glued together by φ_q and this way, we get a new action of Γ_{K_n} on $\mathrm{D}(V)$. Since φ_q is unchanged, this new $(\varphi_q, \Gamma_{K_n})$-module is étale, and therefore corresponds to a representation W of G_{K_n}. This representation W is F-analytic by Theorem 4.2, and its restriction to H_F is isomorphic to V.

The space $\mathrm{Hom}(V, \mathrm{ind}_{G_{K_n}}^{G_K} W)^{H_F}$ is non-empty, and is a finite dimensional representation of Γ_K. Since Γ_K is abelian, we find (possibly extending scalars) a character δ of Γ_K and a non-zero $f \in \mathrm{Hom}(V, \mathrm{ind}_{G_{K_n}}^{G_K} W)^{H_F}$ such that $h(f) = \delta(h) \cdot f$ if $h \in G_K$. This f gives rise to a non-zero G_K-equivariant map $V \otimes \delta \to \mathrm{ind}_{G_{K_n}}^{G_K} W$. Since $\mathrm{ind}_{G_{K_n}}^{G_K} W$ is F-analytic and V is absolutely irreducible, the corollary follows. □

Corollary 4.3 (as well as Theorem 0.6 of [15]) suggests that if we want to attach overconvergent (φ_q, Γ_K)-modules to all F-linear representations of G_K, then we need to go beyond the objects in only one variable. We finish with a conjecture that seems reasonable enough, since it holds for crystalline representations by the work of Kisin and Ren (see also Theorem 0.3 of [15]).

Conjecture 4.4. *If V is F-analytic, then it is overconvergent.*

5. Construction of $\mathcal{R}^+(Y)$-modules

We now explain how to construct some $\mathcal{R}^+(Y)$-modules $\mathrm{M}^+(D)$ that are attached to some filtered φ_q-modules D. Let D be a finite dimensional F-vector space, endowed with an F-linear Frobenius map $\varphi_q : D \to D$, and an action of G_F on D that factors through Γ_F and commutes with φ_q.

For each $0 \leqslant j \leqslant h-1$, let $\mathrm{Fil}_j^\bullet$ be a filtration on $F \otimes_F^{\sigma^j} D \simeq D$ that is stable under Γ_F. If $n \in \mathbf{Z}$, let $\mathbf{B}_{\mathrm{dR}} \otimes_F^{\sigma^n} D$ denote the tensor product of $\mathbf{B}_{\mathrm{dR}}$ and D above F, where F maps to $\mathbf{B}_{\mathrm{dR}}$ via σ^n. We then have $b \otimes a \cdot d = \sigma^n(a) \cdot b \otimes d$. Note that $\mathbf{B}_{\mathrm{dR}} \otimes_F^{\sigma^n} D$ only depends on n mod h. Define $W_{\mathrm{dR}}^{+,j}(D) = \mathrm{Fil}_j^0(\mathbf{B}_{\mathrm{dR}} \otimes_F^{\sigma^j} D)$ so that $W_{\mathrm{dR}}^{+,j}$ is a G_F-stable $\mathbf{B}_{\mathrm{dR}}^+$-lattice of $\mathbf{B}_{\mathrm{dR}} \otimes_F^{\sigma^j} D$.

Example 5.1. If V is an F-linear crystalline representation of G_F of dimension d, then $\mathrm{D}_{\mathrm{cris}}(V)$ is a free $F \otimes_{\mathbf{Q}_p} F$-module of rank d and we have

$$\mathrm{D}_{\mathrm{cris}}(V) = D \oplus \varphi(D) \oplus \cdots \oplus \varphi^{h-1}(D),$$

according to the decomposition of $F \otimes_{\mathbf{Q}_p} F$ as $\prod_{\sigma^i : F \to F} F$. Each $\varphi^j(D)$ comes with the filtration induced from $\mathrm{D}_{\mathrm{cris}}(V)$, and we set $\mathrm{Fil}_j^k D = \varphi^{-j}(\mathrm{Fil}^k \mathrm{D}_{\mathrm{cris}}(V) \cap \varphi^j(D))$.

We now briefly recall some definitions from [21]. The ring $\mathcal{R}^+(Y)$ is a Fréchet–Stein algebra; indeed, its topology is defined by the valuations $\{W(\cdot, [0; s_n])\}_{n \in S}$, where S is any unbounded set of integers, and the ring $\mathcal{R}^{[0;s_n]}(Y)$ is noetherian and flat over $\mathcal{R}^{[0;s_m]}(Y)$ if $m \geqslant n \in S$. Recall that a coherent sheaf is then a family $\{M^{[0;s_n]}\}_{n \in S}$ of finitely generated $\mathcal{R}^{[0;s_n]}(Y)$-modules, such that $\mathcal{R}^{[0;s_n]}(Y) \otimes_{\mathcal{R}^{[0;s_m]}(Y)} M^{[0;s_m]} = M^{[0;s_n]}$ for all $m \geqslant n \in S$. A $\mathcal{R}^+(Y)$-module M is said to be coadmissible if M is the set of global sections of a coherent sheaf $\{M^{[0;s_n]}\}_{n \in S}$. We say that M is a reflexive coadmissible $\mathcal{R}^+(Y)$-module if each $M^{[0;s_n]}$ is a reflexive $\mathcal{R}^{[0;s_n]}(Y)$-module. By Lemma 8.4 of [21], this is the same as requiring that M itself be a reflexive $\mathcal{R}^+(Y)$-module.

Let $\lambda_j = \log_{\mathrm{LT}}(Y_j)/Y_j$ and $\lambda = \lambda_0 \cdots \lambda_{h-1}$, so that for any $n \in \mathbf{Z}$, t is a $\mathbf{Q}_p$-multiple of $\iota_n(\lambda \cdot Y_0 \cdots Y_{h-1})$. Let $f_j = \lambda/\lambda_j$, so that if $k = j$ mod h, then $\iota_k(f_j)$ is a unit of $\mathbf{B}_{\mathrm{dR}}^+$.

If $y = \sum_i y_i \otimes d_i \in \mathcal{R}^+(Y)[1/\lambda] \otimes_F D$, let $\iota_k(y) = \sum_i \iota_k(y_i) \otimes d_i \in \mathbf{B}_{\mathrm{dR}} \otimes_F^{\sigma^{-k}} D$.

Definition 5.2. Let $\mathrm{M}^+(D)$ be the set of $y \in \mathcal{R}^+(Y)[1/\lambda] \otimes_F D$ that satisfy $\iota_k(y) \in W_{\mathrm{dR}}^{+,-k}(D)$ for all $k \geqslant h$.

Theorem 5.3. *If D is a φ_q-module with an action of Γ_F and h filtrations, then*

1. $\mathrm{M}^+(D)$ *is a reflexive coadmissible $\mathcal{R}^+(Y)$-module;*
2. *the $\mathcal{R}^+(Y)[1/f_j]$-module* $\mathrm{M}^+(D)[1/f_j]$ *is free of rank d for $0 \leqslant j \leqslant h-1$;*
3. $\mathrm{M}^+(D) = \cap_{j=0}^{h-1} \mathrm{M}^+(D)[1/f_j]$.

In the remainder of this section, we prove Theorem 5.3. We now establish some preliminary results. Let $S = \{hm + (h-1) \text{ where } m \geqslant 1\}$, and take $n \in S$. Recall that on the ring $\mathcal{R}^{[0;s_n]}(Y)$, the map ι_k is defined for $h \leqslant k \leqslant n$. Let

$$\mathrm{M}(D)^{[0;s_n]} = \{y \in \mathcal{R}^{[0;s_n]}(Y)[1/\lambda] \otimes_F D,\ \iota_k(y) \in W_{\mathrm{dR}}^{+,-k}(D) \text{ for all } h \leqslant k \leqslant n\}.$$

For $0 \leqslant j \leqslant h-1$, recall that $\mathcal{R}^I(Y_j)$ is a ring of power series in one variable. Let

$$N_j^{[0;s_n]} = \{y \in \mathcal{R}^{[0;s_n]}(Y_j)[1/\lambda_j] \otimes_F D,\ \iota_{kh+j}(y) \in W_{\mathrm{dR}}^{+,-j}(D) \text{ for all } 1 \leqslant k \leqslant m\},$$
$$N_j^+ = \{y \in \mathcal{R}^+(Y_j)[1/\lambda_j] \otimes_F D,\ \iota_{kh+j}(y) \in W_{\mathrm{dR}}^{+,-j}(D) \text{ for all } k \geqslant 1\}.$$

Since $\mathcal{R}^+(Y_j) = \varphi^j(\mathcal{R}^+(Y_0))$ if $0 \leqslant j \leqslant h-1$, the construction of N_j^+ is completely analogous to that of $\mathcal{M}(F \otimes_F^{\sigma^{-j}} D)$, given for example in Section 2.2 of [18].

Proposition 5.4. *The $\mathcal{R}^+(Y_j)$-module N_j^+ is free of rank d, for all n we have $N_j^{[0;s_n]} = \mathcal{R}^{[0;s_n]}(Y_j) \otimes_{\mathcal{R}^+(Y_j)} N_j^+$, and the map $\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{R}^+(Y_j)}^{\iota_{kh+j}} N_j^+ \to W_{\mathrm{dR}}^{+,-j}(D)$ is an isomorphism for all $k \geqslant 1$.*

Proof. Since there is only one variable, the proof is a standard argument, analogous to the one which one can find in Section II.1 of [4] or Section 2.2 of [18]. □

Let $M_j^{[0;s_n]} = \mathcal{R}^{[0;s_n]}(Y)[1/f_j] \otimes_{\mathcal{R}^{[0;s_n]}(Y_j)} N_j^{[0;s_n]}$, where $f_j = \lambda/\lambda_j$.

Proposition 5.5. *We have* $\mathrm{M}(D)^{[0;s_n]}[1/f_j] = M_j^{[0;s_n]}$ *and* $\mathrm{M}(D)^{[0;s_n]} = \cap_j M_j^{[0;s_n]}$.

Proof. In the sequel, we use the fact that $Q_1(Y_j)\cdots Q_m(Y_j)$ and λ_j generate the same ideal of $\mathcal{R}^{[0;s_n]}(Y_j)$ (recall that $n = hm + (h-1)$). Let a and b be two integers such that

$$t^a \cdot \mathbf{B}_{\mathrm{dR}}^+ \otimes_F^{\sigma^j} D \subset W_{\mathrm{dR}}^{+,j}(D) \subset t^{-b} \cdot \mathbf{B}_{\mathrm{dR}}^+ \otimes_F^{\sigma^j} D,$$

for all j. We then have $\mathrm{M}(D)^{[0;s_n]} \subset \lambda^{-b} \cdot \mathcal{R}^{[0;s_n]}(Y) \otimes_F D$ by Corollary 3.8.

We have $\varphi^{-(hk+j)}(\mathcal{R}^{[0;s_n]}(Y)[1/f_j]) \subset \mathbf{B}_{\mathrm{dR}}^+$ for all $1 \leqslant k \leqslant m$ so that if $y \in M_j^{[0;s_n]}$, then $\varphi^{-(hk+j)}(y) \in W_{\mathrm{dR}}^{+,-j}(D)$ for all $1 \leqslant k \leqslant m$. On the other hand, if $y \in M_j^{[0;s_n]}$, then $y \in \lambda^{-c} \cdot \mathcal{R}^{[0;s_n]}(Y) \otimes_F D$ for some $c \geqslant 0$, so that $f_j^{a+c} y \in \mathrm{M}(D)^{[0;s_n]}$. This implies that $M_j^{[0;s_n]} \subset \mathrm{M}(D)^{[0;s_n]}[1/f_j]$.

We now prove that $\mathrm{M}(D)^{[0;s_n]} \subset M_j^{[0;s_n]}$. Choose $y \in \mathrm{M}(D)^{[0;s_n]}$. Since

$$\mathrm{M}(D)^{[0;s_n]} \subset \lambda^{-b} \cdot \mathcal{R}^{[0;s_n]}(Y) \otimes_F D,$$

we can write $y = \lambda^{-b} \sum_k z_k \otimes d_k$. By Weierstrass dividing (Proposition 2.1) the z_k's by the polynomial $(Q_1(Y_j)\cdots Q_m(Y_j))^{a+b}$, we can write $y = (Q_1(Y_j)\cdots Q_m(Y_j))^{a+b} z + y_0$ with $y_0 \in \mathcal{R}^{[0;s_n]}(Y)[1/\lambda] \otimes_{\mathcal{R}^{[0;s_n]}(Y_j)} N_j^{[0;s_n]}$.

Note that $(Q_1(Y_j)\ldots Q_m(Y_j))^{a+b} z \in M_j^{[0;s_n]}$ because $t^a \mathbf{B}_{\mathrm{dR}}^+ \otimes_F^{\sigma^j} D \subset W_{\mathrm{dR}}^{+,j}(D)$, so that $(Q_1(Y_j)\ldots Q_m(Y_j))^a \cdot D \subset N_j^{[0;s_n]}$.

Write $y_0 = \sum_{k=1}^d a_k \otimes n_k$ where $a_k \in \mathcal{R}^{[0;s_n]}(Y)[1/\lambda]$ and $n_1, \ldots, n_d$ is a basis of $N_j^{[0;s_n]}$. The element y_0 satisfies $\varphi_q^{-\ell}\varphi^{-j}(y_0) \in W_{\mathrm{dR}}^{+,-j}(D)$ for all $1 \leqslant \ell \leqslant m$. By Proposition 5.4, the map

$$\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{R}^{[0;s_n]}(Y_j)}^{\iota_{h\ell+j}} N_j^{[0;s_n]} \to W_{\mathrm{dR}}^{+,-j}(D)$$

is an isomorphism; this implies that $\varphi_q^{-\ell}\varphi^{-j}(a_k) \in \mathbf{B}_{\mathrm{dR}}^+$ for all $1 \leqslant \ell \leqslant m$. Theorem 3.6 now implies that a_k has no pole at any of the roots of $Q_1(Y_j), \ldots, Q_m(Y_j)$, so that we have $a_k \in \mathcal{R}^{[0;s_n]}(Y)[1/f_j]$. This implies that $y_0 \in M_j^{[0;s_n]}$, and therefore also y. This proves that $\mathrm{M}(D)^{[0;s_n]} \subset M_j^{[0;s_n]}$ and therefore $\mathrm{M}(D)^{[0;s_n]}[1/f_j] = M_j^{[0;s_n]}$.

If $x \in \cap_j M_j^{[0;s_n]}$, and if $k = j \bmod h$ with $0 \leqslant j \leqslant h-1$, then the fact that $x \in \mathrm{M}(D)^{[0;s_n]}[1/f_j] = \mathcal{R}^{[0;s_n]}(Y)[1/f_j] \otimes_{\mathcal{R}^{[0;s_n]}(Y_j)} N_j^{[0;s_n]}$ implies that $\iota_k(x) \in W_{\mathrm{dR}}^{+,-k}(D)$. This is true for all $h \leqslant k \leqslant n$, so that $x \in \mathrm{M}(D)^{[0;s_n]}$ and this proves the second assertion. □

Lemma 5.6. *We have* $\mathrm{M}^+(D)[1/f_j] = \mathcal{R}^+(Y)[1/f_j] \otimes_{\mathcal{R}^+(Y_j)} N_j^+$.

Proof. By combining Propositions 5.4 and 5.5, we find that

$$\mathrm{M}(D)^{[0;s_n]}[1/f_j] = \mathcal{R}^{[0;s_n]}(Y)[1/f_j] \otimes_{\mathcal{R}^+(Y_j)} N_j^+.$$

Since $\mathrm{M}(D)^+ = \cap_j \mathrm{M}(D)^{[0;s_n]}$, we have $\mathrm{M}(D)^+[1/f_j] \subset \cap_j \mathrm{M}(D)^{[0;s_n]}[1/f_j]$. We also have $\mathcal{R}^+(Y)[1/f_j] \otimes_{\mathcal{R}^+(Y_j)} N_j^+ \subset \mathrm{M}^+(D)[1/f_j]$, and those two inclusions imply that $\mathrm{M}^+(D)[1/f_j] = \mathcal{R}^+(Y)[1/f_j] \otimes_{\mathcal{R}^+(Y_j)} N_j^+$. □

Proof of Theorem 5.3. We first prove that the family $\{\mathrm{M}(D)^{[0;s_n]}\}_{n \in S}$ is a coherent sheaf. Take $n \geqslant m \in S$. We have

$$\begin{aligned} &\mathcal{R}^{[0;s_m]}(Y) \otimes_{\mathcal{R}^{[0;s_n]}(Y)} \mathrm{M}(D)^{[0;s_n]} \\ &\qquad = \mathcal{R}^{[0;s_m]}(Y) \otimes_{\mathcal{R}^{[0;s_n]}(Y)} (\cap_j \mathcal{R}^{[0;s_n]}(Y)[1/f_j] \otimes_{\mathcal{R}^{[0;s_n]}(Y_j)} N_j^{[0;s_n]}) \\ &\qquad = \cap_j \mathcal{R}^{[0;s_m]}(Y)[1/f_j] \otimes_{\mathcal{R}^{[0;s_n]}(Y_j)} N_j^{[0;s_n]} = \mathrm{M}(D)^{[0;s_m]}. \end{aligned}$$

This implies that the family $\{\mathrm{M}(D)^{[0;s_n]}\}_{n \in S}$ is a coherent sheaf. It is clear that its global sections are precisely $\mathrm{M}^+(D)$. By Proposition 5.5, we have $\mathrm{M}(D)^{[0;s_n]} = \cap_j \mathrm{M}(D)^{[0;s_n]}[1/f_j]$ where each $\mathrm{M}(D)^{[0;s_n]}[1/f_j]$ is free of rank d over $\mathcal{R}(Y)^{[0;s_n]}[1/f_j]$. The fact that $\mathrm{M}(D)^{[0;s_n]}$ is reflexive now follows from Proposition 6 of VII.4.2 of [8], and this proves (1).

By combining Proposition 5.4 and Lemma 5.6, we get item (2) of the theorem. Suppose now that $x \in \cap_j \mathrm{M}^+(D)[1/f_j]$. If $k = j \bmod h$ with $0 \leqslant j \leqslant h-1$, then the fact that $x \in \mathrm{M}^+(D)[1/f_j] = \mathcal{R}^+(Y)[1/f_j] \otimes_{\mathcal{R}^+(Y_j)} N_j^+$ implies that $\iota_k(x) \in W_{\mathrm{dR}}^{+,-k}(D)$. This being true for all $k \geqslant h$, we have $x \in \mathrm{M}^+(D)$ and this proves item (3) of the theorem. □

Remark 5.7. If $h \leqslant 2$, then the ring $\mathcal{R}^{[0;s_n]}(Y)$ is of dimension $\leqslant 2$, and reflexive $\mathcal{R}^{[0;s_n]}(Y)$-modules are therefore projective. By Lütkebohmert's theorem (see [19], corollary on page 128), the $\mathcal{R}^{[0;s_n]}(Y)$-module $\mathrm{M}(D)^{[0;s_n]}$ is then free of rank d. The system $\{\mathrm{M}(D)^{[0;s_n]}\}_{n \in S}$ then forms a vector bundle over the open unit polydisk. By combining Proposition 2 on page 87 of [16] (note that "A_m" is defined at the bottom of page 82 of loc. cit.), and the main theorem of [1], we get that $\mathrm{M}^+(D)$ is free of rank d over $\mathcal{R}^+(Y)$. If $h \geqslant 3$, I do not know whether this still holds.

6. Properties of $\mathrm{M}^+(D)$

We now prove that $\mathrm{M}(D) = \mathcal{R}(Y) \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$ is a (φ_q, Γ_F)-module over $\mathcal{R}(Y)$, and that if D arises from a crystalline representation V, then $\mathrm{M}^+(D)$ and V are naturally related. It is clear from the definition that $\mathrm{M}^+(D)$ is stable under the action of Γ_F. We also have $\lambda^a \cdot \mathcal{R}^+(Y) \otimes_F D \subset \mathrm{M}^+(D)$ for some $a \geqslant 0$, so that

$$\mathcal{R}^+(Y)[1/\lambda] \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D) = \mathcal{R}^+(Y)[1/\lambda] \otimes_F D.$$

Say that the module D with h filtrations is effective if $\mathrm{Fil}_j^0(D) = D$ for $0 \leqslant j \leqslant h-1$. Recall that $n = hm + (h-1)$ with $m \geqslant 1$.

Lemma 6.1. *If D is effective, then the $\mathcal{R}^+(Y_j)$-module N_j^+ is stable under φ_q, and $N_j^+/\varphi_q^*(N_j^+)$ is killed by $Q_1(Y_j)^{a_j}$ if $a_j \geqslant 0$ is such that $\mathrm{Fil}^{a_j+1} D = \{0\}$.*

Proof. This concerns the construction in one variable, so the proof is standard. See for example Section 2.2 of [18]. □

Proposition 6.2. *If D is effective, then the $\mathcal{R}^+(Y)$-module $\mathrm{M}^+(D)$ is stable under the Frobenius map φ_q, and $\mathrm{M}^+(D)/\varphi_q^*(\mathrm{M}^+(D))$ is killed by $Q_1(Y_0)^{a_0} \cdots Q_1(Y_{h-1})^{a_{h-1}}$.*

Proof. By (2) of Theorem 5.3, we have $\mathrm{M}^+(D) = \cap_j \mathrm{M}^+(D)[1/f_j]$ and by Lemma 5.6, $\mathrm{M}^+(D)[1/f_j] = \mathcal{R}^+(Y)[1/f_j] \otimes_{\mathcal{R}^+(Y_j)} N_j^+$. Lemma 6.1 implies that N_j^+ is stable under φ_q, and so the same is true of $\mathrm{M}^+(D)[1/f_j]$ and hence $\mathrm{M}^+(D)$.

If $x \in \mathrm{M}^+(D)$, then $x \in \mathrm{M}^+(D)[1/f_j] = \mathcal{R}^+(Y)[1/f_j] \otimes_{\mathcal{R}^+(Y_j)} N_j^+$. Note however that at each $k = i \neq j \mod h$, the coefficients of x can have a pole of order at most a_i since $\mathrm{Fil}^{a_i+1} D = \{0\}$. This implies the more precise estimate

$$\mathrm{M}^+(D) \subset \prod_{i \neq j} \lambda_i^{-a_i} \cdot \mathcal{R}^+(Y) \otimes_{\mathcal{R}^+(Y_j)} N_j^+.$$

The $\varphi_q(\mathcal{R}^+(Y))$-module $\mathcal{R}^+(Y)$ is free of rank q^h, with basis $\{Y^\ell, \ell \in \{0, \ldots, q-1\}^h\}$. We therefore have

$$\begin{aligned} Q_1(Y_0)^{a_0} \cdots Q_1(Y_{h-1})^{a_{h-1}} \cdot x &\in \prod_{i \neq j} (\lambda_i / Q_1(Y_i))^{-a_i} \cdot \mathcal{R}^+(Y) \otimes_{\mathcal{R}^+(Y_j)} Q_1(Y_j)^{a_j} \cdot N_j^+ \\ &\subset \oplus_\ell Y^\ell \cdot \varphi_q(\mathcal{R}^+(Y)[1/f_j] \otimes_{\mathcal{R}^+(Y_j)} N_j^+). \end{aligned}$$

This implies that

$$Q_1(Y_0)^{a_0} \cdots Q_1(Y_{h-1})^{a_{h-1}} \cdot x \in \cap_j \oplus_\ell Y^\ell \cdot \varphi_q(\mathrm{M}^+(D)[1/f_j]) = \varphi_q^*(\mathrm{M}^+(D)),$$

which proves the second claim. □

Remark 6.3. Instead of working with a D where the filtrations are defined on D, we could have asked for the filtrations to be defined on $F_n \otimes_F D$ for some $n \geqslant 1$. The construction and properties of $\mathrm{M}^+(D)$ are then basically unchanged, but the annihilator of $\mathrm{M}^+(D)/\varphi_q^*(\mathrm{M}^+(D))$ is possibly more complicated than in Proposition 6.2. This applies in particular to representations of G_F that become crystalline when restricted to G_{F_n} for some $n \geqslant 1$.

Definition 6.4. A (φ_q, Γ_F)-module over $\mathcal{R}(Y)$ is a $\mathcal{R}(Y)$-module M that is of the form $\mathrm{M} = \mathcal{R}(Y) \otimes_{\mathcal{R}^{[s;+\infty[}(Y)} \mathrm{M}^{[s;+\infty[}$ where $\mathrm{M}^{[s;+\infty[}$ is a coadmissible $\mathcal{R}^{[s;+\infty[}(Y)$-module, endowed with a semilinear Frobenius map $\varphi_q : \mathrm{M}^{[s;+\infty[} \to \mathrm{M}^{[qs;+\infty[}$, such that $\varphi_q^*(\mathrm{M}^{[s;+\infty[}) = \mathrm{M}^{[qs;+\infty[}$, and a continuous and compatible action of Γ_F.

Remark 6.5. In the definition above, it would seem natural to impose some additional condition on M, such as "torsion-free." All the (φ_q, Γ_F)-modules over $\mathcal{R}(Y)$ that are constructed in this article are actually reflexive. The definition above should be considered provisional, until we have a better idea of which objects we want to exclude. Note that in the absence of flatness, tensor products may behave badly.

If D is a φ_q-module with an action of Γ_F and h filtrations and if $\ell \in \mathbf{Z}$, let $D(\ell)$ denote the same φ_q-module with an action of Γ_F, but with $\mathrm{Fil}_j^k(D(\ell)) = (\mathrm{Fil}_j^{k-\ell} D)(\ell)$. Note that $D(\ell)$ is effective if $\ell \gg 0$.

Lemma 6.6. *We have* $\mathrm{M}(D(\ell)) = \lambda^{-\ell} \cdot \mathrm{M}(D)$.

Proof. The fact that $\mathrm{M}^+(D(\ell)) = \lambda^{-\ell} \cdot \mathrm{M}^+(D)$ follows from the definition. □

Theorem 6.7. *If D is a φ_q-module with an action of Γ_F and h filtrations as above, then $\mathcal{R}(Y) \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$ is a (φ_q, Γ_F)-module over $\mathcal{R}(Y)$.*

Proof. If D is effective, then this follows from Theorem 5.3 and Proposition 6.2. If D is not effective, then $D(\ell)$ is effective if $\ell \gg 0$, and the theorem follows from the effective case and Lemma 6.6. □

Remark 6.8. In [18], Kisin and Ren construct some (φ_q, Γ_F)-modules $\mathrm{M}^+_{\mathrm{KR}}(D)$ in one variable, over the ring $\mathcal{R}^+(Y_0)$, from the data of a D like ours for which the filtration $\mathrm{Fil}_j^\bullet$ is trivial for $j \neq 0$. For those D, we have $\mathrm{M}^+(D) = \mathcal{R}^+(Y) \otimes_{\mathcal{R}^+(Y_0)} \mathrm{M}^+_{\mathrm{KR}}(D)$. More generally, our construction shows that $\mathrm{M}^+(D)$ comes by extension of scalars from a (φ_q, Γ_F)-module in as many variables as there are non-trivial filtrations among the $\mathrm{Fil}_j^\bullet$.

Proposition 6.9. *If $n = hk + j \geqslant h$, then the map*

$$\mathbf{B}^+_{\mathrm{dR}} \otimes^{\iota_n}_{\mathcal{R}^+(Y)} \mathrm{M}^+(D) \to \mathrm{Fil}^0_{-j}(\mathbf{B}_{\mathrm{dR}} \otimes^{\sigma^{-j}}_F D)$$

is an isomorphism.

Proof. Since $\iota_n(f_j)$ is a unit of $\mathbf{B}^+_{\mathrm{dR}}$, we have

$$\begin{aligned}\mathbf{B}^+_{\mathrm{dR}} \otimes^{\iota_n}_{\mathcal{R}^+(Y)} \mathrm{M}^+(D) &= \mathbf{B}^+_{\mathrm{dR}} \otimes^{\iota_n}_{\mathcal{R}^+(Y)[1/f_j]} \mathrm{M}^+(D)[1/f_j] \\ &= \mathbf{B}^+_{\mathrm{dR}} \otimes^{\iota_n}_{\mathcal{R}^+(Y_j)} N_j^+ \\ &= \mathrm{Fil}^0_{-j}(\mathbf{B}_{\mathrm{dR}} \otimes^{\sigma^{-j}}_F D),\end{aligned}$$

where the last equality follows from Proposition 5.4. □

Suppose now that D comes from an F-linear crystalline representation V of G_F as in Example 5.1. In this case, $\mathrm{Fil}^0_j(\mathbf{B}_{\mathrm{dR}} \otimes^{\sigma^j}_F D) = \mathbf{B}^+_{\mathrm{dR}} \otimes^{\sigma^j}_F V$. Moreover, one recovers V from D by the formula:

$$V = \{y \in (\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_F D)^{\varphi_q = 1},\ \iota_j(y) \in \mathrm{Fil}^0_{-j}(\mathbf{B}_{\mathrm{dR}} \otimes^{\sigma^{-j}}_F D) \text{ for all } 0 \leqslant j \leqslant h-1\}.$$

Recall that we have constructed in Section 3 an injective map $\mathcal{R}^+(Y) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$. This way we get a map

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D) \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_F D \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_F V.$$

Let $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ be the rings defined in Section 2.3 [2]. Recall that $n(r)$ is the smallest n such that $r \leqslant p^{n-1}(p-1)$. We have the following lemma.

Lemma 6.10. *If* $y \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}[1/t]$ *satisfies* $\varphi^{-n}(y) \in \mathbf{B}_{\mathrm{dR}}^{+}$ *for all* $n \geqslant n(r)$*, then* $y \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$*.*

Proof. See Lemma 1.1 of [5] and the proof of Proposition 3.2 in ibid. □

Theorem 6.11. *If* D *comes from a crystalline representation* V*, and if* $r \geqslant p^{h-1}(p-1)$*, then the map above gives rise to an isomorphism*

$$\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D) \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_F V.$$

Proof. We first check that the image of the map above belongs to $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_F V$. If $y \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$, then its image is in $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}[1/t] \otimes_F V$ and satisfies $\varphi^{-n}(y) \in \mathbf{B}_{\mathrm{dR}}^{+} \otimes_F^{\sigma^{-n}} V$ for all $n \geqslant n(r)$, so the assertion follows from Lemma 6.10.

We now prove that $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$ is a free $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$-module of rank d. By (2) of Theorem 5.3, $\mathrm{M}^+(D)[1/f_j]$ is a free $\mathcal{R}^+(Y)[1/f_j]$-module of rank d, and therefore $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}[1/f_j] \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$ is a free $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}[1/f_j]$-module of rank d for all j. The ring $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ is a Bézout ring by Theorem 2.9.6 of [17], and the elements $f_0, \ldots, f_{h-1}$ have no common factor. They therefore generate the unit ideal of $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$, and $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$ is projective of rank d by Theorem 1 of II.5.2 of [8]. Since $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ is a Bézout ring, $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$ is free of rank d. By Proposition 6.9, the map

$$\mathbf{B}_{\mathrm{dR}}^{+} \otimes^{\iota_n}_{\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}} (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)) \to \mathbf{B}_{\mathrm{dR}}^{+} \otimes_F^{\sigma^{-n}} V$$

is an isomorphism if $n \geqslant n(r)$. The two $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$-modules $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_{\mathcal{R}^+(Y)} \mathrm{M}^+(D)$ and $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_F V$ therefore have the same localizations at all $n \geqslant n(r)$, and are both stable under G_F, so that they are equal by the same argument as in the proof of Lemma 2.2.2 of [3] (the idea is to take determinants, so that one is reduced to showing that if $x \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ generates an ideal stable under G_F, and has the property that $\iota_n(x)$ is a unit of $\mathbf{B}_{\mathrm{dR}}^{+}$ for all $n \geqslant n(r)$, then x is a unit of $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$). □

Remark 6.12. If D comes from a crystalline representation V, and if $n \geqslant 0$, then there is likewise an isomorphism $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_{\mathcal{R}^+(Y)}^{\varphi^{-n}} \mathrm{M}^+(D) \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_F^{\sigma^{-n}} V$ for $r \gg 0$.

7. Crystalline (φ_q, Γ_F)-modules

Let M be a (φ_q, Γ_F)-module over $\mathcal{R}(Y)$. In this section, we define what it means for M to be crystalline, and we prove that every crystalline (φ_q, Γ_F)-module M is of the form $\mathrm{M} = \mathrm{M}(D)$, where D is a φ_q-module with h filtrations, on which the action of G_F is trivial. The results are similar to those of [4], which deals with the cyclotomic case.

Lemma 7.1. *We have* $\mathrm{Frac}(\mathcal{R}(Y))^{\Gamma_F} = F$*.*

Proof. If $x \in \mathrm{Frac}(\mathcal{R}(Y))^{\Gamma_F}$, then we can write $x = a/b$ with $a, b \in \mathcal{R}^{[s_n;s_n]}(Y)$ for some $n \gg 0$. By Proposition 3.2, the series $a(u, \ldots, \varphi^{h-1}(u))$ and $b(u, \ldots, \varphi^{h-1}(u))$ converge in $\widetilde{\mathbf{B}}^{[r_n;r_n]}$. We can therefore see $\varphi^{-n}(a)$ and $\varphi^{-n}(b)$ as elements of $\mathbf{B}_{\mathrm{dR}}^{+}$, which satisfy $\varphi^{-n}(a)/\varphi^{-n}(b) \in \mathbf{B}_{\mathrm{dR}}^{G_F}$. The lemma now follows from the fact that $\mathbf{B}_{\mathrm{dR}}^{G_F} = F$. □

If M is a (φ_q, Γ_F)-module over $\mathcal{R}(Y)$, then let $\mathrm{D}_{\mathrm{cris}}(\mathrm{M}) = (\mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} \mathrm{M})^{\Gamma_F}$.

Corollary 7.2. *If* M *is a* (φ_q,Γ_F)*-module over* $\mathcal{R}(Y)$*, then we have* $\dim \mathrm{D}_{\mathrm{cris}}(\mathrm{M}) \leqslant \dim \mathrm{Frac}(\mathcal{R}(Y)) \otimes_{\mathcal{R}(Y)} \mathrm{M}$.

Proof. By a standard argument, Lemma 7.1 implies that the map

$$\mathrm{Frac}(\mathcal{R}(Y)) \otimes_F \mathrm{D}_{\mathrm{cris}}(V) \to \mathrm{Frac}(\mathcal{R}(Y)) \otimes_{\mathcal{R}(Y)} \mathrm{M}$$

is injective. □

Definition 7.3. We say that a (φ_q,Γ_F)-module M over $\mathcal{R}(Y)$ is crystalline if

(1) for some s, $\mathrm{M}^{[s;+\infty[}[1/f_j]$ is a free $\mathcal{R}(Y)^{[s;+\infty[}[1/f_j]$-module of finite rank d;
(2) $\mathrm{M}^{[s;+\infty[} = \cap_{j=0}^{h-1}\mathrm{M}^{[s;+\infty[}[1/f_j]$;
(3) we have $\dim \mathrm{D}_{\mathrm{cris}}(\mathrm{M}) = d$.

For example, if D is a φ_q-module with h filtrations on which the action of G_F is trivial, then $\mathrm{M}(D)$ is a crystalline (φ_q,Γ_F)-module. Note that a crystalline (φ_q,Γ_F)-module is reflexive.

Proposition 7.4. *If* $f \in \mathcal{R}^{[s;+\infty[}(Y)$ *generates an ideal of* $\mathcal{R}^{[s;+\infty[}(Y)$ *that is stable under* Γ_F*, then* $f = u\cdot\prod_{j=0}^{h-1}\prod_{n\geqslant n(s)}(Q_n(Y_j)/p)^{a_{n,j}}$ *where* u *is a unit and* $a_{n,j} \in \mathbf{Z}_{\geqslant 0}$.

Proof. Recall that a power series $f \in \mathcal{R}^I(Y)$ is a unit if and only if it has no zero in the corresponding domain of convergence (by the nullstellensatz, see Section 7.1.2 of [7]).

Let $I = [s;u]$ be a closed subinterval of $[s;+\infty[$, so that $f \in \mathcal{R}^I(Y)$, and let $z = (z_0, z_1, \dots, z_{h-1})$ be a point such that $f(z) = 0$. Let J be the set of indices j such that z_j is not a torsion point of LT_h and let $f_J \in \mathcal{R}^I_{F_k}(\{Y_j\}_{j\in J})$ be the power series that results from evaluation of the Y_m at z_m for all the z_m that are torsion points of LT_h (here k is large enough so that all those z_m belong to F_k). The ideal of $\mathcal{R}^I_{F_k}(\{Y_j\}_{j\in J})$ generated by the power series f_J is stable under $1+p^k\mathcal{O}_F$, so that the set of its zeroes is stable under the action of $1+p^k\mathcal{O}_F$. Furthermore, f_J has a zero none of whose coordinates are torsion points of LT_h. The same argument as in the proof of Proposition 2.4 shows that $f_J = 0$.

If we denote by $Z_I(f)$ the zero set of $f \in \mathcal{R}^I(Y)$, then the preceding argument shows that $Z_I(f)$ is the union of finitely many components of the form $Z_0 \times \cdots \times Z_{h-1}$ where for each j, either Z_j is a torsion point of LT_h or $Z_j = Z_I(\{0\})$. For reasons of dimension, each of these components has precisely one Z_j which is a torsion point, the remaining $h-1$ being $Z_I(\{0\})$. This implies that in $\mathcal{R}^I(Y)$, f is the product of finitely many $Q_n(Y_j)$ by a unit.

The proposition now follows by a standard infinite factorisation argument, by writing $[s;+\infty[= \cup_{u\geqslant s}[s;u]$. □

Corollary 7.5. *If* M *is a crystalline* (φ_q,Γ_F)*-module over* $\mathcal{R}(Y)$*, then the map*

$$\mathcal{R}(Y)[1/t] \otimes_F \mathrm{D}_{\mathrm{cris}}(\mathrm{M}) \to \mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} \mathrm{M}$$

is an isomorphism.

Proof. The map is injective by Lemma 7.1, and its determinant generates an ideal of $\mathcal{R}(Y)[1/t]$ that is stable under Γ_F. Proposition 7.4 implies that this ideal is the unit ideal of $\mathcal{R}(Y)[1/t]$, and therefore that the map is an isomorphism. □

We now consider filtrations on $\mathrm{D}_{\mathrm{cris}}(\mathrm{M})$.

Lemma 7.6. *Let D be an F-vector space, and let W be a $\mathbf{B}_{\mathrm{dR}}^+$-lattice of $\mathbf{B}_{\mathrm{dR}} \otimes_F D$ that is stable under G_F, where G_F acts trivially on D. If we set $\mathrm{Fil}^i D = D \cap t^i \cdot W$, then $W = \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_F D)$.*

Proof. Let $e_1, \dots, e_d$ be a basis of D adapted to its filtration, with $e_i \in \mathrm{Fil}^{h_i} \setminus \mathrm{Fil}^{h_i+1} D$. We then have $\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_F D) = \oplus_{i=1}^d \mathbf{B}_{\mathrm{dR}}^+ \cdot t^{-h_i} e_i$. By definition, we have $t^{-h_i} e_i \in W$, so that $\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_F D) \subset W$. We now prove the reverse inclusion.

If $w \in W$, then we can write $w = a_1 t^{-h_1} e_1 + \cdots + a_d t^{-h_d} e_d$ with $a_i \in \mathbf{B}_{\mathrm{dR}}$ and we need to prove that $a_i \in \mathbf{B}_{\mathrm{dR}}^+$ for all i. If this is not the case, then there exists $n \geqslant 1$ such that if we set $b_i = t^n a_i$, then we have $b_1 t^{-h_1} e_1 + \cdots + b_d t^{-h_d} e_d \in t \cdot W$, with $b_i \in (\mathbf{B}_{\mathrm{dR}}^+)^\times$ for at least one i. Consider the shortest such relation; in particular, $b_i \in (\mathbf{B}_{\mathrm{dR}}^+)^\times$ for all i for which $b_i \neq 0$, and we can assume that $b_i = 1$ for at least one i. If $g \in G_F$, then applying $1 - \chi_{\mathrm{cyc}}(g)^{h_i} g$ to the relation yields a shorter relation. This implies that $(1 - \chi_{\mathrm{cyc}}(g)^{h_i - h_j} g)(b_j) \in t\mathbf{B}_{\mathrm{dR}}^+$ for all $g \in G_F$ and all $1 \leqslant j \leqslant d$. Since $H^0(G_F, \mathbf{C}_p) = F$ and $H^0(G_F, \mathbf{C}_p(h)) = \{0\}$ if $h \neq 0$, we have $b_j \in F + t\mathbf{B}_{\mathrm{dR}}^+$ if $h_i = h_j$ and $b_j \in t\mathbf{B}_{\mathrm{dR}}^+$ otherwise. The relation above therefore reduces to an F-linear combination of those e_j for which $h_j = h_i$, belonging to $D \cap t^{h_i+1} W = \mathrm{Fil}^{h_i+1} D$, and is hence trivial. This proves that $W \subset \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_F D)$. □

Definition 7.7. Let M be a crystalline (φ_q, Γ_F)-module over $\mathcal{R}(Y)$. For $m \gg 0$ and $j = 0, \dots, h-1$ and $n = hm - j$, define

$$\mathrm{Fil}_j^i(F \otimes_F^{\sigma^j} \varphi_q^{-m}(\mathrm{D_{cris}}(\mathrm{M}))) = (F \otimes_F^{\sigma^j} \varphi_q^{-m}(\mathrm{D_{cris}}(\mathrm{M}))) \cap t^i \cdot (\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{R}^{[s;+\infty[}(Y)}^{\varphi^{-n}} \mathrm{M}^{[s;+\infty[}).$$

Proposition 7.8. *The definition of $\mathrm{Fil}_j^i(\mathrm{D_{cris}}(\mathrm{M}))$ does not depend on $m \gg 0$, and we have $\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_F^{\sigma^{-n}} \mathrm{D_{cris}}(\mathrm{M})) = \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{R}^{[s;+\infty[}(Y)}^{\varphi^{-n}} \mathrm{M}^{[s;+\infty[}$.*

Proof. If s is large enough, then $\mathrm{M}^{[qs;+\infty[} = \varphi_q^*(\mathrm{M}^{[s;+\infty[})$ so that

$$\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{R}^{[qs;+\infty[}(Y)}^{\varphi^{-n-h}} \mathrm{M}^{[qs;+\infty[} = \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{R}^{[qs;+\infty[}(Y)}^{\varphi^{-n}\varphi_q^{-1}} \varphi_q^*(\mathrm{M}^{[s;+\infty[}) = \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{R}^{[s;+\infty[}(Y)}^{\varphi^{-n}} \mathrm{M}^{[s;+\infty[},$$

which implies the first statement. The second statement follows from Lemma 7.6, applied to $W = \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{R}^{[s;+\infty[}(Y)}^{\varphi^{-n}} \mathrm{M}^{[s;+\infty[}$. □

Theorem 7.9. *The functors $\mathrm{M} \mapsto \mathrm{D_{cris}}(\mathrm{M})$ and $D \mapsto \mathrm{M}(D)$, between the category of crystalline (φ_q, Γ_F)-modules over $\mathcal{R}(Y)$ and the category of φ_q-modules with h filtrations, are mutually inverse.*

Proof. If D is a φ_q-module with h filtrations, then it is clear that $\mathrm{D_{cris}}(\mathrm{M}(D)) = D$ as φ_q-modules. The fact that $\mathrm{Fil}_j^i(D) = D \cap t^i \cdot \mathrm{Fil}_j^0(\mathbf{B}_{\mathrm{dR}} \otimes_F^{\sigma^{-n}} D)$ follows from taking a basis of D adapted to $\mathrm{Fil}_j^\bullet$ and

$$\mathrm{Fil}_j^0(\mathbf{B}_{\mathrm{dR}} \otimes_F^{\sigma^{-n}} D) = \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{R}^{[s;+\infty[}(Y)}^{\varphi^{-n}} \mathrm{M}^{[s;+\infty[}(D) = \mathrm{Fil}_j^0(\mathbf{B}_{\mathrm{dR}} \otimes_F^{\sigma^{-n}} \mathrm{D_{cris}}(\mathrm{M}(D)))$$

by Propositions 6.9 and 7.8, so that the filtrations on D and $\mathrm{D_{cris}}(\mathrm{M})$ are the same.

We now check that if M is a crystalline (φ_q, Γ_F)-module over $\mathcal{R}(Y)$ and $D = \mathrm{D_{cris}}(\mathrm{M})$ with the filtration given in Definition 7.7, then $\mathrm{M} = \mathrm{M}(D)$. Corollary 7.5 says that we have $\mathcal{R}(Y)[1/t] \otimes_F D = \mathcal{R}(Y)[1/t] \otimes_{\mathcal{R}(Y)} \mathrm{M}$. The theorem now follows from Proposition 7.8 and the fact that if $y \in \mathcal{R}^{[s;+\infty[}(Y)[1/t] \otimes_{\mathcal{R}^{[s;+\infty[}(Y)} \mathrm{M}^{[s;+\infty[}$, then $y \in \mathrm{M}^{[s;+\infty[}$ if and only if $y \in \mathbf{B}_{\mathrm{dR}}^{+} \otimes_{\mathcal{R}^{[s;+\infty[}(Y)}^{\varphi^{-n}} \mathrm{M}^{[s;+\infty[}$ for all n such that $s_n \geqslant s$ by Corollary 3.8 and items (1) and (2) of Definition 7.3. □

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