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DIRICHLET L-FUNCTIONS, ELLIPTIC CURVES, HYPERGEOMETRIC FUNCTIONS, AND RATIONAL APPROXIMATION WITH PARTIAL SUMS OF POWER SERIES

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Abstract. We consider the Diophantine approximation of exponential generating functions at rational arguments by their partial sums and by convergents of their (simple) continued fractions. We establish quantitative results showing that these two sets of approximations coincide very seldom. Moreover, we offer many conjectures about the frequency of their coalescence. In particular, we consider exponential generating functions with real Dirichlet characters and with coefficients of L-functions of elliptic curves, where calculational data provide striking examples showing agreement for certain convergents of high index and gargantuan heights. Finally, we similarly examine hypergeometric functions; note that e is a special case of the latter.

1. Introduction and statements of results

Given a sequence of integer numbers $\{a_n\}$, $0 \leq n < \infty$, consider the exponential generating function $G(s)$ given by

$$G(s) := \sum_{n=0}^{\infty} \frac{a_n}{n!} s^n, \quad s \in \mathbb{C}. \tag{1.1}$$

We focus on the Diophantine approximation of the values of $G(s)$ at rational points $s = r$ by partial sums of the power series expansion (1.1). In the particular case that $a_n = 1$ for all n, $G(s) = e^s$. If, in addition, we choose $r = 1$, then we are asking how well e can be approximated by partial sums $\sum_{n=0}^{N} \frac{1}{n!}$. Sondow [16] conjectured that exactly two of these partial sums are also convergents to the (simple) continued fraction of e. Among other related results, Sondow and Schalm [17, 18] showed that for almost all N, the corresponding partial sum $\sum_{n=0}^{N} \frac{1}{n!}$ is not a convergent to the continued fraction of e.

In the present paper, in broad and diverse settings, we raise many conjectures that generalize or are analogues of Sondow's conjecture. For a wide variety of sequences $\{a_n\}$ of polynomial growth in n and for $s = r$ in (1.1), we examine approximations by partial sums and conjecture that they coincide with approximations by (simple) continued fractions only a finite number of times. We then obtain bounds for the number of times the sequences of partial sums and convergents of the continued fraction coalesce. For those results for which e^r is a special case, our bounds improve

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on those obtained by Sondow and Schalm [17]. We furthermore expand the discussion from convergents of continued fractions to more general Diophantine inequalities. For any rational number $\mu = a/b$, with $(a, b) = 1$, we consider the height of μ, given by $H(\mu) = \max\{|a|, |b|\}$. For any real number α and any positive real number δ, we denote

$$A_{\alpha,\delta} = \left\{ \mu \in \mathbb{Q} : |\alpha - \mu| < \frac{1}{H(\mu)^{1+\delta}} \right\}. \tag{1.2}$$

If $0 < \delta < 1$, the set $A_{\alpha,\delta}$ is much more numerous than the set of convergents to the continued fraction of α. We remark that for many such sets $A_{\alpha,\delta}$, various Diophantine inequalities have been established, showing that these sets contain infinitely many rational numbers of a special form (e.g., a denominator that is a prime number, or a square, or a higher power, etc.). For a basic presentation of the subject, the reader may consult Schmidt [13] and Baker [4]. By contrast, we expect that for the classes of functions (1.1) considered below, only finitely many partial sums of (1.1), with $s = r$, should belong to $A_{G(r),\delta}$.

We first consider the case when

$$G(s) := G_\chi(s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n!} s^n, \qquad s \in \mathbb{C}, \tag{1.3}$$

where χ is a real Dirichlet character modulo q. Writing [7, p. 65]

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) e^{2\pi i n a/q}, \tag{1.4}$$

where $\tau(\overline{\chi})$ is the corresponding Gauss sum, we see that

$$G_\chi(r) = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{e^{2\pi i a n/q}}{n!} r^n = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \left(e^{r e^{2\pi i a/q}} - 1 \right), \tag{1.5}$$

and by the classical theorem of Lindemann [11], [3, Theorem 1.4], $G_\chi(r)$ is transcendental over the rationals.

Conjecture 1.1. *For any real Dirichlet character χ and any nonzero rational number r, there exist only finitely many positive integers n for which the partial sum $\sum_{\ell=0}^{n} \frac{\chi(\ell)}{\ell!} r^n$ is a convergent to the continued fraction of $G_\chi(r)$, given by* (1.3).

In particular, when χ is the trivial character and $r = 1$, Conjecture 1.1 corresponds to Sondow's conjecture, except that here we are only conjecturing that the set is finite. For a given χ and r, one can make conjectures as precise as Sondow's conjecture. In Table 1 below, we let χ denote the Legendre symbol $(\frac{n}{p})$, and set $r = 1$. In each case we list all the numbers that we found to simultaneously be partial sums of the power series and convergents to the corresponding continued fraction. In each case, we calculated the first 10,000 partial sums and the first 200 convergents of the continued fractions. For each row in Table 1 we formulate an associated conjecture, stating that the list provided on that row is complete.

Assuming the validity of Conjecture 1.1, if we were to take any odd prime p, the number of entries in such a line of Table 1 would be finite. A natural problem would be to estimate, for large x, the average length of these lists, or the length of the longest such list, as q runs over all odd primes $\leq x$. One may also raise the same problem,

TABLE 1. Coalescence for the Legendre Symbol

p	Common values	p	Common values
3	$1, \frac{1}{2}, \frac{8}{15}$	43	$\frac{1}{2}, \frac{1}{3}, \frac{3}{8}$
5	$\frac{1}{2}, \frac{1}{3}, \frac{3}{8}$	47	$1, \frac{17}{10}, \frac{5}{3}$
7	$\frac{4}{3}, 1, \frac{11}{8}, \frac{3}{2}$	53	$\frac{1}{2}, \frac{1}{3}, \frac{3}{8}$
11	$1, \frac{2}{3}$	59	$1, \frac{2}{3}$
13	$1, \frac{44}{63}$	61	$1, \frac{2}{3}$
17	$\frac{4}{3}, 1, \frac{11}{8}, \frac{86}{63}, \frac{3}{2}$	67	$\frac{1}{2}, \frac{1}{3}, \frac{3}{8}$
19	$\frac{1}{3}, \frac{1}{2}$	71	$1, \frac{5}{3}$
23	$1, \frac{17}{10}, \frac{5}{3}$	73	$1, \frac{17}{10}, \frac{5}{3}$
29	$\frac{1}{3}, \frac{1}{2}$	79	$1, \frac{11}{8}, \frac{3}{2}, \frac{4}{3}$
31	$1, \frac{11}{8}, \frac{3}{2}, \frac{4}{3}$	83	$1, \frac{7}{10}, \frac{2}{3}$
37	$1, \frac{7}{10}, \frac{2}{3}$	89	$1, \frac{11}{8}, \frac{3}{2}, \frac{4}{3}$
41	$1, \frac{11}{8}, \frac{3}{2}, \frac{4}{3}$	97	$1, \frac{17}{10}, \frac{5}{3}$

with χ running over all real characters of conductor $\leq x$. Similar questions can be asked for other values of r.

In connection with Conjecture 1.1 we offer the following theorem, which improves the result in the aforementioned special case due to Sondow and Schalm [17].

Theorem 1.2. *For any real Dirichlet character χ, any nonzero rational number r, and any real number $\delta > 0$,*

$$\#\left\{1 \leq n \leq N : \sum_{\ell=1}^{n} \frac{\chi(\ell)}{\ell!} r^n \in A_{G_\chi(r),\delta}\right\} = O_{\chi,r,\delta}(\log N), \tag{1.6}$$

for all positive integers N, where G_χ is given by (1.3).

Next, we consider the case of an elliptic curve E defined over $\mathbb{Q}$, and its associated $L-$function

$$L(E,s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}, \tag{1.7}$$

(see, e.g., Silverman [15], Diamond and Shurman [8], and Wiles [19]). In this case $G(s)$ is given by

$$G(s) := G_E(s) := \sum_{n=1}^{\infty} \frac{a_E(n)}{n!} s^n, \tag{1.8}$$

and we seek partial sums $\sum_{\ell=1}^{n} \frac{a_E(\ell)}{\ell!} r^\ell$ that belong to $A_{G_E(r),\delta}$ for a given nonzero rational number r and a given $\delta > 0$. One can similarly raise this type of question in the general context of modular forms. One can pose an analogue of Conjecture 1.1 in which $\chi(\ell)$ (or $a_E(\ell)$ above) is replaced by the integral coefficients of a modular form of positive integral weight on a congruence subgroup of the full modular group. A result as strong as Theorem 1.2 in this context appears to be out of reach at present. For example, consider the Delta function $\Delta(z)$, which is the unique normalized newform of weight 12 on the full modular group,

$$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n, \quad q = e^{2\pi i z}, \quad \operatorname{Im} z > 0, \tag{1.9}$$

where $\tau(n)$ is the Ramanujan tau-function. (For several beautiful properties of $\tau(n)$, see the account of Ramanujan's manuscript on the partition and tau-functions written by the first author and Ono [6].) Consider the corresponding generating function

$$G_\Delta(s) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n!} s^n. \tag{1.10}$$

Assume now that for a large n and a given nonzero rational number r,

$$\sum_{\ell=1}^{n} \frac{\tau(\ell)}{\ell!} r^\ell \in A_{G_\Delta(r),\delta}$$

for a given $\delta > 0$. If such an n is followed by a gap in which the tau-function vanishes, say $\tau(n+1) = \tau(n+2) = \cdots = \tau(n+L) = 0$, then all of the integers $n+1, \ldots, n+L$ will automatically be counted together with n in the set for the analogue of the tau-function on the left-hand side of (1.6). Thus, an analogue of Theorem 1.2 in this context would require that the size of gaps between consecutive nonzero values of the tau-function is at most logarithmic. A well-known conjecture of Lehmer [10] asserts that $\tau(n) \neq 0$ for all $n \geq 1$. Motivated by this conjecture, Serre [14] initiated the study of the gap-function between consecutive nonzero coefficients of modular forms. Returning to the Ramanujan tau-function, we note that by a result of Alkan and one of the authors [2], the gap function is $O(n^{1/4})$, which is far from a logarithmic bound. With this difficulty in mind, in what follows we count only those n for which the corresponding coefficient is nonzero. With this convention, we offer the following result.

Theorem 1.3. *Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication and let $G_E(s)$ be given by* (1.8)*. Then for any nonzero rational number r and any real number $\delta > 0$,*

(1.11)
$$\#\left\{1 \leq n \leq N : a_E(n) \neq 0 \quad \text{and} \quad \sum_{\ell=1}^{n} \frac{a_E(\ell)}{\ell!} r^\ell \in A_{G_E(r),\delta}\right\} = O_\delta(\log N) + O_{E,r,\delta}(1).$$

TABLE 2. Coalescence for Elliptic Curves

Elliptic curve	Common values	Values of M	Values of N
$y^2 = x^3 + 9x + 9$	$1, \frac{991}{1008}, \frac{1868743}{1900800}$	1,7,11	1,6,12
$y^2 = x^3 + 4x + 19$	$1, \frac{59}{60}, \frac{4907389}{4989600}$	1,5,11	1,2,14
$y^2 = x^3 - 9x - 6$	$1, \frac{491}{504}, \frac{19443601}{19958400}$	1,7,11	1,6,14
$y^2 = x^3 - 8x + 7$	$1, \frac{119}{120}, \frac{88215397}{88957440}$	1,5,13	1,3,16
$y^2 = x^3 + 4x - 9$	$1, \frac{59}{60}, \frac{111353129}{113218560}$	1,5,13	1,2,11
$y^2 = x^3 + 9x - 7$	$\frac{41}{40}, \frac{116049007}{113218560}$	5,13	1,11
$y^2 = x^3 + 2x - 12$	$1, \frac{68598251}{69189120}$	1,14	1,8
$y^2 = x^3 + 15x$	$\frac{177843714162239}{177843714048000}$	17	15
$y^2 = x^3 - 5x$	$1, \frac{3420043147403}{3420071424000}$	1,17	1,16
$y^2 = x^3 - 10x$	$1, \frac{44460560901961}{44460928512000}$	1,17	1,18
$y^2 = x^3 - 20$	$1, \frac{1259}{1260}, \frac{15193569601886759}{15205637551104000}$	1,7,19	1,2,22
$y^2 = x^3 - 15$	$\frac{1261}{1260}, \frac{249278515}{249080832}, \frac{9364741849356923}{9357315416064000}$	7,13,19	1,7,23

Conjecture 1.4. *In the notation above, there exist only finitely many positive integers n for which the corresponding partial sums on the left-hand side of* (1.11) *are convergents to the continued fraction of $G_E(r)$.*

For elliptic curves E over $\mathbb{Q}$ given explicitly, one can make more precise conjectures. Table 2 above shows data collected for a few such elliptic curves. In each case, we take $r = 1$, and list all rational numbers that we found to be partial sums of the power series, with the upper index N given, and simultaneously convergents to the corresponding continued fraction, with the index M of the convergents provided. It is remarkable that, for large values of M, there exist convergents with huge heights that are in agreement with values of partial sums. We conjecture that the list provided on each row is complete. As with the case of Dirichlet characters, further questions naturally arise. One can study for instance the maximal length, or the average length of such lists, when E runs over various families of elliptic curves over $\mathbb{Q}$.

We also leave open the problem of finding an analogue of Theorem 1.3 for the Ramanujan tau-function. Concerning the question of which partial sums of $G_\Delta(r)$ are convergents to the corresponding continued fraction, in the case $r = 1$ we make the following conjecture.

Conjecture 1.5. *There are no positive integers n for which the partial sum $\sum_{\ell=1}^{n} \frac{\tau(\ell)}{\ell!}$ is a convergent to the continued fraction of $G_{\Delta}(1)$.*

As far as numerical data for Conjecture 1.5 is concerned, we checked the first 10,000 partial sums and the first 500 convergents to the continued fraction, and did not find any match.

Next, we focus our attention on the Diophantine approximation with partial sums of hypergeometric functions of the form

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; x\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}, \tag{1.12}$$

where p and q are nonnegative integers, with $p \le q$, $x \in \mathbb{C}$, and $(c)_k$ denotes the rising or shifted factorial

$$ (c)_k := \frac{\Gamma(c+k)}{\Gamma(c)} = c(c+1)\cdots(c+k-1), \qquad c \in \mathbb{C}. \tag{1.13}$$

Theorem 1.6. *Let p and q be nonnegative integers, with $p \le q$; let $a_1, a_2, \ldots, a_p$ and $b_1, b_2, \ldots, b_q$ be rational numbers, with none being a negative integer; let r be a nonzero rational number; let $\delta > 0$; and denote*

$$\alpha = {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; r\right].$$

Then there exist a constant $C_1 > 0$, depending only on δ, and a constant $C_2 > 0$, depending on $a_1, a_2, \ldots, a_p$, $b_1, b_2, \ldots, b_q$, r, and δ, such that for all positive integers N,

$$ \#\left\{ n \le N : \sum_{\ell=0}^{n} \frac{(a_1)_\ell (a_2)_\ell \cdots (a_p)_\ell}{(b_1)_\ell (b_2)_\ell \cdots (b_q)_\ell} \frac{r^\ell}{\ell!} \in A_{\alpha,\delta} \right\} \le C_1 \log N + C_2. \tag{1.14}$$

Conjecture 1.7. *In the notation above, there exist only finitely many positive integers n for which the corresponding partial sums on the left-hand side of* (1.14) *are convergents to the continued fraction of*

$$\alpha = {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; r\right].$$

In particular, when $p = q$, $a_1 = b_1, \ldots, a_p = b_p$, and $r = 1$, Conjecture 1.7 reduces to the finiteness part of Sondow's conjecture. Again, more precise versions of Conjecture 1.7 can be made when $a_1, \ldots, a_p, b_1, \ldots, b_q$ and r are given explicitly.

Theorems 1.2, 1.3, and 1.6 above, which are established by a common method, provide logarithmic upper bounds for the lengths of the corresponding lists, and the reader may naturally wonder if one can improve on these bounds. We remark in this connection that if one only uses information on the gap function in combination with information on the size of the integers a_n, of the form $a_n = O_k(n^k)$ say, where k is a given positive integer, one cannot improve on these logarithmic-type results. Indeed, there exist sequences of strictly positive integers a_n (so the gap function is identically zero) such that $a_n = O(n)$, and such that the length of the corresponding list grows at least logarithmically. To be precise, we have the following result.

Theorem 1.8. *For any real number $\delta > 0$, there exist a constant $C_1 > 0$ and a sequence of integers $\{a_n\}$, $n \in \mathbb{N}$, with $1 \le a_n \le n$ for all $n \ge 1$, such that*

$$\# \left\{ n \le N : \sum_{\ell=1}^{n} \frac{a_\ell}{\ell!} \in A_{\alpha,\delta} \right\} \ge C_1 \log N, \tag{1.15}$$

for all N sufficiently large, where $\alpha := \sum_{\ell=1}^{\infty} \frac{a_\ell}{\ell!}$.

Our proof of Theorem 1.8 allows us to take $C_1 > 1$ for a suitable $\delta > 1$. We therefore have the following corollary.

Corollary 1.9. *There exists a sequence of integers $\{a_n\}$, $n \in \mathbb{N}$, with $1 \le a_n \le n$ for all $n \ge 1$, such that the number*

$$\alpha := \sum_{\ell=1}^{\infty} \frac{a_\ell}{\ell!}$$

is transcendental, and for any sufficiently large N there are more than $\log N$ integers $n \le N$ for which the partial sum

$$\sum_{\ell=1}^{n} \frac{a_\ell}{\ell!} \tag{1.16}$$

is also a convergent to the continued fraction of α.

The logarithmic-type limitation established in the corollary above, together with the fact that each of our conjectures, if true, states much stronger results, point to the possible existence of some deep arithmetical phenomena surrounding these conjectures.

2. Proof of Theorem 1.2

Let χ be a real Dirichlet character modulo q, let r be a nonzero rational number, fix a real number $\delta > 0$, and lastly fix a small $\eta > 0$. For each large positive integer N, we let

$$S_{\chi,r,\delta,\eta,N} = \left\{ n \in [N, [(1+\eta)N]] : \sum_{\ell=1}^{n} \frac{\chi(\ell)}{\ell!} r^\ell \in A_{G(r),\delta} \right\}.$$

We shall obtain an upper bound for the number of elements in $S_{\chi,r,\delta,\eta,N}$ that is independent of N. First, we consider a subset $S^*_{\chi,r,\delta,\eta,N}$ of $S_{\chi,r,\delta,\eta,N}$ that is defined as follows. Let n_1 denote the smallest element of $S_{\chi,r,\delta,\eta,N}$, let n_2 denote the smallest element of $S_{\chi,r,\delta,\eta,N}$ that is larger than $q + n_1$, let n_3 denote the smallest element of $S_{\chi,r,\delta,\eta,N}$ that is larger than $q + n_2$, etc. We then set

$$S^*_{\chi,r,\delta,\eta,N} = \{n_1, n_2, n_3, \dots\}.$$

We note that

$$\# S_{\chi,r,\delta,\eta,N} \le q \# S^*_{\chi,r,\delta,\eta,N}. \tag{2.1}$$

Second, let S denote an arbitrary nonempty subset of $S^*_{\chi,r,\delta,\eta,N}$. We write $r = a/b$, with $a, b \in \mathbb{Z}$, $b \ge 1$, and $(a,b) = 1$. For any $n \in S$, let $A_n \in \mathbb{Z}$ be defined by

$$\sum_{\ell=1}^{n} \frac{\chi(\ell)}{\ell!} r^\ell = \frac{A_n}{b^n n!}. \tag{2.2}$$

Now let n and m, $n < m$, be two different elements of S. Then $m \geq n+q$, and not all of the numbers $\chi(\ell)$, with $\ell \in \{n+1, n+2, \dots, m\}$, are zero. Denote

$$\ell^* = \min \{\ell \in \{n+1, n+2, \dots, m\} : \chi(\ell) \neq 0\} .$$

Let us remark that for N sufficiently large, and for any pair n, m, as described above,

$$\left| \frac{\chi(\ell^*)}{\ell^*!} r^{\ell^*} \right| > \sum_{\ell=\ell^*+1}^{m} \left| \frac{\chi(\ell)}{\ell!} r^{\ell} \right|, \tag{2.3}$$

and hence the two partial sums

$$\sum_{\ell=1}^{n} \frac{\chi(\ell)}{\ell!} r^{\ell} \qquad \text{and} \qquad \sum_{\ell=1}^{m} \frac{\chi(\ell)}{\ell!} r^{\ell}$$

represent distinct rational numbers. Thus, the number

$$B_{n,m} := b^{m-n} m(m-1) \cdots (n+1) A_n - A_m \tag{2.4}$$

is a nonzero integer. Also,

$$|B_{n,m}| = \left| b^m m! \sum_{\ell=\ell^*}^{m} \frac{\chi(\ell) a^{\ell}}{\ell! b^{\ell}} \right| \ll \frac{m!}{n!} H(r)^m. \tag{2.5}$$

By (2.4) and (2.5), it follows that the greatest common divisor of A_n and A_m satisfies

$$(A_n, A_m) \ll \frac{m!}{n!} H(r)^m. \tag{2.6}$$

For each n, we set $D_n := (A_n, b^n n!)$. Then, by (2.2),

$$H\left(\sum_{\ell=1}^{n} \frac{\chi(\ell)}{\ell!} r^{\ell} \right) = H\left(\frac{A_n}{b^n n!} \right) = \max \left\{ \frac{|A_n|}{D_n}, \frac{b^n n!}{D_n} \right\} \sim \frac{b^n n!}{D_n} \max\{1, |G(r)|\}, \tag{2.7}$$

as $n \to \infty$. Furthermore, for any $n < m$, $n, m \in S$, $(D_n, D_m) \mid (A_n, A_m)$, and so, by (2.6),

$$(D_n, D_m) \ll \frac{m!}{n!} H(r)^m. \tag{2.8}$$

We now consider the least common multiple L_S of all integers D_n, $n \in S$. Since each such D_n is a divisor of $b^{[(1+\eta)N]}[(1+\eta)N]!$, it follows that

$$L_S \leq b^{[(1+\eta)N]}[(1+\eta)N]! \tag{2.9}$$

On the other hand, a lower bound for L_S is given by

$$L_S \geq \frac{\prod_{n \in S} D_n}{\prod_{\substack{n,m \in S \\ n<m}} (D_n, D_m)}. \tag{2.10}$$

Combining (2.8)–(2.10), setting $v = \#(S)$, and using Stirling's formula, we find that

$$\prod_{n\in S} D_n \ll_v b^{[(1+\eta)N]}[(1+\eta)N]! \prod_{\substack{n,m\in S\\ n<m}} \left(\frac{m!}{n!}H(r)^m\right) \tag{2.11}$$
$$\ll_\eta \left(\frac{2bN}{e}\right)^{(1+\eta)N} H(r)^{v^2(1+\eta)N} \prod_{\substack{n,m\in S\\ n<m}} \frac{m!}{n!}.$$

Next, by reasoning similar to that leading to (2.3), it follows that for each sufficiently large n,

$$\left|G(r) - \sum_{\ell=1}^{n} \frac{\chi(\ell)}{\ell!} r^\ell\right| = \left|\sum_{\ell>n} \frac{\chi(\ell)}{\ell!} r^\ell\right| \sim \frac{|r|^{\ell^*}}{\ell^*!} \geq \frac{|r|^{n+q}}{(n+q)!}, \tag{2.12}$$

as n tends to ∞, where

$$\ell^* = \min\{l > n : \chi(\ell) \neq 0\} \leq n + q.$$

For those $n \in S$, in addition, by (1.2),

$$\left|G(r) - \sum_{\ell=1}^{n} \frac{\chi(\ell)}{\ell!} r^\ell\right| < \left(H\left(\sum_{\ell=1}^{n} \frac{\chi(\ell)}{\ell!} r^\ell\right)\right)^{-1-\delta}. \tag{2.13}$$

From (2.7), (2.12), and (2.13), we deduce that for each $n \in S$,

$$\frac{|r|^{n+q}}{(n+q)!} \ll \left(H\left(\sum_{\ell=1}^{n} \frac{\chi(\ell)}{\ell!} r^\ell\right)\right)^{-1-\delta} \ll_{r,\delta} \left(\frac{D_n}{b^n n!}\right)^{1+\delta}. \tag{2.14}$$

This, in turn, implies that

$$D_n \gg_{r,\delta,q} |a|^{\frac{n}{1+\delta}} |b|^{n\left(1-\frac{1}{1+\delta}\right)} n^{-\frac{q}{1+\delta}} (n!)^{1-\frac{1}{1+\delta}} \tag{2.15}$$
$$\gg_{r,\delta,q} n^{\delta n/2} \geq N^{\delta n/2},$$

for any fixed δ, $0 < \delta < 1$, and N sufficiently large. If we employ (2.15) in (2.11), we find that

$$N^{\delta v N/2} \ll_{r,\delta,q,\eta,v} \left(\frac{2bN}{e}\right)^{(1+\eta)N} H(r)^{v^2(1+\eta)N} \prod_{\substack{n,m\in S\\ n<m}} \frac{m!}{n!}. \tag{2.16}$$

Observe that the product on the right-hand side of (2.16) can be written in the simplified form

$$\prod_{\substack{n,m\in S\\ n<m}} \frac{m!}{n!} = \prod_{n\in S} (n!)^{v-1-2\#\{m\in S:\, m>n\}}. \tag{2.17}$$

On the right-hand side of (2.17), we use the inequality $n! \geq N!$ for those $n \in S$ for which $v - 1 - 2\#\{m \in S : m > n\} < 0$, and the inequality $n! \leq [(1+\eta)N]!$ for those

$n \in S$ for which $v - 1 - 2\#\{m \in S : m > n\} > 0$. It follows that

$$\prod_{\substack{n,m\in S\\ n<m}} \frac{m!}{n!} \le \begin{cases} \left(\dfrac{[(1+\eta)N]!}{N!}\right)^{(v^2-1)/4}, & \text{if } v \text{ is odd,} \\ \left(\dfrac{[(1+\eta)N]!}{N!}\right)^{v^2/4}, & \text{if } v \text{ is even.} \end{cases} \tag{2.18}$$

Using Stirling's formula in (2.18), inserting (2.18) in (2.16), and taking logarithms on both sides, we find that

$$\begin{aligned} \frac{\delta v N}{2}\log N \le (1+\eta)N\log\frac{2bN}{e} + v^2(1+\eta)N\log H(r) \\ + \frac{v^2}{4}\left((1+\eta)N\log\frac{(1+\eta)N}{e} - N\log\frac{N}{e}\right) + O_{r,\delta,q,\eta,v}(1). \end{aligned} \tag{2.19}$$

Dividing both sides of (2.19) by $N\log N$ and letting $N\to\infty$ while keeping the size v of the set S fixed, we find that

$$\frac{\delta v}{2} \le (1+\eta) + \frac{v^2\eta}{4}. \tag{2.20}$$

The discriminant of the quadratic polynomial

$$\frac{\eta}{4}v^2 - \frac{\delta}{2}v + (1+\eta)$$

is equal to

$$D := \frac{\delta^2}{4} - \eta(1+\eta).$$

We now choose η small enough so that $D > 0$ and also so that there is at least one positive integer v for which (2.20) fails. For example, we can take

$$\eta = \frac{\delta^2}{8} \qquad \text{and} \qquad v = \left[\frac{4}{\delta}\right] + 1.$$

We therefore conclude that, with these values for η and v, there are not any large values of N for which $S^*_{\chi,r,\delta,\eta,N}$ contains a subset S of cardinality v. In other words,

$$\#(S^*_{\chi,r,\delta,\eta,N}) \le v - 1 = \left[\frac{4}{\delta}\right],$$

and so by (2.1),

$$\#(S_{\chi,r,\delta,\eta,N}) \ll q\left[\frac{4}{\delta}\right]. \tag{2.21}$$

Lastly, we take a large value of N and subdivide the interval $[1, N]$ into subintervals of the form

$$\left[M, \left(1+\frac{\delta^2}{8}\right)M\right],$$

and apply (2.21) to each subinterval. The number of such subintervals is $O(\log N)$, and so the proof of Theorem 1.2 is complete.

3. Proof of Theorem 1.3

The proof of Theorem 1.3 proceeds along the same lines as that for Theorem 1.2, and so we omit some details and focus on the differences between the two proofs. Let E, G_E, r, and δ, $0 < \delta < 1$, be as in the statement of Theorem 1.3. We fix a small $\eta > 0$, and for each large positive integer N, we let $S_{E,r,\delta,\eta,N}$ denote the set of integers n, in $[N, [(1+\eta)N]]$ for which $a_E(n) \neq 0$ and

$$\sum_{\ell=1}^{n} \frac{a_E(\ell)}{\ell!} r^\ell \in A_{G_E(r),\delta}. \tag{3.1}$$

We write

$$S_{E,r,\delta,\eta,N} = S'_{E,r,\delta,\eta,N} \cup S''_{E,r,\delta,\eta,N}, \tag{3.2}$$

where

$$S'_{E,r,\delta,\eta,N} = \left\{ n \in S_{E,r,\delta,\eta,N} : H\left(\sum_{\ell=1}^{n} \frac{a_E(\ell)}{\ell!} r^\ell \right) > N^{\left(1-\frac{1}{2}\delta\right)N} \right\} \tag{3.3}$$

and

$$S''_{E,r,\delta,\eta,N} = \left\{ n \in S_{E,r,\delta,\eta,N} : H\left(\sum_{\ell=1}^{n} \frac{a_E(\ell)}{\ell!} r^\ell \right) \leq N^{\left(1-\frac{1}{2}\delta\right)N} \right\}. \tag{3.4}$$

To proceed, let n be an arbitrary element of $S'_{E,r,\delta,\eta,N}$. Then, by (3.1) and (3.3),

$$\begin{aligned} \left| \sum_{\ell=n+1}^{\infty} \frac{a_E(\ell)}{\ell!} r^\ell \right| = \left| G_E(r) - \sum_{\ell=1}^{n} \frac{a_E(\ell)}{\ell!} r^\ell \right| &< \left(H\left(\sum_{\ell=1}^{n} \frac{a_E(\ell)}{\ell!} r^\ell \right) \right)^{-1-\delta} \\ &< \frac{1}{N^{\left(1-\frac{1}{2}\delta\right)(1+\delta)N}}. \end{aligned} \tag{3.5}$$

Let $\tilde{n}$ be the smallest integer $> n$ for which $a_E(\tilde{n}) \neq 0$. Then

$$\left| \frac{a_E(\tilde{n})}{\tilde{n}!} r^{\tilde{n}} \right| \geq \frac{|r|^{\tilde{n}}}{\tilde{n}!}.$$

On the other hand, by Hasse's bound [9], for any fixed $\epsilon > 0$, $|a_E(m)| \ll_\epsilon m^{\frac{1}{2}+\epsilon}$, for all positive integers m. Therefore,

$$\begin{aligned} \left| \sum_{\ell=n+1}^{\infty} \frac{a_E(\ell)}{\ell!} r^\ell \right| &= \frac{|a_E(\tilde{n})||r|^{\tilde{n}}}{\tilde{n}!} + O_\epsilon\left(\sum_{m \geq \tilde{n}+1} \frac{m^{\frac{1}{2}+\epsilon}|r|^m}{m!} \right) \\ &= \frac{|a_E(\tilde{n})||r|^{\tilde{n}}}{\tilde{n}!} \left(1 + O_{\epsilon,r}\left(\frac{1}{\tilde{n}^{\frac{1}{2}-\epsilon}} \right) \right) \\ &\gg \frac{|r|^{\tilde{n}}}{\tilde{n}!} \gg \frac{(|r|e)^{\tilde{n}}}{\tilde{n}^{\tilde{n}}\sqrt{\tilde{n}}}, \end{aligned} \tag{3.6}$$

by Stirling's formula. By (3.5) and (3.6),

$$\left(1 + \frac{\delta(1-\delta)}{2} \right) N \log N < \left(\tilde{n} + \frac{1}{2} \right) \log \tilde{n} + O_{|r|}(\tilde{n}). \tag{3.7}$$

The inequality (3.7) clearly cannot hold for N large enough if we choose $\eta < \delta(1-\delta)/2$, since $n < (1+\eta)N$ and, by a result of Balog and Ono [5] or of Alkan [1], the gap $\tilde{n}-n$ is $O(n^c)$, for some $c < 1$. In conclusion, for N sufficiently large, the set $S'_{E,r,\delta,\eta,N}$ is empty.

To bound $\#(S''_{E,r,\delta,\eta,N})$, we proceed as in the proof of Theorem 1.2. We take an arbitrary subset S of $S''_{E,r,\delta,\eta,N}$; set $v = \#(S)$; let $r = a/b$, with $a, b \in \mathbb{Z}$, $b \geq 1$, and $(a, b) = 1$; and for any $n \in S$, let $A_n \in \mathbb{Z}$ be defined by

$$\sum_{\ell=1}^{n} \frac{a_E(\ell)}{\ell!} r^\ell = \frac{A_n}{b^n n!}. \tag{3.8}$$

The reasoning leading to (3.6) above also shows, in particular, that for any two distinct $n, m \in S''_{E,r,\delta,\eta,N}$,

$$\sum_{\ell=1}^{n} \frac{a_E(\ell)}{\ell!} r^\ell \neq \sum_{\ell=1}^{m} \frac{a_E(\ell)}{\ell!} r^\ell.$$

Thus, the integers

$$B_{n,m} := b^{n-m} m(m-1)\cdots(n+1)A_n - A_m \neq 0.$$

As before, denoting $D_n = (A_n, b^n n!)$, we see that

$$(D_n, D_m) \leq (A_n, A_m) \leq |B_{n,m}| = b^m m! \left| \sum_{\ell=n+1}^{m} \frac{a_E(\ell) a^\ell}{\ell! b^\ell} \right| \ll_\epsilon \frac{m!}{n!} m^{\frac{1}{2}+\epsilon} H(r)^m. \tag{3.9}$$

Also, by (3.8), in analogy with (2.7), as $n \to \infty$,

$$H\left(\sum_{\ell=1}^{n} \frac{a_E(\ell)}{\ell!} r^\ell\right) \sim \frac{b^n n!}{D_n} \max\{1, G_E(r)\}, \tag{3.10}$$

which, for $n \in S''_{E,r,\delta,\eta,N}$, along with Stirling's formula and (3.4), gives

$$D_n \gg_r \frac{b^n n!}{N^{\left(1-\frac{1}{2}\delta\right)N}} \geq N^{\delta N/2} \left(\frac{b}{e}\right)^N, \tag{3.11}$$

for sufficiently large N. As in the previous proof, we let L_S denote the least common multiple of the integers D_n, with $n \in S$, and deduce the inequalities

$$\frac{\prod\limits_{n\in S} D_n}{\prod\limits_{\substack{n,m\in S\\ n<m}} (D_n, D_m)} \leq L_S \leq b^{[(1+\eta)N]}[(1+\eta)N]! \ll \left(\frac{2bN}{e}\right)^{(1+\eta)N}. \tag{3.12}$$

By (3.9), (3.11), and (3.12), it follows that

$$N^{\delta v N/2} \left(\frac{b}{e}\right)^{vN} \ll_{v,\epsilon,r} \left(\frac{2bN}{e}\right)^{(1+\eta)N} N^v H(r)^{v^2(1+\eta)N} \prod_{\substack{n,m\in S\\ n<m}} \left(\frac{m!}{n!}\right), \tag{3.13}$$

which we combine with (2.18) and Stirling's formula to obtain

(3.14)
$$\frac{\delta v N}{2}\log N + vN\log\frac{b}{e} \le (1+\eta)N\log\frac{2bN}{e} + v\log N + v^2(1+\eta)N\log H(r) + \frac{v^2}{4}\left((1+\eta)N\log\frac{(1+\eta)N}{e} - N\log\frac{N}{e}\right) + O_{r,v,\epsilon}(1).$$

For large N, (3.14) implies the same inequality as in (2.20), namely,

$$\frac{\delta v}{2} \le (1+\eta) + \frac{v^2\eta}{4}. \tag{3.15}$$

By the same sort of reasoning that gave (2.21), we conclude that

$$\#(S''_{E,r,\delta,\eta,N}) = O_\delta(1),$$

for N sufficiently large. Since $S'_{E,r,\delta,\eta,N}$ is empty for N sufficiently large, we finally deduce that

$$\#(S_{E,r,\delta,\eta,N}) = O_\delta(1),$$

for all $N \ge N_{E,r,\delta}$ for some $N_{E,r,\delta}$, which depends only on E, r, and δ, where η is a fixed small positive number depending only on δ. We now subdivide the interval $[1, N]$, just as we did in the proof of Theorem 1.2, to deduce (1.11) and complete the proof of Theorem 1.3.

4. Proof of Theorem 1.6

Our goal in this section is to prove Theorem 1.6.

For brevity, set $\boldsymbol{a} = \{a_1, a_2, \ldots, a_p\}$ and $\boldsymbol{b} = \{b_1, b_2, \ldots, b_q\}$. Let $\boldsymbol{a}$, $\boldsymbol{b}$, r, δ, and α be given as in the statement of Theorem 1.6. First, fix a small $\eta > 0$, which will depend only on δ. Let N be a large integer, and denote by $S_{\boldsymbol{a},\boldsymbol{b},r,\delta,\eta,N}$ the set of integers n in the interval $[N, (1+\eta)N]$ for which

$$\sum_{\ell=0}^{n} \frac{(a_1)_\ell (a_2)_\ell \cdots (a_p)_\ell}{(b_1)_\ell (b_2)_\ell \cdots (b_q)_\ell}\frac{r^\ell}{\ell!} \in A_{\alpha,\delta}.$$

Let S be an arbitrary nonempty subset of $S_{\boldsymbol{a},\boldsymbol{b},r,\delta,\eta,N}$ and let $v = \#(S)$. Uniformly, for $n \ge N$,

$$\left|\alpha - \sum_{\ell=0}^{n} \frac{(a_1)_\ell (a_2)_\ell \cdots (a_p)_\ell}{(b_1)_\ell (b_2)_\ell \cdots (b_q)_\ell}\frac{r^\ell}{\ell!}\right| = \left|\frac{(a_1)_{n+1}(a_2)_{n+1}\cdots(a_p)_{n+1}}{(b_1)_{n+1}(b_2)_{n+1}\cdots(b_q)_{n+1}}\frac{r^{n+1}}{(n+1)!}\right|\left(1 + O_{\boldsymbol{a},\boldsymbol{b},r}\left(\frac{1}{N^{1+q-p}}\right)\right), \tag{4.1}$$

from which it follows that

$$\left|\alpha - \sum_{\ell=0}^{n} \frac{(a_1)_\ell (a_2)_\ell \cdots (a_p)_\ell}{(b_1)_\ell (b_2)_\ell \cdots (b_q)_\ell}\frac{r^\ell}{\ell!}\right| = \frac{1}{n^{n(q-p+1+o(1))}}. \tag{4.2}$$

As in our previous arguments, it follows that for N sufficiently large, and any distinct integers m and n with $m, n \ge N$, the partial sums with upper indices n and m are distinct. Again, as in our foregoing proofs, for $n \in S$ and n sufficiently large,

$$(4.3)\quad H\left(\sum_{\ell=0}^{n} \frac{(a_1)_\ell (a_2)_\ell \cdots (a_p)_\ell}{(b_1)_\ell (b_2)_\ell \cdots (b_q)_\ell} \frac{r^\ell}{\ell!}\right) < n^{\frac{q-p+1+o(1)}{1+\delta} n}$$
$$< N^{\frac{(1+\eta)(q-p+1+o(1))}{1+\delta} N} < N^{\left(1-\frac{1}{2}\delta\right)(q-p+1)N},$$

for $0 < \delta < 1$, η sufficiently small in terms of δ, and N sufficiently large.

Next, for each ℓ, $1 \le \ell \le N$, we write

$$(4.4)\qquad \frac{(a_1)_\ell (a_2)_\ell \cdots (a_p)_\ell}{(b_1)_\ell (b_2)_\ell \cdots (b_q)_\ell} \frac{r^\ell}{\ell!} = \frac{E_\ell}{F_\ell},$$

where $E_\ell, F_\ell \in \mathbb{Z}$, $F_\ell > 0$, and $(E_\ell, F_\ell) = 1$. For any positive integer n, such that $\ell \le n$, we set

$$(4.5)\qquad B_n := [F_0, F_1, \ldots, F_n].$$

Then the partial sum with upper index n can be written as a fraction (not necessarily irreducible) of the form

$$(4.6)\qquad \sum_{\ell=0}^{n} \frac{(a_1)_\ell (a_2)_\ell \cdots (a_p)_\ell}{(b_1)_\ell (b_2)_\ell \cdots (b_q)_\ell} \frac{r^\ell}{\ell!} = \frac{A_n}{B_n},$$

with $A_n \in \mathbb{Z}$. Note that by (4.5), each $B_n \mid B_m$ for $m > n$. Therefore, the numbers

$$(4.7)\qquad T_{n,m} := A_m - \frac{B_m A_n}{B_n} = B_m \sum_{\ell=n+1}^{m} \frac{(a_1)_\ell (a_2)_\ell \cdots (a_p)_\ell}{(b_1)_\ell (b_2)_\ell \cdots (b_q)_\ell} \frac{r^\ell}{\ell!}$$

are nonzero integers.

It now behoves us to produce upper bounds for $T_{n,m}$. Let us remark that in the particular case when $b_1 = a_1, \ldots, b_p = a_p$, $b_{p+1} = \cdots = b_q = 1$, and $r = 1$, the left-hand side of (4.4) reduces to $(\ell!)^{p-q-1}$. Thus, in this case, $E_\ell = 1$ $F_\ell = (\ell!)^{q-p+1}$, (4.5) becomes $B_n = (n!)^{q-p+1}$, and (4.7) reduces to

$$(4.8)\quad T_{n,m} = A_m - (m(m-1)\cdots(n+1))^{q-p+1} A_n = (m!)^{q-p+1} \sum_{\ell=n+1}^{m} \frac{1}{(\ell!)^{q-p+1}},$$

in which case

$$(4.9)\qquad |T_{n,m}| < \left(\frac{m!}{n!}\right)^{q-p+1}.$$

We will derive bounds almost as accurate as (4.9) in the general case. In order to achieve this, we show that B_n is always close to $(n!)^{q-p+1}$ in the sense that

$$\frac{B_n}{(n!)^{q-p+1}}$$

is a rational number of relatively small height.

To proceed, let us set

$$a_1 = \frac{c_1}{d_1}, \ldots, a_p = \frac{c_p}{d_p}, \quad b_1 = \frac{e_1}{f_1}, \ldots, b_q = \frac{e_q}{f_q}, \quad r = \frac{a}{b},$$

with $d_1, \ldots, d_p, f_1, \ldots, f_q, b \geq 1$, $(c_j, d_j) = 1$, $1 \leq j \leq p$, $(e_j, f_j) = 1$, $1 \leq j \leq q$, and $(a, b) = 1$. Set

$$K := \max\{H(a_1), \ldots, H(a_p), H(b_1), \ldots, H(b_q), H(r)\}.$$

For each positive integer ℓ, denote

$$M(\ell) := \operatorname{lcm}\{1, 2, \ldots, \ell\}.$$

Note that by the prime number theorem,

$$M(\ell) < \ell^{\pi(\ell)} = e^{\ell(1+O(1/\log \ell))}. \tag{4.10}$$

For each $\ell \in \mathbb{N}$ and each $j \in \{1, 2, \ldots, q\}$, the denominator of $(b_j)_\ell$, in irreducible form, equals f_j^ℓ. The numerator of $(b_j)_\ell$, which is $e_j(e_j + f_j) \cdots (e_j + (\ell - 1)f_j)$, is "almost" divisible by $\ell!$, in the sense that it is divisible by the largest divisor of $\ell!$ that is relatively prime to f_j. We also see that $e_j(e_j + f_j) \cdots (e_j + (\ell - 1)f_j)$ is a divisor of $(\ell!)M(K\ell)$. Therefore, the rational number $(b_j)_\ell/\ell!$, in its irreducible form has a numerator that divides $M(K\ell)$ and a denominator with each of its prime factors dividing f_j. This denominator divides $f_j^{2\ell}$. Applying the foregoing reasoning to each b_j, $1 \leq j \leq q$, and to each a_j, $1 \leq j \leq p$, we find that the rational number

$$(a_1)_\ell \cdots (a_p)_\ell ((b_1)_\ell)^{-1} \cdots ((b_q)_\ell)^{-1} (\ell!)^{q-p},$$

in irreducible form, has a denominator that divides $M(K\ell)^q d_1^{2\ell} \cdots d_p^{2\ell}$. Multiplying this rational number by $r^\ell (\ell!)^{p-q-1}$, and putting the result in irreducible form, we obtain E_ℓ/F_ℓ, as originally defined in (4.4). Therefore,

$$E_\ell \mid a^\ell M(K\ell)^p f_1^{2\ell} \cdots f_q^{2\ell} \tag{4.11}$$

and

$$F_\ell \mid b^\ell M(K\ell)^q d_1^{2\ell} \cdots d_p^{2\ell} (\ell!)^{q-p+1}. \tag{4.12}$$

Using (4.12) for each $\ell \leq n$, in conjunction with (4.5), we find that

$$B_n \mid b^n M(Kn)^q d_1^{2n} \cdots d_p^{2n} (n!)^{q-p+1}. \tag{4.13}$$

Next, for ℓ sufficiently large, by Stirling's formula,

$$|(a_i)_\ell| < \ell!\, \ell^{\lceil a_i \rceil}, \quad \text{for each } i \in \{1, \ldots, p\}, \tag{4.14}$$

and

$$|(b_j)_\ell| > \ell!\, \ell^{\lfloor b_j \rfloor - 2}, \quad \text{for each } j \in \{1, \ldots, q\}. \tag{4.15}$$

Therefore, for N sufficiently large and any $n, m \in S$ with $n < m$,

$$\begin{aligned} \left| \sum_{\ell=n+1}^{m} \frac{(a_1)_\ell \ldots (a_p)_\ell r^\ell}{(b_1)_\ell \ldots (b_q)_\ell \ell!} \right| &< \sum_{\ell=n+1}^{m} \frac{|r|^\ell \ell^{2q+\sum_{i=1}^p \lceil a_i \rceil - \sum_{j=1}^q \lfloor b_j \rfloor}}{(\ell!)^{q-p+1}} \\ &\leq \frac{2|r|^n n^{2q+\sum_{i=1}^p \lceil a_i \rceil - \sum_{j=1}^q \lfloor b_j \rfloor}}{(n!)^{q-p+1}}. \end{aligned} \tag{4.16}$$

Combining (4.7), (4.10), (4.13), and (4.16), we find that

$$
\begin{aligned}
(4.17)\qquad |T_{n,m}| &\le 2b^m M(Km)^q d_1^{2m}\cdots d_p^{2m}|r|^n n^{2q+\sum_{i=1}^{p}\lceil a_i\rceil-\sum_{j=1}^{q}\lfloor b_j\rfloor}\left(\frac{m!}{n!}\right)^{q-p+1}\\
&\le \left(\frac{m!}{n!}\right)^{q-p+1} e^{K_1N},
\end{aligned}
$$

for some constant K_1 depending only on $a_1,\dots,a_p$, $b_1,\dots,b_q$, r, and η.

Let us denote for each n, $D_n=(A_n,B_n)$, and let L_S denote the least common multiple of those D_n with $n\in S$. Then

$$
(4.18)\qquad L_S \ge \frac{\prod\limits_{n\in S} D_n}{\prod\limits_{\substack{n,m\in S\\ n<m}}(D_n,D_m)}.
$$

Also, if n^* denotes the largest element of S, then since for each $n\in S$, $D_n\mid B_n$, and since $B_n\mid B_{n^*}$, by (4.13),

$$
\begin{aligned}
(4.19)\qquad L_S &\le b^{n^*}M(Kn^*)^q d_1^{2n^*}\cdots d_p^{2n^*}(n^*!)^{q-p+1}\\
&\le N^{(1+\eta)(q-p+1)N}e^{K_2N},
\end{aligned}
$$

for some constant K_2 depending only on $a_1,\dots,a_p$, $b_1,\dots,b_q$, r, and η.

For each $n,m\in S$ with $n<m$, (D_n,D_m) divides (A_n,A_m), which divides $A_m-(B_m/B_n)A_n$, and hence by (4.7) and (4.17),

$$
(4.20)\qquad (D_n,D_m)\le |T_{n,m}|\le \left(\frac{m!}{n!}\right)^{q-p+1} e^{K_1N}.
$$

From (4.20) and (2.18), we deduce that

$$
\begin{aligned}
(4.21)\qquad \prod_{\substack{n,m\in S\\ n<m}}(D_n,D_m) &\le e^{\frac12 v(v-1)K_1N}\left(\prod_{\substack{n,m\in S\\ n<m}}\frac{m!}{n!}\right)^{q-p+1}\\
&\le e^{\frac12 v(v-1)K_1N}\left(\frac{\lfloor(1+\eta)N\rfloor!}{N!}\right)^{\frac14 v^2(q-p+1)}\\
&\le N^{\frac14\eta v^2(q-p+1)N}e^{v^2K_3N},
\end{aligned}
$$

for some constant K_3 depending only on $a_1,\dots,a_p$, $b_1,\dots,b_q$, r, and η.

Lastly, since

$$
H\left(\frac{A_n}{B_n}\right)=\max\left\{\left|\frac{A_n}{D_n}\right|,\left|\frac{B_n}{D_n}\right|\right\},
$$

by (4.6) and (4.3), it follows that, for each $n\in S$,

$$
(4.22)\qquad \frac{B_n}{D_n}\le H\left(\frac{A_n}{B_n}\right)\le N^{\left(1-\frac12\delta\right)(q-p+1)N}.
$$

Also, recall that the reasoning that led us to (4.11)–(4.12) showed that E_ℓ/F_ℓ equals $r^\ell(\ell!)^{p-q-1}$ times a fraction whose numerator divides $M(K\ell)^p f_1^{2\ell}\cdots f_q^{2\ell}$ and whose denominator divides $M(K\ell)^q d_1^{2\ell}\cdots d_p^{2\ell}$. In the particular case when $\ell = n$, this gives

$$\frac{E_n}{F_n} = \frac{\mu_n}{(n!)^{q-p+1}}, \tag{4.23}$$

with $\mu_n \in \mathbb{Q}$ and $H(\mu_n) \le e^{K_4 n}$, for some constant K_4 depending only on $a_1, \dots, a_p$, $b_1, \dots, b_q$, and r. Since F_n and E_n are relatively prime, (4.23) implies that

$$F_n e^{K_4 n} \ge (n!)^{q-p+1}.$$

Since $F_n \le B_n$, we conclude that, for any $n \ge N$,

$$B_n \ge \frac{(n!)^{q-p+1}}{e^{K_4 n}} \ge \frac{(N!)^{q-p+1}}{e^{K_4 N}} \ge \frac{N^{(q-p+1)N}}{e^{K_5 N}}, \tag{4.24}$$

for some constant K_5 depending only on $a_1, \dots, a_p$, $b_1, \dots, b_q$, and r. By (4.22) and (4.24), we see that, for each $n \in S$,

$$D_n \ge N^{\frac{1}{2}\delta(q-p+1)N} e^{-K_5 N}. \tag{4.25}$$

Finally, by (4.18), (4.19), (4.21), and (4.25), we deduce that

$$\frac{N^{\frac{1}{2}v\delta(q-p+1)N} e^{-vK_5 N}}{N^{\frac{1}{4}\eta v^2(q-p+1)N} e^{v^2 K_3 N}} \le L_S \le N^{(1+\eta)(q-p+1)N} e^{K_2 N}. \tag{4.26}$$

Taking logarithms on both sides of (4.26), dividing both sides by $(q-p+1)N\log N$, and letting $N \to \infty$ while keeping v fixed, we find that

$$\frac{v\delta}{2} \le \frac{\eta v^2}{4} + (1+\eta). \tag{4.27}$$

The inequality (4.27) fails, for example, for

$$\eta = \frac{\delta^2}{8} \quad \text{and} \quad v = \left\lfloor \frac{4}{\delta} \right\rfloor + 1.$$

Thus, there are at most $\lfloor 4/\delta \rfloor$ elements in $S_{\boldsymbol{a},\boldsymbol{b},r,\delta,\eta,N}$ for the chosen value of η, and by the same kind of argument that we have used previously in this paper, this completes the proof of the theorem.

5. Proof of Theorem 1.8

Fix a real number $\delta > 0$. Next fix a number

$$\delta_1 > \max\{\delta, 1\}, \tag{5.1}$$

and define a sequence of integers $1 = N_1 < N_2 < N_3 < \cdots$ by

$$N_k := \lfloor (1+\delta_1)^{k-1} \rfloor, \qquad k = 1, 2, \dots. \tag{5.2}$$

Next, we define a sequence of integers $\{a_n\}$, $n \in \mathbb{N}$, with $1 \le a_n \le n$, as follows. For any $n \ge 1$, we let

$$a_n = \begin{cases} n, & \text{if } n = N_k \text{ for some } k, \\ n-1, & \text{otherwise.} \end{cases} \tag{5.3}$$

It follows that for any $k > 1$,

$$\sum_{\ell=N_{k-1}+1}^{N_k} \frac{a_\ell}{\ell!} = \frac{1}{(N_k-1)!} + \sum_{\ell=N_{k-1}+1}^{N_k-1} \frac{\ell-1}{\ell!} = \frac{1}{N_{k-1}!}. \tag{5.4}$$

Therefore the partial sum

$$\sum_{\ell=1}^{N_k} \frac{a_\ell}{\ell!} = \frac{B_k}{N_{k-1}!} \in \mathbb{Q},$$

where B_k and $N_{k-1}!$ are not necessarily relatively prime. Thus, its height satisfies the inequality

$$H\left(\sum_{\ell=1}^{N_k} \frac{a_\ell}{\ell!}\right) \ll N_{k-1}!. \tag{5.5}$$

On the other hand,

$$\alpha - \sum_{\ell=1}^{N_k} \frac{a_\ell}{\ell!} = \sum_{\ell=N_k+1}^{\infty} \frac{a_\ell}{\ell!} \le \sum_{\ell=N_k+1}^{\infty} \frac{1}{(\ell-1)!} \ll \frac{1}{N_k!}. \tag{5.6}$$

So, by (5.1), (5.2), (5.5), (5.6), and Stirling's formula, for k sufficiently large,

$$\alpha - \sum_{\ell=1}^{N_k} \frac{a_\ell}{\ell!} < \frac{1}{(N_k/e)^{N_k}} < \frac{1}{(N_{k-1}!)^{1+\frac{1}{2}(\delta+\delta_1)}} < \left(H\left(\sum_{\ell=1}^{N_k} \frac{a_\ell}{\ell!}\right)\right)^{-1-\delta}. \tag{5.7}$$

Hence, we can conclude that

$$\sum_{\ell=1}^{N_k} \frac{a_\ell}{\ell!} \in A_{\alpha,\delta}.$$

Lastly, for N sufficiently large, there are $C_1 \log N$ numbers N_k in the interval $[1, N]$, where $C_1 > 0$ is a constant depending only on δ. This completes the proof of Theorem 1.8.

In order to prove Corollary 1.9, we choose δ and δ_1 such that $1 < \delta < \delta_1 < e - 1$, and follow the proof of Theorem 1.8. Then the rate of growth of $\{N_k\}$ is slower than that of e^k, and for each sufficiently large N, there will be more than $C \log N$ such N_k's in the interval $[1, N]$. Also, since the corresponding partial sums

$$\sum_{\ell=1}^{N_k} \frac{a_\ell}{\ell!} \in A_{\alpha,\delta},$$

they will be convergents to the continued fraction of α. Lastly, the transcendence of α follows from the Thue–Siegel–Roth Theorem [12]. By that theorem, if α is algebraic, then each set $A_{\alpha,\delta}$, with $\delta > 1$, is finite, which is not the case here.

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