

ALEXANDROV IMMERSED MINIMAL TORI IN S^3

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ABSTRACT. We show that any minimal torus in S^3 which is Alexandrov immersed must be rotationally symmetric. An analogous result holds for surfaces of constant mean curvature.

1. Introduction

In a recent paper [2], we showed the Clifford torus is the only embedded minimal surface in S^3 of genus 1, thereby confirming a conjecture of Lawson [8]. In this note, we classify minimal tori in S^3 that are immersed in the sense of Alexandrov.

Theorem 1. *Let $F : \Sigma \rightarrow S^3$ be an immersed minimal surface in S^3 of genus 1. Moreover, we assume that F is an Alexandrov immersion; this means that there exists a compact manifold N and an immersion $\bar{F} : N \rightarrow S^3$ such that $\partial N = \Sigma$ and $\bar{F}|_\Sigma = F$. Then Σ is rotationally symmetric. That is, we can find an anti-symmetric matrix $Q \in \mathfrak{so}(4)$ of rank 2 such that $QF(x) \in \text{span}\{\frac{\partial F}{\partial x_1}(x), \frac{\partial F}{\partial x_2}(x)\}$ for all $x \in \Sigma$.*

There is a complete classification of all rotationally symmetric minimal tori in S^3 ; for details, we refer to [9] or [4], Theorem 1.4. Besides the Clifford torus, there is a large class of additional examples which are Alexandrov immersed but fail to be embedded.

We will present the proof of Theorem 1 in Section 2. The argument is similar in spirit to the case of embedded surfaces studied in [2], and we will only indicate the necessary modifications.

It was observed by Andrews and Li [1] that the arguments in [2] can be extended to the setting of constant mean curvature surfaces. As a result, they showed that every embedded constant mean curvature surface in S^3 is rotationally symmetric. More recently, we showed that the arguments in [2] can be generalized to a class of Weingarten surfaces (see [3]).

Our proof of Theorem 1 also extends to the setting of constant mean curvature surfaces. This yields the following result:

Theorem 2. *Let $F : \Sigma \rightarrow S^3$ be an immersed constant mean curvature surface in S^3 of genus 1. Suppose that F extends to an immersion $\bar{F} : N \rightarrow S^3$ where $\partial N = \Sigma$ and that ∂N is mean convex with respect to the pull-back of the standard metric on S^3 under \bar{F} . Then Σ is rotationally symmetric.*

The proof of Theorem 2 is similar to Theorem 1. The condition that the surface is Alexandrov immersed is natural in light of the work of Korevaar et al. [5], Korevaar, Kusner, and Solomon [6], and Kusner et al. [7], where Alexandrov immersed constant mean curvature surfaces in \mathbb{R}^3 have been studied.

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2. Proof of Theorem 1

For convenience, we put a Riemannian metric on N so that \bar{F} is a local isometry. In particular, there exists a real number $\delta > 0$ so that $\bar{F}(x) \neq \bar{F}(y)$ for all points $x, y \in N$ satisfying $d_N(x, y) \in (0, \delta)$.

For each point $x \in \Sigma$ and any number $\alpha \geq 1$, we define

$$D_\alpha(x) = \left\{ p \in S^3 : \frac{\alpha}{\sqrt{2}} |A(x)| (1 - \langle F(x), p \rangle) + \langle \nu(x), p \rangle \leq 0 \right\}.$$

Note that $D_\alpha(x)$ is a geodesic ball in S^3 whose boundary passes through the point $F(x)$. Moreover, the outward-pointing unit normal vector to $\partial D_\alpha(x)$ at the point $F(x)$ is given by $\nu(x)$.

Let I denote the set of all points $(x, \alpha) \in \Sigma \times [1, \infty)$ with the property that there exists a smooth map $G : D_\alpha(x) \rightarrow N$ such that $\bar{F} \circ G = \text{id}_{D_\alpha(x)}$ and $G(F(x)) = x$.

Lemma 3. *Let us fix a pair $(x, \alpha) \in I$. Then there is a unique map $G : D_\alpha(x) \rightarrow N$ such that $\bar{F} \circ G = \text{id}_{D_\alpha(x)}$ and $G(F(x)) = x$.*

Proof. Suppose that there are two maps $G, \tilde{G} : D_\alpha(x) \rightarrow N$ such that $\bar{F} \circ G = \bar{F} \circ \tilde{G} = \text{id}_{D_\alpha(x)}$ and $G(F(x)) = \tilde{G}(F(x)) = x$. For each point $p \in D_\alpha(x)$, we have $\bar{F}(G(p)) = \bar{F}(\tilde{G}(p))$, hence $d_N(G(p), \tilde{G}(p)) \notin (0, \delta)$. By continuity, we either have $G(p) = \tilde{G}(p)$ for all $p \in D_\alpha(x)$ or we have $G(p) \neq \tilde{G}(p)$ for all $p \in D_\alpha(x)$. The second case can be ruled out, as $G(F(x)) = \tilde{G}(F(x))$. Thus, we conclude that $G(p) = \tilde{G}(p)$ for all $p \in D_\alpha(x)$. This shows that G is unique. \square

Lemma 4. *The set I is closed. Furthermore, the map G depends continuously on the pair (x, α) .*

Proof. Consider a sequence of pairs $(x^{(m)}, \alpha^{(m)}) \in I$ such that $\lim_{m \rightarrow \infty} (x^{(m)}, \alpha^{(m)}) = (\hat{x}, \hat{\alpha})$. By definition of I , we can find a sequence of smooth maps $G^{(m)} : D_{\alpha^{(m)}}(x^{(m)}) \rightarrow N$ such that $\bar{F} \circ G^{(m)} = \text{id}_{D_{\alpha^{(m)}}(x^{(m)})}$ and $G^{(m)}(F(x^{(m)})) = x^{(m)}$. Since \bar{F} is a smooth immersion, the maps $G^{(m)}$ are uniformly bounded in C^2 norm. Hence, after passing to a subsequence, the maps $G^{(m)}$ converge in C^1 to a map $G : D_{\hat{\alpha}}(\hat{x}) \rightarrow N$. The map G satisfies $\bar{F} \circ G = \text{id}_{D_{\hat{\alpha}}(\hat{x})}$ and $G(F(\hat{x})) = \hat{x}$. From this, we deduce that G is smooth. Thus, $(\hat{x}, \hat{\alpha}) \in I$, and the assertion follows. \square

Lemma 5. *We have $(x, \alpha) \in I$ if α is sufficiently large.*

Proof. By a result of Lawson, we have $|A(x)| > 0$ for all $x \in \Sigma$. Hence, if we choose α sufficiently large, the radius of the geodesic ball $D_\alpha(x) \subset S^3$ can be made arbitrarily small. Therefore, if α is sufficiently large, the implicit function theorem guarantees the existence of a smooth map $G : D_\alpha(x) \rightarrow N$ such that $\bar{F} \circ G = \text{id}_{D_\alpha(x)}$ and $G(F(x)) = x$.

After these preparations, we now describe the proof of Theorem 1. Let us define

$$\kappa = \inf\{\alpha : (x, \alpha) \in I \text{ for all } x \in \Sigma\}.$$

Clearly, $\kappa \in [1, \infty)$. For each point $x \in \Sigma$, there is a unique map $G_x : D_\kappa(x) \rightarrow N$ such that $\bar{F} \circ G_x = \text{id}_{D_\kappa(x)}$ and $G_x(F(x)) = x$. For each point $x \in \Sigma$, the map G_x and the map $\bar{F}|_{G_x(D_\kappa(x))}$ are one-to-one. \square

We next define a smooth function $Z : \Sigma \times \Sigma \rightarrow \mathbb{R}$ by

$$Z(x, y) = \frac{\kappa}{\sqrt{2}} |A(x)| (1 - \langle F(x), F(y) \rangle) + \langle \nu(x), F(y) \rangle$$

for $x, y \in \Sigma$. In contrast to [2], the function $Z(x, y)$ might be negative somewhere.

As in [2], we distinguish two cases:

Case 1: Suppose first that $\kappa = 1$.

Lemma 6. *We have $Z(x, y) \geq 0$ if x and y are sufficiently close.*

Proof. The proof is by contradiction. Suppose that there exist two sequences of points $x^{(m)}, y^{(m)} \in \Sigma$ such that $\lim_{m \rightarrow \infty} x^{(m)} = \lim_{m \rightarrow \infty} y^{(m)}$ and $Z(x^{(m)}, y^{(m)}) < 0$ for all m . Since $Z(x^{(m)}, y^{(m)}) < 0$, the point $F(y^{(m)})$ lies in the interior of the geodesic ball $D_\kappa(x^{(m)})$. Since $G_{x^{(m)}}$ is an immersion, the point $\tilde{y}^{(m)} := G_{x^{(m)}}(F(y^{(m)}))$ must lie in the interior of N . Since $y^{(m)}$ lies on the boundary Σ , it follows that

$$\tilde{y}^{(m)} \neq y^{(m)}.$$

On the other hand, we have

$$\bar{F}(\tilde{y}^{(m)}) = F(y^{(m)})$$

and

$$\begin{aligned} \lim_{m \rightarrow \infty} \tilde{y}^{(m)} &= \lim_{m \rightarrow \infty} G_{x^{(m)}}(F(y^{(m)})) = \lim_{m \rightarrow \infty} G_{x^{(m)}}(F(x^{(m)})) = \lim_{m \rightarrow \infty} x^{(m)} \\ &= \lim_{m \rightarrow \infty} y^{(m)}. \end{aligned}$$

This is impossible since \bar{F} is an immersion. \square

Lemma 7. *Fix a point $x \in \Sigma$, and let $\{e_1, e_2\}$ be an orthonormal basis of $T_x \Sigma$ such that $h(e_1, e_1) > 0$, $h(e_1, e_2) = 0$, and $h(e_2, e_2) < 0$. Then $e_1(|A|) = 0$.*

Proof. The proof is analogous to arguments in [2]. Let $\gamma : \mathbb{R} \rightarrow \Sigma$ be a geodesic such that $\gamma(0) = x$ and $\gamma'(0) = e_1$. Since the function Z is non-negative when x and y are sufficiently close, the function $f(t) = Z(x, \gamma(t))$ is non-negative when t is sufficiently small. Since $\kappa = 1$, we have $f(0) = f'(0) = f''(0) = 0$. Since the function $f(t)$ is non-negative in a neighborhood of 0, we conclude that $f'''(0) = 0$. From this, we deduce that $(D_{e_1}^\Sigma h)(e_1, e_1) = 0$. From this, the assertion follows. \square

Lemma 7 implies that the function $|A|$ is constant along one family of curvature lines on Σ . We claim that F is rotationally symmetric. To see this, we define a vector field V on Σ by $V = |A|^{-\frac{1}{2}} e_1$. The identity $e_1(|A|) = 0$ implies that $[V, e_1] = 0$. Moreover, we compute

$$D_{e_1}^\Sigma e_2 = -\frac{1}{2|A|} e_2(|A|) e_1$$

and

$$D_{e_2}^\Sigma e_1 = 0.$$

This implies

$$[e_1, e_2] = D_{e_1}^\Sigma e_2 - D_{e_2}^\Sigma e_1 = -\frac{1}{2|A|} e_2(|A|) e_1,$$

hence

$$[V, e_2] = |A|^{-\frac{1}{2}} [e_1, e_2] - e_2(|A|^{-\frac{1}{2}}) e_1 = 0.$$

Using the identities $[V, e_1] = [V, e_2] = 0$, it is straightforward to check that $\mathcal{L}_V g = \mathcal{L}_V h = 0$. Therefore, V is the restriction of an ambient Killing vector field on S^3 . Hence, we can find a constant matrix $Q \in \mathfrak{so}(4)$ such that $V(x) = Q F(x)$ for all $x \in \Sigma$. We now differentiate this relation along e_2 . Since $g(e_2, V) = h(e_2, V) = 0$, we obtain the relation $D_{e_2}^\Sigma V = Q e_2$. Since $D_{e_2}^\Sigma V = e_2(|A|^{-\frac{1}{2}}) e_1$, we conclude that

$$Q \left(e_2 + \frac{1}{2|A|} e_2(|A|) F(x) \right) = Q e_2 + \frac{1}{2|A|^{\frac{3}{2}}} e_2(|A|) e_1 = 0.$$

Therefore, the matrix Q has non-trivial nullspace. Thus, Q has rank 2 and F is rotationally symmetric.

Case 2: Suppose next that $\kappa > 1$.

Lemma 8. *Fix a point $x \in \Sigma$. Then there exists a constant $\beta > 0$ such that $d_N(G_x(p), \Sigma) \geq \beta |p - F(x)|^2$ for all points $p \in \partial D_\kappa(x)$ that are sufficiently close to $F(x)$.*

Proof. Let us fix a point $x \in \Sigma$. Moreover, we consider the function

$$\varphi_x : \partial D_\kappa(x) \rightarrow \mathbb{R}, \quad p \mapsto d_N(G_x(p), \Sigma).$$

Clearly, $\varphi_x(F(x)) = 0$, and the gradient of the function φ_x at the point $F(x)$ vanishes. Moreover, since $\kappa > 1$, the Hessian of the function φ_x at the point $F(x)$ is positive definite. Hence, we can find a positive constant $\beta > 0$ such that $\varphi_x(p) \geq \beta |p - F(x)|^2$ for all points $p \in \partial D_\kappa(x)$ that are sufficiently close to $F(x)$. \square

Lemma 9. *There exists a point $\hat{x} \in \Sigma$ such that $\Sigma \cap G_{\hat{x}}(D_\kappa(\hat{x})) \neq \{\hat{x}\}$.*

Proof. Suppose this is false. Then $\Sigma \cap G_x(D_\kappa(x)) = \{x\}$ for all $x \in \Sigma$. This implies that $d_N(G_x(p), \Sigma) > 0$ for all $x \in \Sigma$ and all points $p \in \partial D_\kappa(x) \setminus \{F(x)\}$. Using Lemma 8, we conclude that there exists a positive constant $\gamma > 0$ such that $d_N(G_x(p), \Sigma) \geq \gamma |p - F(x)|^2$ for all $x \in \Sigma$ and all $p \in \partial D_\kappa(x)$. By the implicit function theorem, there exists a small number $\varepsilon > 0$ such that $(x, \kappa - \varepsilon) \in I$ for all $x \in \Sigma$. This contradicts the definition of κ . \square

Let \hat{x} be chosen as in Lemma 9. Moreover, let us pick a point $\hat{y} \in \Sigma \cap G_{\hat{x}}(D_\kappa(\hat{x}))$ such that $\hat{x} \neq \hat{y}$. Since $\hat{y} \in G_{\hat{x}}(D_\kappa(\hat{x}))$, we conclude that $F(\hat{y}) \in D_\kappa(\hat{x})$ and $G_{\hat{x}}(F(\hat{y})) = \hat{y}$. Moreover, we claim that $F(\hat{x}) \neq F(\hat{y})$; indeed, if $F(\hat{x}) = F(\hat{y})$, then $\hat{x} = G_{\hat{x}}(F(\hat{x})) = G_{\hat{x}}(F(\hat{y})) = \hat{y}$, which contradicts our choice of \hat{y} .

We next consider the function Z defined above. We claim that the function Z is non-negative in a neighborhood of the point (\hat{x}, \hat{y}) .

Lemma 10. *We have $Z(x, y) \geq 0$ if (x, y) is sufficiently close to (\hat{x}, \hat{y}) .*

Proof. We argue by contradiction. Suppose that there exist sequences of points $x^{(m)}, y^{(m)} \in \Sigma$ such that $\lim_{m \rightarrow \infty} x^{(m)} = \hat{x}$, $\lim_{m \rightarrow \infty} y^{(m)} = \hat{y}$, and $Z(x^{(m)}, y^{(m)}) < 0$. Since $Z(x^{(m)}, y^{(m)}) < 0$, the point $F(y^{(m)})$ lies in the interior of the ball $D_\kappa(x^{(m)})$. Since $G_{x^{(m)}}$ is an immersion, we conclude that the point $\tilde{y}^{(m)} := G_{x^{(m)}}(F(y^{(m)}))$ lies in the interior of N . In particular, we have

$$\tilde{y}^{(m)} \neq y^{(m)}.$$

\square

On the other hand, we have

$$\bar{F}(\tilde{y}^{(m)}) = F(y^{(m)})$$

and

$$\lim_{m \rightarrow \infty} \tilde{y}^{(m)} = \lim_{m \rightarrow \infty} G_{x^{(m)}}(F(y^{(m)})) = G_{\hat{x}}(F(\hat{y})) = \hat{y} = \lim_{m \rightarrow \infty} y^{(m)}.$$

This contradicts the fact that \bar{F} is an immersion. Thus, $Z(x, y) \geq 0$ for (x, y) close to (\hat{x}, \hat{y}) .

Therefore, we can find disjoint open sets $U, V \subset \Sigma$ such that $\hat{x} \in U$, $\hat{y} \in V$, and $Z(x, y) \geq 0$ for all points $(x, y) \in U \times V$. As in [2], we define

$$\Omega = \{x \in U : \text{there exists a point } y \in V \text{ such that } Z(x, y) = 0\}.$$

Since $F(\hat{y}) \in D_\kappa(\hat{x})$, we have $Z(\hat{x}, \hat{y}) \leq 0$. This implies that $Z(\hat{x}, \hat{y}) = 0$, hence $\hat{x} \in \Omega$. We can now use the calculation in [2] to conclude that Z is a supersolution of a degenerate elliptic equation. More precisely, suppose that (\bar{x}, \bar{y}) is an arbitrary point in $U \times V$. Then, we can find a system of geodesic normal coordinates (x_1, x_2) around \bar{x} and a system of geodesic normal coordinates (y_1, y_2) around \bar{y} such that

$$\begin{aligned} & \sum_{i=1}^2 \frac{\partial^2 Z}{\partial x_i^2}(\bar{x}, \bar{y}) + 2 \sum_{i=1}^2 \frac{\partial^2 Z}{\partial x_i \partial y_i}(\bar{x}, \bar{y}) + \sum_{i=1}^2 \frac{\partial^2 Z}{\partial y_i^2}(\bar{x}, \bar{y}) \\ & \leq -\frac{\kappa^2 - 1}{\sqrt{2}\kappa} \frac{|A(\bar{x})|}{1 - \langle F(\bar{x}), F(\bar{y}) \rangle} \sum_{i=1}^2 \left\langle \frac{\partial F}{\partial x_i}(\bar{x}), F(\bar{y}) \right\rangle^2 \\ & + \Lambda \left(Z(\bar{x}, \bar{y}) + \sum_{i=1}^2 \left| \frac{\partial Z}{\partial x_i}(\bar{x}, \bar{y}) \right| + \sum_{i=1}^2 \left| \frac{\partial Z}{\partial y_i}(\bar{x}, \bar{y}) \right| \right), \end{aligned}$$

where Λ is a positive constant. Using Bony's version of the strict maximum principle, we conclude that the set Ω contains an open neighborhood of \hat{x} . Moreover, the gradient of $|A|$ vanishes on the set Ω . By analytic continuation, $|A|$ is a constant function on Σ . This implies that F is congruent to the Clifford torus. This completes the proof of Theorem 1.

3. Sketch of the proof of Theorem 2

Finally, let us sketch the proof of Theorem 2. Let $F : \Sigma \rightarrow S^3$ be an immersed constant mean curvature surface in S^3 of genus 1. We assume that F extends to an immersion $\bar{F} : N \rightarrow S^3$ where $\partial N = \Sigma$ and that ∂N is mean convex with respect to the pull-back of the standard metric on S^3 under \bar{F} . Given a point $x \in \Sigma$ and a real number $\alpha \geq 1$, one defines

$$D_\alpha(x) = \left\{ p \in S^3 : \left(\frac{H}{2} + \frac{\alpha}{\sqrt{2}} |\mathring{A}(x)| \right) (1 - \langle F(x), p \rangle) + \langle \nu(x), p \rangle \leq 0 \right\}$$

(cf. [1]). Here, H is the mean curvature (i.e., the sum of the principal curvatures) and \mathring{A} denotes the trace-free part of the second fundamental form. As above, let I denote the set of all points $(x, \alpha) \in \Sigma \times [1, \infty)$ with the property that there exists a smooth map $G : D_\alpha(x) \rightarrow N$ such that $\bar{F} \circ G = \text{id}_{D_\alpha(x)}$ and $G(F(x)) = x$. It is well known that a constant mean curvature torus in S^3 has no umbilic points, so $|\mathring{A}(x)| > 0$ for all points $x \in I$. Hence, if α is sufficiently large, then the radius of the geodesic ball

$D_\alpha(x)$ will be very small. From this, we deduce that $(x, \alpha) \in I$ if α is sufficiently large. We then define

$$\kappa = \inf\{\alpha : (x, \alpha) \in I \text{ for all } x \in \Sigma\}.$$

If $\kappa = 1$, we can argue as above to conclude that $e_1(|A|) = 0$, where e_1 is one of the eigenvectors of the second fundamental form. This implies that F is rotationally symmetric. On the other hand, if $\kappa > 1$, we can combine the arguments above with the calculations in [1] and [2] to arrive at a contradiction.

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