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ON THE TORSION OF CHOW GROUPS OF TWISTED SPIN-FLAGS

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Abstract. In the present paper, we provide a uniform bound for the annihilators of torsion of Chow groups of the variety of Borel subgroups of a strongly inner linear algebraic group of orthogonal type.

1. Introduction

Let X be the variety of Borel subgroups of a semisimple linear algebraic group G over an arbitrary field k. One way to study the geometry of X is to study its Chow group $\mathrm{CH}^d(X)$ of codimension d cycles modulo the rational equivalence relation. Since the rank of the free part of $\mathrm{CH}^d(X)$ can be easily determined by counting the number of cells in the Bruhat decomposition over the closure and then analyzing action of the Galois group, the problem of describing $\mathrm{CH}^d(X)$ reduces to the problem of determining its torsion part.

The latter seems to be a highly non-trivial question. Only very few partial results are known and most of them concern small codimensions ($d \leq 4$). For strongly inner groups and $d = 2$, it is shown in [4] that the torsion part of $\mathrm{CH}^2(X)$ is cyclic of order dividing the Dynkin index of G, and for $d = 3, 4$, an annihilator of the torsion of $\mathrm{CH}^d(X)$ is provided in [1]. For quadrics, [8] and [9] contain several results on the torsion of CH^d, where $d = 2, 3, 4$. For example, it is shown [9, Section 5, Section 6] that the torsion of CH^4 has order at most 4 if the dimension of the quadric is ≥ 9, and that of CH^3 has order at most 2; on the other hand, there is an example of quadric of dimension 5 with infinitely generated torsion part of CH^4.

In the present paper we provide a uniform bound for the annihilator of the torsion of $\mathrm{CH}^d(X)$ for any d, $1 \leq d \leq 2n-3$, and strongly inner orthogonal group G of rank n, i.e., the *Spin*-group of a quadratic form with trivial discriminant and trivial Clifford invariant. Observe that X doesn't depend on the isogeny class of G. Namely, we prove the following:

Theorem 1.1. *Let G be a strongly inner group of an orthogonal type of rank n over an arbitrary field. Let X be the respective variety of Borel subgroups. Then for all $1 \leq d \leq 2n-3$ the integer*

$$M_d = (d-1)! \prod_{i=2}^{d} (i-1)! \cdot [i/2]! \cdot 2^{i+1}$$

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annihilates the torsion part of $\mathrm{CH}^d(X)$.

Observe that for groups of types A and C there are no non-split strongly inner forms, so the torsion part of $\mathrm{CH}^d(X)$ is trivial. Observe also that there is only finite number of exceptional types. Hence, the natural case to consider is that of orthogonal groups (types B and D).

In our proofs, we essentially rely on the fact that the $K_0(X)$ of a strongly inner form X does not change after going over the closure [12], this is why we restrict ourselves to strongly inner orthogonal groups. Observe that the only known estimate for the annihilator of the torsion of $\mathrm{CH}^d(X)$ is given by the torsion index t_G of G [5, Definition 3]. For orthogonal groups, e.g., for *Spin groups*, it was computed in [14, Theorem 01] and has the property that $t_G \to \infty$ as the rank of G grows. We would like to stress that our bound M_d does not depend on the rank but on the codimension d only.

The paper is organized as follows. In Sections 2 and 3, we prove several technical facts (Proposition 2.9 and Corollary 3.3) concerning the ideals of generalized invariants and symmetric functions. These facts are used in Section 4 to relate the kernel of the characteristic map with the ideal of invariants (Propositions 4.5 and 4.7). In Section 5, we extend the results of the paper [1] by providing a uniform upper bound for all the exponents of the Weyl group action (Proposition 5.8). In Section 6, we combine the obtained results to prove the main theorem (Theorem 6.2).

2. Divided differences and ideals of invariants

In the present section, we provide several basic facts concerning the ring of symmetric polynomials over an arbitrary commutative ring and the associated invariant ideals. We refer to [7, 10] for details.

2.1. Consider a polynomial ring $R = A[e_1, \dots, e_n]$ over a commutative ring A. The symmetric group S_n acts on R by permutations of variables $\{e_1, \dots, e_n\}$. The subring of invariants R^{S_n} is a polynomial ring in elementary symmetric functions

$$s_1 = e_1 + \cdots + e_n,\ s_2 = \sum_{i<j} e_i e_j,\ s_3 = \sum_{i<j<k} e_i e_j e_k,\ \dots, s_n = \prod_{i=1}^{n} e_i.$$

Let $J = (s_1, s_2, \dots, s_n)$ denote the ideal of R generated by symmetric functions. We denote by $\epsilon\colon R \to A$ the augmentation map $e_i \mapsto 0$. Observe that ϵ restricts to $\epsilon\colon R^{S_n} \to A$.

2.2. Following [7, Section 0] consider divided difference operators Δ_σ, $\sigma \in S_n$. Each of them is an A-linear operator $\Delta_\sigma\colon R^{(m)} \to R^{(m-l(\sigma))} \cup \{0\}$ decreasing the degree m of a homogeneous polynomial in $e_1, \dots, e_n$ by the length $l(\sigma)$ of permutation σ. It is defined as follows:

We set $\Delta_1 = \mathrm{id}$. If $m < l(\sigma)$, then we set $\Delta_\sigma = 0$. For a (non-trivial) transposition $\tau = (ij)$, we set $\Delta_\tau(f) = (f - f^\tau)/(e_i - e_j)$ for $f \in R^{(m)}$, $m \geq 1$. If σ is a product of transpositions, we define Δ_σ to be the composite of the respective Δ_τ. This does not depend on the choice of a reduced decomposition of σ. Observe that if s is a symmetric polynomial, then

$$\Delta_\sigma(s \cdot f) = s \cdot \Delta_\sigma(f).$$

By definition we have

$$R^{S_n} = \{f \in R \mid \Delta_\tau(f) = 0 \text{ for all non-trivial transpositions } \tau\}.$$

Definition 2.3. Following [7, p.239] we define the ideal of *generalized invariants* I as

$$I = \{f \in R \mid \Delta_\sigma(f) = 0 \quad \forall \sigma \in S_n \text{ with } l(\sigma) = \deg(f)\}.$$

Lemma 2.4. *We have $J \subseteq I$.*

Proof. Follows from the fact that $\Delta_\sigma(s_i \cdot f) = s_i \cdot \Delta_\sigma(f) = 0$ for any $s_i \cdot f \in J^{(m)}$ and $l(\sigma) = m$. □

Definition 2.5. Following [10, Section 1] we define the ideal of *stable invariants* J_∞ inductively as:

Set $R_1 := R$ and $R_{m+1} := R_m \otimes_{R_m^{S_n}} A$ for $m \geq 1$, where A is the $R_m^{S_n}$-module via the augmentation ϵ. Observe that S_n acts on R_{m+1} via the action on R_m and there is a canonical S_n-equivariant surjection $R_m \to R_{m+1}$. Let J_m denote the kernel of the composite $R_1 \to \cdots \to R_m \to R_{m+1}$, i.e., $R_{m+1} = R/J_m$. We then set

$$J_\infty := \bigcup_{m \geq 1} J_m.$$

Remark 2.6. The ideal J_m can be also defined inductively as follows:

$$J_m = \begin{cases} (0) & \text{if } m = 0, \\ \{f \in R \mid \Delta_\sigma(f) \in J_{m-1}, \ \forall \sigma \in S_n\} & \text{if } m \geq 1. \end{cases}$$

Lemma 2.7 (*cf.* **[10, Lemma 3.2]**)**.** *We have $I \subseteq J_\infty$.*

Proof. We show by induction on m that $I^{(m)} \subseteq J_m$.

If $m = 1$ and $f \in I^{(1)}$, then $\Delta_\tau(f) = 0$ for all $\tau \neq 1$ with $l(\tau) = 1$, implies that $f \in R^{S_n} \cap \ker \epsilon$. Therefore, $f \in J_1$.

Suppose that $I^{(m)} \subseteq J_m$. For $m+1$ and $f \in I^{(m+1)}$, if $\Delta_\sigma(f) = 0$ for any reduced decomposition $\sigma = \tau_1\tau_2 \ldots \tau_{m+1}$, then $\Delta_{\tau_1 \ldots \tau_m} \Delta_{\tau_{m+1}}(f) = 0$. So by induction $\Delta_{\tau_{m+1}}(f) \in I^{(m)} \subseteq J_m$ and, therefore, $\Delta_\tau(f) \in J_m$ for all $\tau \neq 1$ with $l(\tau) = 1$. By the remark above, $f \in J_{m+1}$. □

The ideal J_∞ is universal in the following sense

Lemma 2.8 (*cf.* **[10 Lemma 2.1]**)**.** *Let $J' \subset R$ be an S_n-stable ideal with $\epsilon(J') = 0$. If $(R/J')^{S_n} = A$, then $J_\infty \subseteq J'$.*

Proof. We prove $J_m \subset J'$ by induction on m. If $m = 1$, since $(R/J')^{S_n} = A$, the compositions $R^{S_n} \hookrightarrow R \to R/J'$ and $R^{S_n} \xrightarrow{\epsilon} A \hookrightarrow R/J'$ coincide, hence, there is a map $R/J_1 = R \otimes_{R^{S_n}} A \to R/J'$, which shows that $J_1 \subset J'$.

Now assume that $J_m \subset J'$. Repeating the above arguments after replacing R (resp. J') by $R_m = R/J_m$ (resp. by the ideal $J'_m = (J')$ in R_m), we see that $J_{m+1} \subset J$. This completes the proof. □

Proposition 2.9. *We have $J = I = J_\infty$ in the polynomial ring $R = A[e_1, \ldots, e_n]$.*

Proof. Since $R^{S_n} = A \otimes \mathbb{Z}[e_1, \ldots, e_n]^{S_n}$, $(R/J)^{S_n} = A$. Therefore, by Lemma 2.8, $J_\infty \subseteq J$. The proposition then follows by combining Lemmas 2.4 and 2.7. □

3. Elementary symmetric functions and power sums

In the present section, we prove several technical lemmas that will be used in the subsequent section.

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ denote a partition with $1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_m$ of an integer $|\alpha| = \alpha_1 + \cdots + \alpha_m$. We set $|(0)| = 0$. Let $s_\alpha = \prod_{i=1}^m s_{\alpha_i}$ denote the product of respective elementary symmetric functions.

Lemma 3.1. *Let d be a positive integer and $d_0 = \min\{d, n\}$. Consider a homogeneous polynomial in variables $e_1, \ldots, e_n$ of degree d*

$$P_m = \sum_{\{\alpha \mid d_0 - m \le |\alpha| \le d_0\}} f^\alpha s_\alpha,$$

where $0 \le m < d_0$, $f^\alpha \in \mathbb{Z}[e_1, \ldots, e_n]$, and $\deg(f^\alpha) = d - |\alpha|$.

If $M \mid P_m$ for some positive integer M, then for each α there exists a homogeneous polynomial v^α of degree $d - |\alpha|$ such that

$$\sum_{\{\alpha | d_0 - m \le |\alpha| \le d_0\}} v^\alpha s_\alpha = 0 \text{ and } M \mid (f^\alpha + v^\alpha).$$

Proof. Assume that $d \le n$, then $d_0 = d$. We proceed by induction on $m \ge 0$.

If $m = 0$, then $P_0 = \sum_{|\alpha| = d} f^\alpha s_\alpha$, $f^\alpha \in \mathbb{Z}$. If $M \mid P_0$, then $M \mid f^\alpha$ for each α as $\{s_\alpha\}_{|\alpha|=d}$ are linearly independent in $A[e_1, \ldots, e_n]$ for $A = \mathbb{Z}/M\mathbb{Z}$. So we can take $v^\alpha = 0$ for each α.

Assume it is true for $m-1$, $m \ge 1$. Now we prove it for m. We use β to denote those partitions α with $|\alpha| = d - m$. Observe that $\deg(f^\beta) = m$. Applying to P_m a divided difference operator Δ of length m, we obtain:

$$M \mid \Delta(P_m) = \sum_\beta \Delta(f^\beta) s_\beta, \text{ where } \Delta(f^\beta) \in \mathbb{Z}.$$

Similar to $m = 0$ case this implies that $M \mid \Delta(f^\beta)$ for every Δ of length m. By Definition 2.3 this means that $f^\beta \in I$, where I is the ideal of generalized invariants for $A = \mathbb{Z}/M\mathbb{Z}$. By Proposition 2.9, we have $J = I$ and, therefore,

$$f^\beta \equiv \sum_{j=1}^m g^\beta_{m-j} s_j \mod M, \text{ for some polynomial } g^\beta_{m-j} \text{ of degree } m - j.$$

Plugging it into the original expression for P_m, we obtain

$$P_m \equiv \sum_{\{\alpha | d-m+1 \le |\alpha| \le d\}} (f^\alpha + g^{\beta_\alpha}_{d-|\alpha|}) s_\alpha \equiv 0 \mod M,$$

where β_α is the unique β such that $s_\alpha = s_{|\alpha| - |\beta_\alpha|} s_{\beta_\alpha}$. By induction, for each α such that $|\alpha| \ge d - m + 1$ there exists a polynomial v^α such that

$$\sum_{\{\alpha | d-m+1 \le |\alpha| \le d\}} v^\alpha s_\alpha = 0 \text{ and } M \mid (f^\alpha + g^{\beta_\alpha}_{d-|\alpha|} + v^\alpha).$$

Now we set $\tilde{v}^\beta = -\sum_{j=1}^m g^\beta_{m-j} s_j$ and $\tilde{v}^\alpha = g^{\beta_\alpha}_{d-|\alpha|} + v^\alpha$ for $|\alpha| \ge d - m + 1$. Then $\sum_{d-m \le |\alpha| \le d} \tilde{v}^\alpha s_\alpha = 0$. So these $\tilde{v}^\alpha$ satisfy the condition of the lemma.

If $d > n$, then $d_0 = n$, and the proof is similar. This completes the proof. □

3.1. Let $q_i = e_1^i + e_2^i + \cdots + e_n^i$, $i \geq 1$ denote the power sum symmetric function. Given a partition $\alpha = (\alpha_1, \ldots, \alpha_m)$, let $q_\alpha = \prod_{i=1}^m q_{\alpha_i}$ denote the product of respective power sum functions.

According to [11, Chapter I. 2.11] the elementary symmetric function s_i can be written in terms of q_j, $j \leq i$ as $s_i = \frac{1}{i!}\sum_{|\alpha|=i} a_\alpha q_\alpha$, $a_\alpha \in \mathbb{Z}$. Since $i! = \max_{\sum i_j = i}\{i_j!\}$, we have

$$s_\alpha = \frac{1}{|\alpha|!}\sum_{\{\beta,\ |\beta|=|\alpha|\}} a_{\alpha,\beta} q_\beta\ , \text{ for some } a_{\alpha,\beta} \in \mathbb{Z}.$$

Multiplying by the respective denominators, we obtain the following version of Lemma 3.1 for power sum functions:

Corollary 3.2. *Assume that $d_0 = \min\{d, n\}$ and $d_0! \mid M$. Consider a homogeneous polynomial $P = \sum_{i=1}^{d_0} f_{d-i} q_i$ of degree d with integer coefficients ($\deg(f_{d-i}) = d - i$).*

If $M \mid P$, then there exist $\hat{f}_{d-i}$, $i = 1 \ldots d_0$ such that

$$\sum_{i=1}^{d_0} \hat{f}_{d-i} q_i = P \text{ and } \tfrac{M}{d_0!} \mid \hat{f}_{d-i}.$$

Proof. Let $q_i = \sum_{|\alpha|=i} c_\alpha s_\alpha$, $c_\alpha \in \mathbb{Z}$, so we get $M \mid P = \sum_{|\alpha|=1}^{|\alpha|=d_0} c_\alpha f_{d-|\alpha|} s_\alpha$. By Lemma 3.1, there exist v^α with $P = \sum (c_\alpha f_{d-|\alpha|} + v^\alpha) s_\alpha$ and $(c_\alpha f_{d-|\alpha|} + v^\alpha)$ all divisible by M. Expressing s_α in terms of q_β using the formula from 3.1 we obtain

$$P = \tfrac{1}{d_0!}\sum_\alpha (c_\alpha f_{d-|\alpha|} + v^\alpha) \sum_{|\beta|=|\alpha|} a_{\alpha,\beta} q_\beta = \tfrac{1}{d_0!}\sum_{i=1}^{d_0} \tilde{f}_{d-i} q_i \text{ for some } \tilde{f}_{d-i}.$$

Now all these $\tilde{f}_{d-i}$ are integral polynomials in $c_\alpha f_{d-|\alpha|} + v^\alpha$ and, hence, are divisible by M. Therefore, $\hat{f}_{d-i} \stackrel{\text{def}}{=} \frac{1}{d_0!}\tilde{f}_{d-i}$ is divisible by $\frac{M}{d_0!}$. □

Restricting to power sums of even degree only we obtain

Corollary 3.3. *Assume that $d_0 = \min\{2n, d\}$ and let $[d_0/2]! \mid M$. Consider a homogeneous polynomial $P = \sum_{i=1}^{[d_0/2]} f_{d-2i} q_{2i}$ of degree d with integer coefficients.*

If $M \mid P$, then there exist $\hat{f}_{d-2i}$, $i = 1, \ldots, [d_0/2]$ such that

$$\sum_{i=1}^{[d_0/2]} \hat{f}_{d-2i} q_{2i} = P \text{ and } \tfrac{M}{[d_0/2]!} \mid \hat{f}_{d-2i}.$$

Proof. For each i we express f_{d-2i} as a linear combination

$$f_{d-2i} = \sum_\delta e^\delta f_{d-2i}^\delta,$$

where $\delta = (\delta_1, \ldots, \delta_n)$ with $\delta_i = 0, 1$, $e^\delta = \prod_{i=1}^n e_i^{\delta_i}$ and f_{d-2i}^δ is a linear combination of even monomials $e_1^{2i_1} e_2^{2i_2} \cdots e_n^{2i_n}$ only. Denote $|\delta| = \sum \delta_j$, $d^\delta = \frac{d-|\delta|}{2}$ and $d_0^\delta = \min\{d^\delta, n\} \leq [d_0/2]$. Collecting the terms with e^δ we obtain

$$P = \sum_\delta e^\delta \sum_{i=1}^{d_0^\delta} f_{d-2i}^\delta q_{2i} \equiv 0 \mod M.$$

This implies that $M \mid \sum_{i=1}^{d_0^\delta} f_{d-2i}^\delta q_{2i}$ for each δ. We apply Corollary 3.2 to the polynomial $P_\delta = \sum_{i=1}^{d_0^\delta} f_{d-2i}^\delta q_{2i}$ viewed as a polynomial in variables e_j^2 of degree d^δ. We obtain polynomials $\hat{f}_{d-2i}^\delta$ divisible by $\frac{M}{d_0^\delta!}$ and $\sum_{i=1}^{d_0^\delta} f_{d-2i}^\delta q_{2i} = \sum_{i=1}^{d_0^\delta} \hat{f}_{d-2i}^\delta q_{2i}$.

We then set

$$\hat{f}_{d-2i} = \sum_\delta e^\delta \hat{f}_{d-2i}^\delta.$$

Since $d_0^\delta \leq [\frac{d_0}{2}]$ for all δ, the proof is completed. $\square$

4. Invariants and characteristic map

In the present section, we investigate the relationships between the kernel of the characteristic map and the ideal of invariants. We refer to [2] for basic definitions and details.

4.1. Consider a crystallographic root system of Dynkin type $\mathfrak{D}$ with the weight lattice Λ. Let G be the associated split simple simply-connected linear algebraic group with a split maximal torus T and a Borel subgroup $B \supset T$. Observe that Λ can be identified with the group of characters of T with a basis given by the fundamental weights $\omega_1, \ldots, \omega_n$.

Consider the variety G/B of Borel subgroups of G. To every character $\lambda \in \Lambda$ we may associate the line bundle $\mathcal{L}(\lambda)$ over G/B. It induces the ring homomorphism from the symmetric algebra $S^*(\Lambda)$ to the Chow ring of G/B called the *characteristic map* [2, Section 8]

$$c_a \colon S^*(\Lambda) \to \mathrm{CH}^*(G/B), \quad \lambda \mapsto c_1(\mathcal{L}(\lambda)).$$

The order of the cokernel of $c_a^N : S^N(\Lambda) \to \mathrm{CH}^N(G/B)$, where $N = \dim G/B$, is called the *torsion index* of G [2, Section 5]. Its prime divisors are called the *torsion primes* of G. In particular, the torsion index of a group of type A and C is 1, and of type B and D is a power of 2.

4.2. The Weyl group W of G acts on the weight lattice Λ by means of simple reflections and, hence, on $S^*(\Lambda)$. Let I_a^W be the ideal of $S^*(\Lambda)$ generated by non-constant W-invariants. According to [2, Section 4 Corollary 2], $\ker c_a \subset I_a^W$ and

$$\ker c_a \otimes_{\mathbb{Z}} \mathbb{Q} \cong I_a^W \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Moreover, for each homogeneous degree d there exists an smallest integer b_d such that the prime decomposition of b_d consists only of torsion primes of G and

$$b_d \cdot (\ker c_a)^{(d)} \subseteq (I_a^W)^{(d)}.$$

The ideal $(\ker c_a) \otimes \mathbb{Z}[\frac{1}{t_G}] = I_a^W \otimes \mathbb{Z}[\frac{1}{t_G}]$, where t_G is the torsion index of G, is freely generated by the basic polynomial invariants, which are homogeneous polynomials in fundamental weights.

The purpose of the present section is to get an upper bound for the integers b_d.

Example 4.3. For the type $\mathfrak{D} = \mathrm{A}_n$ the torsion index is 1. This implies that $b_d = 1$ for all d and the characteristic map c_a is surjective with the kernel $\ker c_a = I_a^W$.

Example 4.4. According to [6, p. 68] the basic polynomial invariants for the root system of type $\mathfrak{D} = \mathrm{B}_n$ are given by even power sums

$$q_{2i} = \sum_{j=1}^{n} e_j^{2i}, \quad 1 \le i \le n,$$

where $e_1 = \omega_1$, $e_j = \omega_j - \omega_{j-1}$ for $2 \le j < n$ and $e_n = 2\omega_n - \omega_{n-1}$. Therefore,

$$q_{2i} = \omega_1^{2i} + (\omega_2 - \omega_1)^{2i} + \cdots + (\omega_{n-1} - \omega_{n-2})^{2i} + (2\omega_n - \omega_{n-1})^{2i}.$$

The basic polynomial invariants for type D_n are given by q_{2i}, $1 \le i \le n-1$ and $p_n = e_1 e_2 \dots e_n$, where $e_1 = \omega_1$, $e_j = \omega_j - \omega_{j-1}$ for $2 \le j \le n-1$ and $e_{n-1} = \omega_n - \omega_{n-1}, e_n = \omega_n + \omega_{n-1} - \omega_{n-2}$.

Assume now that $\mathfrak{D} = \mathrm{B}_n$. In this case, the torsion index of G is a power of 2.

Proposition 4.5. *Let $\mathfrak{D} = \mathrm{B}_n$ and $d_0 = \min\{d, 2n\}$. Then $2^d[d_0/2]!\cdot(\ker c_a)^{(d)} \subset I_a^W$, i.e., $b_d \mid 2^d[d_0/2]!$.*

Proof. Since $(\ker c_a) \otimes \mathbb{Z}[\frac{1}{2}] = I_a^W \otimes \mathbb{Z}[\frac{1}{2}]$ is generated by q_{2i}, $i = 1, \dots, n$, given a polynomial $f \in (\ker c_a)^{(d)}$, we can write it as

$$2^r f = \sum_{i=1}^{[d_0/2]} f_{d-2i} q_{2i} \in I_a^W, \text{ for some } f_{d-2i} \in \mathbb{Z}[\omega_1, \dots, \omega_n] \text{ and } r \ge 0.$$

Suppose r is the smallest such integer. To finish the proof it suffices to show that $r \le v_2([d_0/2]!) + d$, where v_2 denotes the 2-adic valuation.

Assume the contrary, i.e., that $r \ge v_2([d_0/2]!) + d + 1$. Expressing ω_j's in terms of e_j's, we obtain $f = \frac{1}{2^d}\tilde{f}$ and $f_{d-2i} = \frac{1}{2^{d-2i}}\tilde{f}_{d-2i}$ for some $\tilde{f}, \tilde{f}_{d-2i} \in \mathbb{Z}[e_1, \dots, e_n]$. So that

$$M = 2^{d+1}[d_0/2]! \mid 2^r s \cdot \tilde{f} = \sum_{i=1}^{[d_0/2]} (2^{2i} s \tilde{f}_{d-2i}) \cdot q_{2i}, \text{ where } s = \tfrac{[d_0/2]!}{2^{v_2([d_0/2]!)}}.$$

By Corollary 3.3, there exists $\tilde{h}_{d-2i} \in \mathbb{Z}[e_1, \dots, e_n]$ divisible by $\frac{M}{[d_0/2]!} = 2^{d+1}$ such that $2^r s \cdot \tilde{f} = \sum_{i=1}^{[d_0/2]} \tilde{h}_{d-2i} q_{2i}$. Expressing e_j's in terms of ω_j's back, we obtain $2^d 2^r s \cdot f = \sum_{i=1}^{[d_0/2]} \tilde{h}_{d-2i} q_{2i}$, which implies

$$2^{r-1} f = \sum_{i=1}^{[d_0/2]} \left(\tfrac{1}{2^{d+1}} \tilde{h}_{d-2i} - \tfrac{s-1}{2} f_{d-2i}\right) \cdot q_{2i}.$$

Since $\tilde{h}_{d-2i}$ are divisible by 2^{d+1}, we have $\frac{1}{2^{d+1}}\tilde{h}_{d-2i} \in \mathbb{Z}[\omega_1, \dots, \omega_n]$. This contradicts to the minimality assumption on r. □

Assume now that $\mathfrak{D} = \mathrm{D}_n$. In this case, we have an additional basic polynomial invariant p_n in degree n, however, this does not change the situation much in view of the following slight modifications of Corollaries 3.2 and 3.3:

Lemma 4.6. *Suppose that $d \ge n$.*

(1) *Assume that $c \le n-1$ and $c!|M$. If $M \mid \sum_{i=1}^{c} f_{d-i} q_i + g_{d-n} p_n$, then there exists v_{d-i} and u_{d-n} such that $\sum_{i=1}^{c} v_{d-i} q_i + u_{d-n} p_n = 0$ and $\frac{M}{c!} \mid g.c.d.(f_{d-i} + v_{d-i}, g_{d-n} + u_{d-n})$.*

(2) *Denote* $d_0 = \min\{d, 2n-2\}$ *and suppose that* $[d_0/2]!|M$. *If* $M \mid \sum_{i=1}^{[d_0/2]} f_{d-2i}q_{2i} + g_{d-n}p_n$, *then we can find* v_{d-2i}, u_{d-n} *such that*

$$\sum_{i=1}^{[d_0/2]} v_{d-2i}q_{2i} + u_{d-n}p_n = 0 \text{ and } \frac{M}{[d_0/2]!} \mid g.c.d.(f_{d-2i} + v_{d-2i}, g_{d-n} + u_{d-n}).$$

Proof. The proof of (1) follows by the same arguments as the proof of Corollary 3.2, noticing that $p_n = e_1e_2 \ldots e_n = s_n$.

As for (2), multiplying by p_n we obtain

$$M \mid \sum_{i=1}^{[d_0/2]} h_{d+n-2i}q_{2i} + g_{d-n}p_n^2, \text{ where } h_{d+n-2i} = f_{d-2i}p_n.$$

Following the proof of 3.3, we can rewrite this as

$$M \mid \sum_{\delta} e^{\delta} \left(\sum_{i=1}^{[d_0/2]} h^{\delta}_{d+n-2i}q_{2i} + g^{\delta}_{d-n}p_n^2 \right)$$

and reduce it to (1) (replacing by $e'_j = e_j^2$). $\square$

Proposition 4.7. *Let* $\mathfrak{D} = D_n$. *Denote* $d_0 = \min\{d, 2n-2\}$. *Then*

$$2^d[d_0/2]! \cdot (\ker c_a)^{(d)} \subset I_a^W.$$

Proof. If $d < n$, then it is similar to the B_n-case. If $d \geq n$, consider the equation

$$2^r f = \sum_{i=1}^{[d_0/2]} f_{d-2i}q_{2i} + g_{d-n}p_n \in I_a^W, f_{d-2i}, g_{d-n} \in \mathbb{Z}[\omega_1, \ldots, \omega_n].$$

Following the proof of Proposition 4.5 and using Lemma 4.6, we show that the smallest 2^r satisfying the equation will divide $2^d[d_0/2]!$. $\square$

5. Exponents of types $\mathbf{B}_n$ and $\mathbf{D}_n$

5.1. Let $\psi : \mathfrak{g} \to \mathfrak{sl}(V)$ be a linear representation of a simple Lie algebra $\mathfrak{g}$ of G. Then, for any $x, y \in \mathfrak{g}$ there exists a unique positive integer j such that $\mathrm{tr}(\psi(x), \psi(y)) = j(x, y)$, where $(\cdot, \cdot)$ is the Killing form on $\mathfrak{g}$ normalized in such a way that $(\alpha, \alpha) = 2$ for any long root α. The g.c.d. of all such j's of all linear representations of $\mathfrak{g}$ is called the *Dynkin index* of $\mathfrak{g}$. For example, the Dynkin index of a group of orthogonal type is 2 [1, p.142].

5.2. Following the notation of [1] let $I_m := \ker(\mathbb{Z}[\Lambda] \to \mathbb{Z})$ and $I_a := \ker(S^*(\Lambda) \to \mathbb{Z})$ be the augmentation ideals, where $\mathbb{Z}[\Lambda] \to \mathbb{Z}$ (resp. $S^*(\Lambda) \to \mathbb{Z}$) is the map sending e^{λ} to 1 (resp. any element of positive degree to 0). For any $d \geq 0$, we consider the ring homomorphism

$$\phi^{(d)} : \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda]/I_m^{d+1} \to S^*(\Lambda)/I_a^{d+1} \to S^d(\Lambda),$$

where the first and the last maps are projections and the middle map sends $e^{\sum_{i=1}^n a_i\omega_i}$ to $\prod_{i=1}^n (1-\omega_i)^{-a_i}$. The dth *exponent* of a root system (denoted by τ_d) is the g.c.d. of all non-negative integers N_d satisfying

$$N_d \cdot (I_a^W)^{(d)} \subseteq \phi^{(d)}(I_m^W),$$

where $I_m^W := \langle \mathbb{Z}[\Lambda]^W \cap I_m \rangle$ (resp. $I_a^W := \langle S^*(\Lambda)^W \cap I_a \rangle$) denotes the W-invariant augmentation ideal of $\mathbb{Z}[\Lambda]$ (resp. $S^*(\Lambda)$).

It is shown in [1] that the dth exponent divides the Dynkin index for any $d \leq 4$. In the present section, we show that the dth exponent of an orthogonal type divides the corresponding Dynkin index (which is 2) for every d.

5.3. For any $\lambda \in \Lambda$, we denote by $W(\lambda)$ the W-orbit of λ. For any finite set S of weights, we denote $-S$ the set of opposite weights. By the action of Weyl groups of types B_n and D_n, one has the following decomposition of W-orbits: if $\mathfrak{D} = \mathrm{B}_n$ (resp. $\mathfrak{D} = \mathrm{D}_n$), then for any $1 \leq k \leq n-1$ (resp. $1 \leq k \leq n-2$)

$$W(\omega_k) = W_+(\omega_k) \cup -W_+(\omega_k), \tag{5.1}$$

where $W_+(\omega_k) = \{e_{i_1} \pm \cdots \pm e_{i_k}\}_{1 \leq i_1 < \cdots < i_k \leq n}$. If n is even, then the W-orbits of the last two fundamental weights of D_n are given by

$$W(\omega_{n-1}) = W_+(\omega_{n-1}) \cup -W_+(\omega_{n-1}) \text{ and } W(\omega_n) = W_+(\omega_n) \cup -W_+(\omega_n), \tag{5.2}$$

where $W_+(\omega_{n-1})$ (resp. $W_+(\omega_n)$) is the subset of $W(\omega_{n-1})$ (resp. $W(\omega_n)$) containing elements of the positive sign of e_1.

For any $\lambda = \sum_{i=1}^n a_i \omega_i \in \Lambda$ and any integer $m \geq 0$, we set $\lambda(m) = \sum_{i=1}^n a_i \omega_i^m$. We shall need the following lemma:

Lemma 5.4. *Let p be a positive integer and $m_1, \ldots, m_p$ non-negative integers.*

(i) *If $\mathfrak{D} = \mathrm{B}_n$ (resp. D_n), then for odd p and any $1 \leq k \leq n-1$ (resp. any $1 \leq k \leq n-2$), we have*
$$\sum_{\lambda \in W(\omega_k)} \lambda(m_1) \cdots \lambda(m_p) = 0.$$

(ii) *If $\mathfrak{D} = \mathrm{D}_n$, then for even p and odd n, we have*
$$\sum_{\lambda \in W(\omega_n)} \lambda(m_1) \cdots \lambda(m_p) = \sum_{\lambda \in W(\omega_{n-1})} \lambda(m_1) \cdots \lambda(m_p).$$
Moreover, if p is odd and less than n, then they are all zeroes.

Proof. (i) It follows from (5.1) that
$$\sum_{\lambda \in W(\omega_k)} \lambda(m_1) \cdots \lambda(m_p) = \sum_{\lambda \in W_+(\omega_k)} \lambda(m_1) \cdots \lambda(m_p) - \sum_{\lambda \in W_+(\omega_k)} \lambda(m_1) \cdots \lambda(m_p) = 0.$$

(ii) If n is odd, then we have $W(\omega_n) = -W(\omega_{n-1})$. Hence, the result immediately follows from the assumption that p is even. If n is even, then the result follows from (5.2) by a similar argument.

□

Let p be an even integer and $q \geq 2$ an integer. For any non-negative integers $m_1, \ldots, m_p$, we define
$$\Lambda(p,q)(m_1, \ldots, m_p) := \sum \lambda_{j_1}(m_1) \cdots \lambda_{j_p}(m_p),$$
where the sum ranges over all different $\lambda_{i_1}, \ldots, \lambda_{i_q} \in W_+(\omega_1)$ and all $\lambda_{i_1}, \ldots, \lambda_{i_p} \in \{\lambda_{i_1}, \ldots, \lambda_{i_q}\}$ such that the numbers of $\lambda_{i_1}, \ldots, \lambda_{i_q}$ appearing in $\lambda_{i_1}, \ldots, \lambda_{i_p}$ are all

non-negative even solutions of $x_1+\cdots+x_q = p$. If $p < 2q$, we set $\Lambda(p,q)(m_1,\dots,m_p) = 0$. We simply write $\Lambda(p,q)$ for $\Lambda(p,q)(m_1,\dots,m_p)$. For instance, $\Lambda(4,2)$ is the sum of $\lambda_{j_1}(m_1)\lambda_{j_2}(m_2)\lambda_{j_3}(m_3)\lambda_{j_4}(m_4)$ for all $j_1,j_2,j_3,j_4 \in \{i,j\}$ and all $1 \le i \ne j \le n$ such that two i's and two j's appear in j_1,j_2,j_3,j_4.

Lemma 5.5. *If* $\mathfrak{D} = \mathrm{B}_n$ *(resp.* D_n*), then for any* $2 \le k \le n-1$ *(resp.* $2 \le k \le n-2$*), any even* p*, and any non-negative integers* $m_1,\dots m_p$ *we have*

$$\sum_{W(\omega_k)} \lambda(m_1)\cdots\lambda(m_p) = 2^{k-1}\binom{n-1}{k-1}\sum_{W(\omega_1)}\lambda(m_1)\cdots\lambda(m_p) + \sum_{j=2}^{k} 2^k\binom{n-j}{k-j}\Lambda(p,j).$$

Proof. Let L be the left-hand side (LHS) of the above equation. For any $\lambda \in W(\omega_1)$, there are $2^k\binom{n-1}{k-1}$ choices of the element containing λ in $W(\omega_k)$, thus we have the term

$$2^{k-1}\binom{n-1}{k-1}\sum_{W(\omega_1)}\lambda(m_1)\cdots\lambda(m_p) \text{ in } L.$$

If an element $\lambda \in W(\omega_1)$ appears odd times in a term $\lambda_{i_1}(m_1)\cdots\lambda_{i_p}(m_p)$ of L, where $\lambda_{i_1},\dots,\lambda_{i_p} \in W(\omega_1)$, then from the orbit this term vanishes in L. Hence, the remaining terms in L are linear combinations of $\Lambda(p,j)$ for all $2 \le j \le k$ such that $p \ge 2k$. As each term $\Lambda(p,j)$ appears $2^k\binom{n-j}{k-j}$ times in L, the result follows. $\square$

For any $\lambda \in \Lambda$, we denote by $\rho(\lambda)$ the sum of all elements $e^\mu \in \mathbb{Z}[\Lambda]$ over all elements μ of $W(\lambda)$. By the recursive formulas in [1, Section 1], we can let $d!\cdot\phi^{(d)}(e^\lambda) = \lambda^d + Q_d$ for any $d \ge 1$, where Q_d is the sum of remaining terms in $d!\cdot\phi^{(d)}(e^\lambda)$. Hence, for any fundamental weight ω_k we have

$$d!\cdot\phi^{(d)}(\rho(\omega_k)) = \sum_{W(\omega_k)}\lambda^d + \sum_{W(\omega_k)} Q_d. \tag{5.3}$$

We view $d!\cdot\phi^{(d)}(e^\lambda)$ as a polynomial in variables $\lambda, \lambda(m_1),\dots,\lambda(m_j)$ for some non-negative integers $m_1,\dots,m_j$. Let T_d be the sum of monomials in Q_d whose degrees are even.

If $\mathfrak{D} = \mathrm{B}_n$ (resp. D_n), then by Lemma 5.4(i) equation (5.3) reduces to

$$d!\cdot\phi^{(d)}(\rho(\omega_k)) = \sum_{W(\omega_k)}\lambda^d + \sum_{W(\omega_k)} T_d \tag{5.4}$$

for any $1 \le k \le n-1$ (resp. $1 \le k \le n-2$).

Given p and q, we define $\Omega(p,q) := \sum \Lambda(p,q)(m_1,\dots,m_p)$, where the sum ranges over all $m_1,\dots,m_p$, which appear in all monomials of T_d.

Example 5.6. If $\mathfrak{D} = \mathrm{B}_n$ $(n \geq 4)$ or D_n $(n \geq 5)$ and $d = 6$, then by (5.4) and Lemma 5.5 we have

$$6!\phi^{(6)}(\rho(\omega_1)) = \sum_{W(\omega_1)} \lambda^6 + \sum_{W(\omega_1)} T_6,$$

$$6!\phi^{(6)}(\rho(\omega_2)) = \sum_{W(\omega_2)} \lambda^6 + 2(n-1) \sum_{W(\omega_1)} T_6 + 4\Omega(4,2),$$

$$6!\phi^{(6)}(\rho(\omega_3)) = \sum_{W(\omega_3)} \lambda^6 + 4\binom{n-1}{2} \sum_{W(\omega_1)} T_6 + 8(n-2)\Omega(4,2),$$

which implies that

$$\phi^{(6)}(\rho(\omega_3)) - 2(n-2)\phi^{(6)}(\rho(\omega_2)) + 2(n-1)(n-2)\phi^{(6)}(\rho(\omega_1)) = \sum_{i<j<k} e_i^2 e_j^2 e_k^2.$$

Lemma 5.7. *Let $\mathfrak{D} = \mathrm{D}_n$ and let $1 \leq p \leq n-1$ and $m_1, \ldots, m_p$ be non-negative integers. Then we have*

$$\sum_{W(\omega_n)} \lambda(m_1) \cdots \lambda(m_p) = \sum_{W(\omega_{n-1})} \lambda(m_1) \cdots \lambda(m_p).$$

Moreover, we have $\sum_{W(\omega_n)} \lambda^n - \sum_{W(\omega_{n-1})} \lambda^n = n! e_1 \cdots e_n$.

Proof. Let L be the LHS of the the second equation. As $|W(\omega_n)| = |W(\omega_{n-1})| = 2^{n-1}$, we have

$$(n!/2^n)2^{n-1}e_1 \cdots e_n - (-(n!/2^n)2^{n-1}e_1 \cdots e_n) = n!e_1 \cdots e_n$$

in L. If one of the exponents $i_1, \ldots, i_n$ in $e_1^{i_1} \cdots e_n^{i_n}$ (except the case $i_1 = \cdots = i_n = 1$) is odd, then from the orbits $W(\omega_n)$ and $W(\omega_{n-1})$ this monomial vanishes in each sum of L. If $n \geq 4$ is even, then the terms $2^{n-2}\sum_{j=1}^{n} e_j^n$, $\Lambda(n,2) \cdots, \Lambda(n, n/2)$ with $m_1 = \cdots = m_n = 1$ are in both $\sum_{W_+(\omega_n)} \lambda^n$ and $\sum_{W_+(\omega_{n-1})} \lambda^n$. This completes the proof of the second equation.

By Lemma 5.4(ii), it is enough to consider the case where both n and p are even. Let $X = \sum_{W_+(\omega_n)} \lambda(m_1) \cdots \lambda(m_p)$ and $Y = \sum_{W_+(\omega_{n-1})} \lambda(m_1) \cdots \lambda(m_p)$. By the orbits, any term $\lambda_{i_1}(m_1) \cdots \lambda_{i_p}(m_p)$ with $\lambda_{i_j} \in W(\omega_1)$ appearing odd times in either X or Y vanishes. Since the terms $2^{n-2}\sum_{W_+(\omega_1)} \lambda(m_1) \cdots \lambda(m_p), \Lambda(p,2), \ldots, \Lambda(p, p/2)$ appear in both X and Y, this completes the proof. □

Proposition 5.8. *If $\mathfrak{D} = \mathrm{B}_n$ (resp. D_n), then for any $d \geq 3$ and any $n \geq [d/2]+1$ (resp. $n \geq [d/2]+2$) the exponent τ_d divides the Dynkin index $\tau_2 = 2$.*

Proof. As $\mathrm{B}_2 = \mathrm{C}_2$ and $\mathrm{D}_3 = \mathrm{A}_3$, we have $1 = \tau_3 \mid 2$ by [1, Theorem 5.4]. If $\mathfrak{D} = \mathrm{D}_n$ for any $n \geq 4$, then by Lemma 5.7 we have

$$p_n = \phi^{(n)}(\rho(\omega_n)) - \phi^{(n)}(\rho(\omega_{n-1})),$$

which implies that the invariant p_n is in the ideal generated by the image of $\phi^{(n)}$. As there are no invariants of odd degree except p_n, we have

$$\tau_{2d+1} \mid \tau_{2d}$$

for all $d \geq 1$. Therefore, it suffices to show that $\tau_{2d} \mid \tau_2$ for any $d \geq 2$.

By Lemma 5.5 together with the same argument as in Example 5.6, we have

$$\phi^{(2d)}(\rho(\omega_d))+\sum_{j=1}^{d-1}a_j\phi^{(2d)}(\rho(\omega_{d-j}))=\sum_{j_1<\cdots<j_d}e_{j_1}^2\cdots e_{j_d}^2, \tag{5.5}$$

where the integers $a_1,\ldots,a_{d-1}$ satisfy

$$\left(\sum_{j=k}^{d-2}2^{j+1}\binom{n-1-k}{j-k}a_{j+1}\right)+2^d\binom{n-1-k}{d-1-k}=0,$$

for $0\le k\le d-2$. Let r_d be the right-hand side (RHS) of (5.5). Then this equation implies that r_d is in the image of $\phi^{(2d)}$.

We show that the invariant q_{2d} is in the ideal $\phi^{(2d)}(I_m^W)$ for any $d\ge 2$. We proceed by induction on d. By [1, Lemma 5.3], the case $d=2$ is obvious. By Newton's identities we have

$$(-1)^{d-1}q_{2d}=dr_d-\sum_{j=1}^{d-1}(-1)^{j-1}r_{d-j}q_{2j}. \tag{5.6}$$

By the induction hypothesis, the sum of (5.6) is in $\phi^{(2d)}(I_m^W)$. Hence, q_{2d} is in $\phi^{(2d)}(I_m^W)$. □

6. Annihilators of torsion of twisted spin-flags

In the present section, we apply Propositions 4.5, 4.7 and 5.8 to prove the main result of the paper.

6.1. Let Y be a smooth projective variety over k. The γ-filtration of Y is defined as follows [4, Section 1A]: the ith term is the subgroup

$$\gamma^i(Y)=\langle c_{n_1}^{K_0}(\mathcal{E}_1)\cdot\ldots\cdot c_{n_r}^{K_0}(\mathcal{E}_r)\mid \textstyle\sum n_j\ge i \text{ and } \mathcal{E}_j\in K_0(Y)\rangle,$$

and the associated quotient is $\gamma^{(i)}(Y)=\gamma^i(Y)/\gamma^{i+1}(Y)$. Here $c_j^{K_0}$ is the jth characteristic class in K_0. In particular, since $K_0(G/B)$ is generated by line bundles, we can define $\gamma^i(G/B)$ using only $c_1^{K_0}$ of line bundles [4, Section 1D].

The topological filtration of Y is defined as follows [4, Section 1A]: the ith term is the subgroup

$$\tau^i(Y)=\langle[\mathcal{O}_V]\mid V \text{ is a closed subvariety of } Y \text{ and } \operatorname{codim} V\ge i\rangle.$$

Define $\tau^{(i)}(Y)=\tau^i(Y)/\tau^{i+1}(Y)$. In particular, we have $\gamma^i(Y)\subseteq\tau^i(Y)$ for any Y.

For any G split simple simply connected linear algebraic group over k consider a composite of maps:

$$H^1(k,G)\to H^1(k,G^{\mathrm{ad}})\to H^1(k,\mathrm{Aut}(G)),$$

where G^{ad} is the adjoint group isogeneous to G, the first map is induced by taking quotient modulo the center and the second map is induced by taking inner automorphism. If the class of $\xi\in Z^1(k,\mathrm{Aut}(G))$ belongs to the image of $H^1(k,G^{\mathrm{ad}})$ (resp. $H^1(k,G)$), we say that ξ is an *inner form* (resp. a *strongly inner* form). In particular, according to [12, Theorem 2.2.(2)] for a strongly inner ξ we have an isomorphism $K_0(G/B)\simeq K_0({}_\xi(G/B))$, which commutes with the γ-filtrations, i.e., we have $\gamma^{(i)}(G/B)\simeq\gamma^{(i)}({}_\xi(G/B))$.

Theorem 6.2. *Let G be a split simple simply-connected linear algebraic group of Dynkin type B_n ($n \geq 3$) or D_n ($n \geq 4$), i.e., a Spin group. Let $X = {}_{\xi}(G/B)$ be a twisted form of the variety of Borel subgroups of G by means of a cocycle $\xi \in Z^1(k, G)$.*

If G is of type B_n (resp. of type D_n), then for all $2 \leq d \leq 2n-1$ (resp. $2 \leq d \leq 2n-3$), the integer $M_d = (d-1)! \prod_{i=2}^{d} (i-1)! \cdot [i/2]! \cdot 2^{i+1}$ annihilates the torsion part of $\mathrm{CH}^d(X)$, i.e.,

$$M_d \cdot \operatorname{Tors} \mathrm{CH}^d(X) = 0.$$

Remark 6.3. Observe that since 2 is the only torsion prime of G we can replace the integer M_d by its 2-primary part. Note also that the integer M_d depends only on the codimension d but not on the rank n of G.

Proof. We follow the arguments of [1, Section 6]. We have the following diagram:

$$\begin{tikzcd} I_m^i / I_m^{i+1} \arrow[r, "(-1)^{i-1}(i-1)!\phi^{(i)}"] \arrow[d, "c_m"'] & S^i(\Lambda) \arrow[d, "c_a"] \\ \gamma^{(i)}(G/B) \arrow[r, "c_i"] & \mathrm{CH}^i(G/B). \end{tikzcd}$$

Here c_a is the characteristic map and c_i is the ith Chern class. The map c_m is the surjection induced by the characteristic map $c_m : \mathbb{Z}[\Lambda] \to K_0(G/B)$, since $K_0(G/B)$ is generated by line bundles. Moreover, we have $\ker c_m = I_m^W$. Since $\mathrm{CH}^i(G/B)$ is torsion free, using Propositions 4.5, 4.7 and 5.8, by diagram chasing, we see that the integer $m_i := (i-1)! \cdot [i/2]! \cdot 2^{i+1}$ annihilates the torsion part of $\gamma^{(i)}(G/B)$ for $i \geq 2$.

By [12, Theorem 2.2.(1)], we have that $\gamma^{(d)}(X) \simeq \gamma^{(d)}(G/B)$, hence, its torsion part is annihilated by m_d as well. Moreover, we have the following short exact sequences of abelian groups:

$$\gamma^{(d)} \hookrightarrow \tau^d/\gamma^{d+1} \twoheadrightarrow \tau^d/\gamma^d \text{ and } \tau^{d+1}/\gamma^{d+1} \hookrightarrow \tau^d/\gamma^{d+1} \twoheadrightarrow \tau^{(d)},$$

where $\tau^d = \tau^d(X)$ and $\gamma^d = \gamma^d(X)$. So, we have

$$e(\tau^d/\gamma^{d+1}) | e(\gamma^{(d)}) \cdot e(\tau^d/\gamma^d) | e(\gamma^{(d)}) \cdot e(\tau^{d-1}/\gamma^d),$$

where $e(-)$ is the exponent of the torsion part of an abelian group. Recursively, we see that

$$e(\tau^{(d)}) | e(\tau^d/\gamma^{d+1}) | e(\gamma^{(d)}) \cdot e(\gamma^{(d-1)}) \cdot \cdots \cdot e(\gamma^{(1)}) = \prod_{j=2}^{d} m_j$$

(notice that $e(\gamma^{(1)}) = 1$). Finally, by the Riemann–Roch theorem [3, Ex.15.3.6], the composition $\mathrm{CH}^d(X) \to \tau^{(d)} \to \mathrm{CH}^d(X)$ is multiplication by $(-1)^{d-1}(d-1)!$ Therefore, $M_d := (d-1)! \prod_{j=2}^{d} m_d$ annihilates the torsion part of $\mathrm{CH}^d(X)$. $\square$

Remark 6.4. Using the motivic decomposition of [13] one immediately extends this result to any generically split twisted form $X = {}_{\xi}G/P$, where P is a parabolic subgroup of G. In particular, it holds for any maximal orthogonal Grassmannian of a quadratic form with trivial discriminant and trivial Clifford invariant.

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