

NEW CONSTRUCTIONS OF CREMONA MAPS

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ABSTRACT. One defines two ways of constructing rational maps derived from other rational maps, in a characteristic-free context. The first introduces the Newton complementary dual of a rational map. One main result is that this dual preserves birationality and gives an involutorial map of the Cremona group to itself that restricts to the monomial Cremona subgroup and preserves de Jonquières maps. In the monomial restriction, this duality commutes with taking inverse in the group, but is not a group homomorphism. The second construction is an iterative process to obtain rational maps in increasing dimension. Starting with birational maps, it leads to rational maps whose topological degree is under control. Making use of monoids, the resulting construct is in fact birational if the original map is so. A variation of this idea is considered in order to preserve properties of the base ideal, such as Cohen–Macaulayness. Combining the two methods, one is able to produce explicit infinite families of Cohen–Macaulay Cremona maps with prescribed dimension, codimension, and degree.

1. Introduction

The purpose of this paper is to introduce a couple of constructions of rational maps out of given ones. The core of the results consists in that the constructions preserve birationality and often a few other properties. The source and the target of any rational map considered here are projective spaces over a field k . Such a map will typically be denoted $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, where the field k is self-understood. For the geometric purpose k is usually taken to be algebraically closed. However, we stress that the constructions and the main theorems are characteristic-free.

The first of these constructions is a large extension of an operation introduced in [13] for rational maps defined by monomials. It associates to $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ another rational map with same source and target which will be birational (onto the image) if \mathfrak{F} is so. In particular, if \mathfrak{F} is a Cremona map of \mathbb{P}^n , so is its associated construct. Since the set of Cremona maps of \mathbb{P}^n form a group under composition, it is obvious that one can produce new Cremona maps galore out of given ones by means of group theoretic operations, such as generation and conjugation. However, the present construction is of a different nature and resembles to a “duality” on the Cremona group — in that applying twice successively yields back the original map, — and to a group homomorphism, when restricted to the monomial Cremona subgroup — in that it preserves inverses (but not the group operation).

The second construction is an inductive procedure associating to $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ a rational map $\mathfrak{G} : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{m+1}$. A germ of this idea has been given in [3], but it is possible that examples of this construction have been considered in the classical

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references. Nonetheless, our interest is as to how the properties of the given rational map transfer over to the construct. The procedure preserves birationality and, in addition, properties of the base ideal. A main purpose of this construction here is to explicitly produce, for arbitrarily given integers $n \geq 2$, $d \geq 1$ and r in the integer interval $[2, n]$, infinitely many Cremona maps of \mathbb{P}^n of degree $d \geq 1$ and codimension r ; in addition, if $d \geq n + 1 - r$, there are infinitely many such maps whose base ideal is Cohen–Macaulay. To our knowledge, such families do not seem to have been explicitly stated in the literature.

Our central results are stated as Theorems 2.9, 3.7, and 3.9. They are better understood within the context of the respective section. We accordingly give a brief description of the corresponding sections.

Section 2 starts with a swift recap of birational maps, after which we introduce the concept of the Newton matrix $N(\mathbf{f})$ of a set $\mathbf{f} = \{f_0, \dots, f_m\}$ of forms of the same degree in the standard graded polynomial ring $R = k[x_0, \dots, x_n]$ over a field k . This matrix is briefly described as the concatenation of the log-matrices of all terms appearing in these forms. One should be careful not to abuse too much handling the ideals generated by the forms or its various parts, as it is rather the subalgebra generated by those that will play a distinctive role. The subsequent main construct is what one calls the Newton complementary dual matrix $\widehat{N}(\mathbf{f})$ of the Newton matrix $N(\mathbf{f})$. This in turn gives rise, in a sort of “coefficient-frame” operation, to a new set of forms $\widehat{\mathbf{f}}$ in the same ring R , called similarly the Newton complementary dual set of \mathbf{f} . From the definitions follows that $\widehat{\widehat{\mathbf{f}}} = \mathbf{f}$. The first relevant result is the statement that if \mathbf{f} are monomials, then \mathbf{f} defines a birational map (onto the image) if and only if $\widehat{\mathbf{f}}$ does too. The proof is entirely of a field-theoretic flavor, based on the characteristic property of the construction in the monomial case (see Remark 2.2 (3)). In particular, it restricts to a well-defined “duality” of the monomial subgroup of the entire Cremona group of dimension n . Another meaningful property of this duality in the case of monomials is that it commutes with taking inverses. However, this property no longer subsists for arbitrary birational maps, as is shown by examples.

At any rate, the main point of this section is proving that birationality is preserved for arbitrary forms and, in addition, there is a k -isomorphism of the graded k -subalgebras $k[\mathbf{f}]$ and $k[\widehat{\mathbf{f}}]$. As a non-trivial illustration of the theory, we show that Newton complementary duality preserves plane de Jonquières maps. However, the degree of the resulting map is given by a rather involved expression in terms of the defining forms of the original map — perhaps not too surprising, as the degree of a plane Cremona map is dependent upon the multiplicities of the infinitely near base points of the map. Unfortunately, Newton complementary duality does not preserve the composition of Cremona maps. Therefore, it is less natural to ask what group theoretic properties are preserved by this construction.

In the second section we deal with a procedure of associating to a rational map $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ a rational map $\mathfrak{G} : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{m+1}$, by viewing \mathbb{P}^n as a cone and letting the last coordinate of \mathfrak{G} be a monoid with respect to the cone variable. This will have the effect that the image of \mathfrak{G} will be a cone over the image of \mathfrak{F} . Such rigid features of the construction will entail the preservation of birationality and typical properties of the base ideal.

In a more explicit way, let $\mathbf{f} \subset R = k[x_0, \dots, x_{n-1}]$ be forms of degree d defining \mathfrak{F} . Pick $g = g_{d-1}x_n + g_d \in R[x_n] = k[x_0, \dots, x_n]$, where $g_{d-1} \neq 0$ and g_d are forms in R of degrees $d-1$ and d , respectively. Then the forms \mathbf{f}, g define \mathfrak{G} . We show that \mathfrak{G} is birational along with \mathfrak{F} . Moreover, if $g_d \in I$ and g_{d-1} is a non-zero-divisor on R/I then the Cohen–Macaulay property transfers as well. This is obtained as a particular case of a more general principle when one takes g to be a form of higher degree in the new variable x_n , by which the topological degree \mathfrak{G} may turn out to be non-trivial depending on that degree, even when \mathfrak{F} is birational. In the sequel, we give a slight variation of this iterative procedure in order to preserve the condition that the base ideal is Cohen–Macaulay of codimension 2. In this form, the construction resembles the de Jonquières parametrizations of [9]. However, the resemblance is superficial since there the purpose is implicitization, while here the main use is in the Cremona case. Combining the two methods developed in this section, we are able to prove the existence of infinite families of Cohen–Macaulay Cremona transformations with prescribed dimension, codimension and degree (bounded below in a natural way).

At the end of the section we show how to produce codimension 2 Cohen–Macaulay Cremona maps that are very close to the determinantal polar maps introduced in [2]. These examples may bear a meaning for the theory of determinantal polar maps, a rather narrow habitat.

2. The Newton complementary dual of a rational map

2.1. Recap of birational maps. Our reference for the basics in this part is [12], which contains enough of the introductory material in the form we use here (see also [5] for a more general overview).

Let k denote an arbitrary infinite field which will be assumed to be algebraically closed for the geometric purpose. A rational map $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is defined by $m+1$ forms $\mathbf{f} = \{f_0, \dots, f_m\} \subset R := k[\mathbf{x}] = k[x_0, \dots, x_n]$ of the same degree $d \geq 1$, not all null. We often write $\mathfrak{F} = (f_0 : \dots : f_m)$ to underscore the projective setup. Any rational map can without loss of generality be brought to satisfy the condition that $\gcd\{f_0, \dots, f_m\} = 1$, (i.e., the linear system spanned by \mathbf{f} has no fixed part). One also assumes that every variable x_i appears effectively in some f_j . If the rational map satisfies both conditions we will say that it satisfies the *canonical restrictions*. These conditions are usually hand-waved when considering Cremona maps of dimension n . The reason is that one can rather take equivalence classes of sets \mathbf{f} as above, of the same cardinality, by multiplication with an arbitrary form; clearly, every such class has a unique representative satisfying $\gcd\{f_0, \dots, f_m\} = 1$. Composition in the Cremona group is well-defined for these classes, which allows for the slight get-away of working with these unique representatives (see [5] for a detailed account of such preliminaries, but of course this is well-known in the classical subject).

The common degree d of the f_j is the *degree* of \mathfrak{F} , not to be confused with its topological (or field) degree. The ideal $I_{\mathfrak{F}} := (f_0, \dots, f_m) \subset R$ is called the *base ideal* of \mathfrak{F} . The *image* of \mathfrak{F} is the projective subvariety $W \subset \mathbb{P}^m$ whose homogeneous coordinate ring is the k -subalgebra $k[\mathbf{f}] \subset R$ after degree renormalization. Write $S := k[\mathbf{f}] \simeq k[\mathbf{y}]/I(W)$, where $I(W) \subset k[\mathbf{y}] = k[y_0, \dots, y_m]$ is the homogeneous defining ideal of the image in the embedding $W \subset \mathbb{P}^m$.

We say that \mathfrak{F} is *birational onto the image* if there is a rational map backwards $\mathbb{P}^m \dashrightarrow \mathbb{P}^n$ such that the residue classes $\mathbf{f}' = \{f'_0, \dots, f'_n\} \subset S$ of a set of defining coordinates do not simultaneously vanish and satisfy the relations

$$(\mathbf{f}'_0(\mathbf{f}) : \dots : \mathbf{f}'_n(\mathbf{f})) = (x_0 : \dots : x_n) \quad \text{and} \quad (\mathbf{f}_0(\mathbf{f}') : \dots : \mathbf{f}_m(\mathbf{f}')) = (y_0 : \dots : y_m),$$

where the second equality is to be understood modulo the ideal $I(W)$. Let K denote the field of fractions of $S = k[\mathbf{f}]$. The coordinates $\{f'_0, \dots, f'_n\}$ defining the “inverse” map are not uniquely defined; any other set $\{f''_0, \dots, f''_n\}$ related to the first through requiring that it define the same element of the projective space $\mathbb{P}^n_K = \mathbb{P}^n_k \otimes_k \text{Spec}(K)$ will do as well — both tuples are called *representatives* of the rational map (see [12] for details). One can see that, if k is algebraically closed, these relations translate into the geometric definition in terms of invertibility of the map on a dense Zariski open set.

An alternative definition of a birational map is in terms of field extensions. Letting the common degree of the f ’s be d , one has an inclusion of rings $S = k[\mathbf{f}] \subset R^{(d)} := k[\mathbf{x}_d]$. Then \mathfrak{F} is birational onto the image if and only if this inclusion triggers an equality $K = k(\mathbf{x}_d)$ at the level of the respective fields of fractions. The inverse map is given by extracting one representative from the inverse field isomorphism.

2.2. The Newton complementary dual. The following notion extends to arbitrary forms of the same degree in $R = k[x_0, \dots, x_n]$ the concept of a certain dual construction of an integer matrix (see [3], [7], [13]).

Definition 2.1. Let f be d -form in R , for some $d \geq 1$. Denote by $N(f)$ (as a reminder of the Newton polygon) the so-called log-matrix of the set of nonzero terms of f , where we assume that the terms are lexicographically ordered. Thus, $N(f)$ is the matrix whose columns are the exponents vectors of the nonzero terms of f lexicographically ordered. One may call $N(f)$ the NEWTON LOG MATRIX (or simply the NEWTON MATRIX) of f .

Given an ordered set $\mathbf{f} := \{f_0, \dots, f_m\}$ of such forms of the same degree $d \geq 1$, let $N(\mathbf{f})$ denote the concatenation of the Newton matrices $N(f_0), \dots, N(f_m)$; accordingly, we call $N(\mathbf{f})$ the NEWTON MATRIX of the set \mathbf{f} . Note that $N(\mathbf{f})$ is an integer stochastic matrix.

The row vector \mathbf{c}_f whose entries are the nonzero coefficients of a form f , ordered by the lexicographic monomial order, is the *coefficient frame* of f . We write symbolically

$$(2.1) \quad f = \langle \mathbf{c}_f, \mathbf{x}^{N(f)} \rangle$$

as the inner product of the coefficient frame by the set of the corresponding monomials.

The NEWTON COMPLEMENTARY DUAL MATRIX (or simply the COMPLEMENTARY DUAL MATRIX) of $N(\mathbf{f}) = (a_{i,\ell})$ is the matrix

$$\widehat{N(\mathbf{f})} = (\alpha_i - a_{i,\ell}),$$

where, for every $0 \leq i \leq n$, $\alpha_i = \max_{\ell} \{a_{i,\ell}\}$, with ℓ indexing the set of all nonzero terms of all forms in the set \mathbf{f} .

In other words, denoting $\underline{\alpha} := (\alpha_0, \dots, \alpha_n)^t$, one has

$$\widehat{N(\mathbf{f})} = [\underline{\alpha} | \dots | \underline{\alpha}]_{(n+1) \times (r_0 + \dots + r_m)} - N(\mathbf{f}),$$

where r_j denotes the number of nonzero terms of f_j , $j = 0, \dots, m$.

For every $j = 0, \dots, m$, let $\widehat{N(\mathbf{f})}_j$ denote the submatrix of $\widehat{N(\mathbf{f})}$ whose columns come from f_j . Finally consider the set of forms

$$(2.2) \quad \widehat{\mathbf{f}} := \{\widehat{f}_0 := \langle \mathbf{c}_0, \mathbf{x}^{\widehat{N(\mathbf{f})}_0} \rangle, \dots, \widehat{f}_m := \langle \mathbf{c}_m, \mathbf{x}^{\widehat{N(\mathbf{f})}_m} \rangle\}$$

where $\mathbf{c}_j = \mathbf{c}_{f_j}$ stands for the coefficient frame of f_j .

- Remark 2.2.** (1) It is extremely important to note that the submatrix $\widehat{N(\mathbf{f})}_j$ is in general different from the Newton complementary dual matrix $\widehat{N(f_j)}$ of the set consisting solely of f_j . In this regard the above notation \widehat{f}_j may lead to confusion, but it will always mean the j th element of the set $\widehat{\mathbf{f}}$ by applying the Newton procedure to the whole set \mathbf{f} and not step by step to its individual elements. The process is always controlled by the “global” vector $\underline{\alpha}$ above which is clearly different for the whole of \mathbf{f} as for one of its subsets.
- (2) By the same token, the notion is not “recurrent” in the sense that we cannot in general inductively reconstruct $\widehat{\mathbf{f}}$ from $\widehat{\mathbf{g}}$, where $\mathbf{g} = \{f_0, \dots, f_{m-1}\}$. Actually, without further restrictions the procedure can lead to very uninteresting situations; for example, if $\mathbf{f} = \{x_0\}$ then $\widehat{\mathbf{f}} = \{1\}$!
- (3) If M is a monomial in the list of nonzero terms of \mathbf{f} then $M \cdot \widehat{M} = \underline{x}^\alpha$, with $\underline{x}^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$, where the α ’s are as above.
- (4) The common degree of the forms in $\widehat{\mathbf{f}}$ is the *dual degree* $\hat{d} := \alpha_0 + \cdots + \alpha_n - d$. Note the immediate bounds $n + 1 - d \leq \hat{d} \leq nd$, where the first inequality follows from the non-degeneracy of the forms in \mathbf{f} (and tells us literally nothing when $d > n + 1$). Anyway, both are clearly easily attainable in the family of arbitrary sets \mathbf{f} . However, for special relevant subfamilies these bounds can be tightened. Thus, for instance, if \mathbf{f} defines a Cremona map of \mathbb{P}^n of degree $d \geq 2$ then a slightly better upper bound is $nd - 1$. It is challenging to lower this upper bound for families of Cremona maps of \mathbb{P}^n enjoying special properties.

Definition 2.3. We will call $\widehat{\mathbf{f}}$ the NEWTON COMPLEMENTARY DUAL set of \mathbf{f} .

The terms “Newton” and “complementary” seem to be appropriate. The following easy result justifies the term “dual”.

Lemma 2.4. *If \mathbf{f} satisfies the canonical restrictions then so does $\widehat{\mathbf{f}}$ and one has $\widehat{\widehat{\mathbf{f}}} = \mathbf{f}$.*

Proof. Since \mathbf{f} has no fixed part, in particular no variable belongs to the support of every term of \mathbf{f} . Therefore, for any given $i = 0, \dots, n$, some $a_{ii} = 0$. On the other hand, \mathbf{f} has also the property that for every $i = 0, \dots, n$, some $a_{ik} \neq 0$. Collecting the two gives that for any i , $\alpha_i \geq 1$. We may assume that this is the maximum. It follows that $\alpha_i - a_{ik} = 0$. We thus conclude that $\widehat{\mathbf{f}}$ satisfies the canonical restrictions.

Next, write $\widehat{N(\mathbf{f})} = (b_{ij})$. By definition, $b_{ij} = \alpha_i - a_{ij}$, for every $j = 0, \dots, n$, where $\alpha_i = \max_j \{a_{ij}\}$. Let $\beta_i = \max_j \{b_{ij}\} = \max_j \{\alpha_i - a_{ij}\}$. Then $\beta_i = \alpha_i - \min_j \{a_{ij}\} = \alpha_i$, as $\min_j \{a_{ij}\} = 0$. Thence

$$(\widehat{\widehat{N(\mathbf{f})}})_{ij} = \beta_i - b_{ij} = \alpha_i - (\alpha_i - a_{ij}) = a_{ij},$$

the last equality following from the hypothesis that \mathbf{f} has no fixed part. Therefore, $\widehat{\widehat{N(\mathbf{f})}} = N(\mathbf{f})$, thus showing the contention. \square

We will call the vector $\underline{\alpha} := (\alpha_1, \dots, \alpha_n)^t$ the DIRECTRIX VECTOR of \mathbf{f} . Note that the directrix vectors of \mathbf{f} and of $\widehat{\mathbf{f}}$ coincide, as has been seen in the proof of the lemma.

Example 2.5. If $\mathbf{f} = \{x_0, \dots, x_n\}$ then its directrix vector is $(1, \dots, 1)^t$ and

$$\widehat{\mathbf{f}} = \{x_1 \cdots x_n, \dots, x_0 \cdots \widehat{x_i} \cdots x_n, \dots, x_0 \cdots x_{n-1}\}.$$

In terms of the rational maps defined by the respective forms, this says that the Newton complementary dual of the identity map of \mathbb{P}^n is the Magnus reciprocal involution.

Note the relations $f_j \widehat{f_j} = x_0 \cdots x_n$ for every $j = 0, \dots, n$. The example, along with this property, is propaedeutic for the result in the next section.

2.3. The main result. For convenience, we often say that a set \mathbf{f} of forms of the same degree satisfying the canonical restrictions is birational if the associated rational map is birational onto its image. Note that this notion is independent of the ordering of the forms.

We first deal with the case where the forms in \mathbf{f} are monomials.

Proposition 2.6. *Let $\mathbf{f} = \{f_0, \dots, f_m\} \subset R = k[x_0, \dots, x_n]$ be monomials of the same degree satisfying the canonical restrictions. If \mathbf{f} defines a birational map onto its image then so does its Newton complementary dual set $\widehat{\mathbf{f}}$.*

Proof. Let d be the common degree of the monomials in \mathbf{f} .

Note that $k(\underline{x}_d) = k(x_0^d, x_1/x_0, \dots, x_n/x_0)$, hence the birationality of \mathbf{f} is tantamount to the equality

$$k(f_0, \dots, f_m) = k\left(x_0^d, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right).$$

Therefore, for every $i = 1, \dots, n$ one has an expression

$$\frac{x_i}{x_0} = \frac{g_i(f_0, \dots, f_m)}{g'_i(f_0, \dots, f_m)},$$

where g_i, g'_i are forms in the polynomial ring $k[y_0, \dots, y_m]$ of the same degree, say, s .

By the same token, it suffices to show the equality

$$k(\widehat{f_0}, \dots, \widehat{f_m}) = k\left(x_0^{|\underline{\alpha}|-d}, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) (= k(\underline{x}_{|\underline{\alpha}|-d})),$$

where $\widehat{\mathbf{f}} = \{\widehat{f_0}, \dots, \widehat{f_m}\}$ and $\underline{\alpha}$ is the directrix vector and $|\underline{\alpha}| = \alpha_0 + \dots + \alpha_m$. Once more we insist on the meaning of $\widehat{f_j}$ as explained at the beginning of the section.

Now, since the forms in \mathbf{f} are monomials, one has $\widehat{f_j} = \underline{x}^\alpha / f_j$ for every $j = 0, \dots, m$, where $\underline{x}^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. It follows that, for every $i \in \{1, \dots, n\}$, one has

$$\begin{aligned} \frac{x_i}{x_0} &= \frac{g_i(f_0, \dots, f_m)}{g'_i(f_0, \dots, f_m)} \frac{(\underline{x}^\alpha)^s}{(\underline{x}^\alpha)^s} = \frac{g_i(f_0, \dots, f_m)}{(\underline{x}^\alpha)^s} \frac{1}{\frac{g'_i(f_0, \dots, f_m)}{(\underline{x}^\alpha)^s}} \\ &= \frac{g_i(\frac{1}{f_0}, \dots, \frac{1}{f_m})}{g'_i(\frac{1}{f_0}, \dots, \frac{1}{f_m})} = \frac{h_i(\widehat{f_0}, \dots, \widehat{f_m})}{h'_i(\widehat{f_0}, \dots, \widehat{f_m})}, \end{aligned}$$

for suitable forms $h_i, h'_i \in k[y_0, \dots, y_m]$ of the same degree. It remains to deal with the power $x_0^{|\underline{\alpha}|-d}$, for which we claim likewise that it belongs to $k(\widehat{f_0}, \dots, \widehat{f_m})$. Since \mathbf{f} is a set of monomials satisfying the canonical restrictions, there is an index j such that x_0 does not divide f_j . Without loss of generality, say, $j = 0$. Write accordingly $f_0 = x_{i_1} \cdots x_{i_d}$ with $0 < i_1 \leq i_2 \leq \dots \leq i_d$. Using the above expressions, one has

$$\frac{f_0}{x_0^d} = \frac{x_{i_1}}{x_0} \cdots \frac{x_{i_d}}{x_0} = \frac{h_{i_1}(\widehat{f_0}, \dots, \widehat{f_m})}{h'_{i_1}(\widehat{f_0}, \dots, \widehat{f_m})} \cdots \frac{h_{i_d}(\widehat{f_0}, \dots, \widehat{f_m})}{h'_{i_d}(\widehat{f_0}, \dots, \widehat{f_m})}.$$

On the other hand,

$$\begin{aligned} \frac{f_0}{x_0^d} &= \frac{f_0}{x_0^d} \frac{x_0^{|\underline{\alpha}|}}{x_0^{|\underline{\alpha}|}} = x_0^{|\underline{\alpha}|-d} \frac{f_0}{x_0^{|\underline{\alpha}|}} = x_0^{|\underline{\alpha}|-d} \frac{f_0}{\underline{x}^\alpha} \frac{\underline{x}^\alpha}{x_0^{\alpha_0} \cdots x_0^{\alpha_n}} = x_0^{|\underline{\alpha}|-d} \frac{1}{\widehat{f_0}} \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{x_0^{\alpha_1} \cdots x_0^{\alpha_n}} \\ &= x_0^{|\underline{\alpha}|-d} \frac{1}{\widehat{f_0}} \left(\frac{h_{i_1}(\widehat{f_0}, \dots, \widehat{f_m})}{h'_{i_1}(\widehat{f_0}, \dots, \widehat{f_m})} \right)^{\alpha_1} \cdots \left(\frac{h_{i_d}(\widehat{f_0}, \dots, \widehat{f_m})}{h'_{i_d}(\widehat{f_0}, \dots, \widehat{f_m})} \right)^{\alpha_n}. \end{aligned}$$

Equating the above two expressions yields the required result. \square

Besides preserving birationality in the case of monomials, the dual process has an additional crucial purely algebraic property which holds true for any rational data satisfying the canonical restrictions:

Lemma 2.7. *Let $\mathbf{f} = \{f_0, \dots, f_m\} \subset R = k[x_0, \dots, x_n]$ be monomials of the same degree satisfying the canonical restrictions, and let $\widehat{\mathbf{f}}$ denote its Newton complementary dual set. Then the identity map of $k[y_0, \dots, y_m]$ induces a k -algebra isomorphism $k[\mathbf{f}] \simeq k[\widehat{\mathbf{f}}]$.*

Proof. Recall the basic relationship between the two in the monomial case: $f_j = \mathbf{x}^\alpha / \widehat{f_j}$, for $j = 0, \dots, m$, where α gives the coordinates of the directrix vector $\underline{\alpha}$.

It suffices to show that any polynomial relation of \mathbf{f} is one of $\widehat{\mathbf{f}}$, and conversely too. Since both algebras are toric and homogeneous, it suffices to consider homogeneous binomial relations. Thus, let $\mathbf{y}^\beta - \mathbf{y}^\gamma \in k[y_0, \dots, y_m]$, with $|\beta| = |\gamma|$. This binomial is a relation of \mathbf{f} if and only if any one of the following equivalent conditions hold:

$$\begin{aligned} \prod_j f_j^{\gamma_j} = \prod_j f_j^{\beta_j} &\iff \frac{(\underline{x}^\alpha)^{|\gamma|}}{\prod_j f_j^{\gamma_j}} = \frac{(\underline{x}^\alpha)^{|\beta|}}{\prod_j f_j^{\beta_j}} \iff \frac{\prod_j (\underline{x}^\alpha)^{\gamma_j}}{\prod_j f_j^{\gamma_j}} = \frac{\prod_j (\underline{x}^\alpha)^{\beta_j}}{\prod_j f_j^{\beta_j}} \\ &\iff \prod_j \left(\frac{\underline{x}^\alpha}{f_j} \right)^{\gamma_j} = \prod_j \left(\frac{\underline{x}^\alpha}{f_j} \right)^{\beta_j} \iff \prod_j (\widehat{f_j})^{\gamma_j} = \prod_j (\widehat{f_j})^{\beta_j}. \end{aligned}$$

Since the last equality above means that the binomial $\mathbf{y}^\beta - \mathbf{y}^\gamma$ is a relation of $\widehat{\mathbf{f}}$, we are through. \square

Remark 2.8. Note that the basic relations $f_j = \mathbf{x}^\alpha / \widehat{f_j}$, for $j = 0, \dots, m$, imply an isomorphism of the algebras $k[\mathbf{f}]$ and $k[1/\widehat{f_0}, \dots, 1/\widehat{f_m}]$. However, for arbitrary forms g_0, \dots, g_m of the same degree, there may not be an isomorphism between $k[g_0, \dots, g_m]$ and $k[1/g_0, \dots, 1/g_m]$ induced by the identity map of $k[y_0, \dots, y_m]$. An example is provided by noting that the Grassmann–Plücker quadratic relation of the 2-minors of a 2×4 matrix is not a relation of the reciprocals of these minors.

One is now ready for the main result of this part.

Theorem 2.9. *Let $\mathbf{f} = \{f_0, \dots, f_m\} \subset R = k[x_0, \dots, x_n] (n \geq 1)$ be arbitrary forms of the same degree satisfying the canonical restrictions and let $\widehat{\mathbf{f}}$ denote its Newton complementary dual set. One has:*

- (a) *There is an isomorphism $k[\mathbf{f}] \simeq k[\widehat{\mathbf{f}}]$ by gluing and restriction of the isomorphism at the level of the terms.*
- (b) *If \mathbf{f} defines a birational map onto its image then so does $\widehat{\mathbf{f}}$.*

Proof. Keeping the notation of (2.1), write $f_j = \langle \mathbf{c}_{f_j}, \mathbf{x}^{N(f_j)} \rangle$, for $j = 0, \dots, m$, where $\mathbf{c}_{f_j} = (c_{j,1}, \dots, c_{j,r_j})$, so that $f_j = \langle \mathbf{c}_{f_j}, \mathbf{x}^{N(f_j)} \rangle = \sum_{\ell=1}^{r_j} c_{j,\ell} M_{j,\ell}$, for suitable monomials $M_{j,\ell}$. In this notation, one has $\widehat{f_j} = \sum_{\ell=1}^{r_j} c_{j,\ell} \widehat{M_{j,\ell}}$. Let $\{z_{j,1}, \dots, z_{j,r_j}\}$, with $j = 0, \dots, m$, denote $r_0 + \dots + r_m$ mutually independent indeterminates,

By Lemma 2.7, for each $0 \leq j \leq m$ the identity map of $k[z_{j,1}, \dots, z_{j,r_j}]$ induces an isomorphism of $k[M_{j,1}, \dots, M_{j,r_j}]$ onto $k[\widehat{M_{j,1}}, \dots, \widehat{M_{j,r_j}}]$. Gluing together, the identity map of the entire polynomial ring $k[z_{j,1}, \dots, z_{j,r_j} \mid 0 \leq j \leq m]$ induces an isomorphism of $k[\mathbf{M}] := k[M_{j,1}, \dots, M_{j,r_j} \mid 0 \leq j \leq m]$ onto $k[\widehat{\mathbf{M}}] := k[\widehat{M_{j,1}}, \dots, \widehat{M_{j,r_j}} \mid 0 \leq j \leq m]$.

Setting $y_j = \sum_{\ell=1}^{r_j} c_{j,\ell} z_{j,\ell}$, for $0 \leq j \leq m$, yields algebraically independent elements y_0, \dots, y_m over k . By restriction to the polynomial ring $k[y_0, \dots, y_m]$ we clearly obtains that the identity map of the polynomial ring $k[y_0, \dots, y_m]$ induces an isomorphism of $k[\mathbf{f}]$ onto $k[\widehat{\mathbf{f}}]$. This shows part (a).

Part (b) goes as follows: by part (a), passing to the respective fields of fractions on both sides, yields a diagram of isomorphisms and inclusions:

$$\begin{array}{ccc} k(\mathbf{x}_d) & & k(\mathbf{x}_{|\underline{\alpha}|-d}), \\ \cup & & \cup \\ k(\mathbf{M}) & \simeq & k(\widehat{\mathbf{M}}), \\ \cup & & \cup \\ k(\mathbf{f}) & \simeq & k(\widehat{\mathbf{f}}), \end{array}$$

where α was explained earlier. The birationality assumption on \mathbf{f} implies that $k(\mathbf{f}) = k(\mathbf{x}_d)$. In particular, $k(\mathbf{f}) = k(\mathbf{M})$. Since all horizontal maps are induced by the identity map on the polynomial ring $k[z_{j,1}, \dots, z_{j,r_j} \mid 0 \leq j \leq m]$, it follows that $k(\widehat{\mathbf{f}}) = k(\widehat{\mathbf{M}})$. Finally, by the monomial case in Lemma 2.6, one has $k(\widehat{\mathbf{M}}) = k(\mathbf{x}_{|\underline{\alpha}|-d})$. Therefore, $k(\widehat{\mathbf{f}}) = k(\mathbf{x}_{|\underline{\alpha}|-d})$, as was required to show. \square

2.4. Notable Cremona maps.

2.4.1. Monomial Cremona maps. The operations of taking the Newton dual and the inverse map commute in the case of monomial Cremona maps, according to [3, Theorem 3.1.3]. For convenience we restate this result along with its proof, which depends on [14, Theorem 2.2] (see also [4, Theorem 2.2] for a restatement stressing the canonical restrictions).

For light reading, if \mathbf{f} defines a Cremona map, we let \mathbf{f}^{-1} denote its inverse map. Recall that if \mathbf{f} is a set of monomials of the same degree, its Newton matrix is just the log-matrix $L(\mathbf{f})$ of these monomials. In the case of a monomial Cremona map of \mathbb{P}^n , there is a meaningful vector of \mathbb{N}^{n+1} called the *inversion vector* (see [4, Section 2]).

Along with the directrix vector $\underline{\alpha}^t \in \mathbb{N}^{n+1}$ it constitutes a key for many structural results in monomial Cremona theory.

Proposition 2.10. *Let $\mathbf{f} \subset R = k[x_0, \dots, x_n]$ be a set of $n+1$ monomials of the same degree satisfying the canonical restrictions and defining a Cremona map of \mathbb{P}^n . Then $(\widehat{\mathbf{f}})^{-1}$ and $\widehat{\mathbf{f}^{-1}}$ coincide as maps. More exactly,*

$$(2.3) \quad L((\widehat{\mathbf{f}})^{-1}) = \widehat{L(\mathbf{f}^{-1})} = [\beta_i - b_{ij}],$$

where $L(\mathbf{f}^{-1}) = [b_{ij}]$ and $\beta_i = \max\{b_{ij} \mid j = 0, \dots, n\}$.

Proof. By Proposition 2.6, $\widehat{\mathbf{f}}$ defines a Cremona map, so it remains to show equality (2.3). Consider the log-matrices $L(\mathbf{f})$ and $L(\mathbf{f}^{-1})$. By [14, Theorem 2.2], there is a unique vector $\gamma \in \mathbb{N}^{n+1}$ (the inversion vector) such that

$$(2.4) \quad L(\mathbf{f}) \cdot L(\mathbf{f}^{-1}) = \Gamma + I_{n+1},$$

where $\Gamma = \underbrace{[\gamma \mid \dots \mid \gamma]}_{n+1}$.

Set $L(\mathbf{f}) = (a_{ij})$ and $L(\mathbf{f}^{-1}) = (b_{ij})$ and let $\alpha_i = \max\{a_{i0}, \dots, a_{in}\}$ and $\beta_i = \max\{b_{i0}, \dots, b_{in}\}$ denote the i th coordinates of the respective directrix vectors.

Define:

$$(2.5) \quad \begin{aligned} \widehat{b}_{ij} &:= \beta_i - b_{ij}, \\ \widehat{\gamma}_i &:= \gamma_i + \alpha_i \left(\sum_{l=0}^n \beta_l - d' \right) - \sum_{l=0}^n a_{il} \beta_l, \\ &= \gamma_i + \left(\alpha_i \deg(\widehat{\mathbf{f}^{-1}}) - (L(\mathbf{f}) \cdot \underline{\beta}^t)_i \right). \end{aligned}$$

where $d' = \deg(\mathbf{f}^{-1})$ and $\underline{\beta}^t$ is the directrix vector of \mathbf{f}^{-1} . Note that $\widehat{\gamma}_i$ depends solely upon the i th rows of $L(\mathbf{f})$ and $L(\mathbf{f}^{-1})$.

Letting \widehat{c}_{ij} denote the i, j entry of the product $\widehat{L(\mathbf{f})} \cdot (\widehat{b}_{ij})$, one has

$$\begin{aligned} \widehat{c}_{ij} &= \sum_{l=0}^n \widehat{a}_{il} \widehat{b}_{lj} = \sum_{l=0}^n (\alpha_i - a_{il})(\beta_l - b_{lj}) \\ &= \sum_{l=0}^n \alpha_i \beta_l - \sum_{l=0}^n \alpha_i b_{lj} - \sum_{l=0}^n a_{il} \beta_l + \sum_{l=0}^n a_{il} b_{lj}. \end{aligned}$$

From this, using (2.4), follows

$$\widehat{c}_{ij} = \alpha_i \left(\sum_{l=0}^n \beta_l - d' \right) - \sum_{l=0}^n a_{il} \beta_l + \gamma_i + \delta_{ij} = \widehat{\gamma}_i + \delta_{ij}.$$

Therefore, setting $\widehat{\gamma} = (\widehat{\gamma}_0 \dots \widehat{\gamma}_n)$ and noting that $\widehat{L(\mathbf{f}^{-1})} = [\widehat{b}_{ij}]$, one obtains

$$\widehat{L(\mathbf{f})} \widehat{L(\mathbf{f}^{-1})} = \underbrace{[\widehat{\gamma}^t \mid \dots \mid \widehat{\gamma}^t]}_{n+1} + I_{n+1},$$

and hence, again by [14, Theorem 2.2], $\widehat{L(\mathbf{f}^{-1})} = L((\widehat{\mathbf{f}})^{-1})$. □

Remark 2.11. Note that γ and $\hat{\gamma}$ are additively related by a summand that can be positive or negative. It is worth observing that the monomial whose log-matrix is the vector γ above is exactly the so-called target inversion factor of the Cremona map defined by \mathbf{f} . This points to a potential meaningful relation between the respective inversion factors of \mathbf{f} and its Newton complementary dual, at least in the monomial case. Inversion factors play a role in the realm of symbolic powers (see [11]).

Unfortunately, Proposition 2.10 does not extend to arbitrary Cremona maps. The following is one of the simplest examples where it fails.

Example 2.12. Consider the polar map in \mathbb{P}^2 defined by the cubic form $f := x(xz - y^2) \in k[x, y, z]$ (conic and a tangent line). It is well known that this polynomial is homaloidal and the associated polar map is a de Jonquières map of degree 2, hence an involution (up to a change of variables both in the source and the target).

An easy calculation shows that the Newton complementary dual of the set of the partial derivatives $\{2xz - y^2, -2xy, x^2\}$ is $\{x^2z - 2xy^2, xyz, y^2z\}$. The claim is that the Cremona map defined by these cubic forms is not an involution (up to a change of variables both in the source and the target). The inverse map to the latter Cremona map is defined by the forms $\{u^3 - tuv, u^2v - tv^2, 2uv^2\}$ in the target variables t, u, v . We cannot obviously argue by trying every change of variables in source and target. Thus, we need an effective criterion. We use the one stated in [5, Section 2.2], by which, in the particular case of rational maps defined on the ambient \mathbb{P}^n , there is a unique representative without fixed part. Since the tuple $(x^2z - 2xy^2, xyz, y^2z)$ has no fixed part, and neither does the tuple $(u^3 - tuv, u^2v - tv^2, 2uv^2)$ even after change of variables and coordinates, the only way would be that they give the same tuple up to renaming variables. This would imply, in particular, that the corresponding base ideals would coincide. But this is not the case as the first has three minimal primes, while the second has only two minimal primes (in particular, using a more geometric language, the maps have sets of proper and infinitely near points base points of different nature).

2.4.2. Plane de Jonquières maps. The so-called de Jonquières maps are at the heart of classical plane Cremona map theory. The next result enhances the role of the Newton dual for these maps.

We briefly recall a few preliminaries on these maps. A plane de Jonquières map can be defined by three forms of degree d in $k[x, y, z]$ of the shape xq, yq, f , where both q and f are z -monoids, with f irreducible (see [8, Section 4.1, especially Proposition 4.3]). (A z -monoid is a polynomial of the shape $f = f_0 + zf_1$, where f_0, f_1 are forms in $k[x, y]$. Irreducibility of such an f is equivalent to having $\gcd(f_0, f_1) = 1$ — see Section 3 for a more encompassing use of this notion.

The next result shows that de Jonquières maps are preserved under the Newton dual transform.

Proposition 2.13. *The Newton complementary dual of a plane de Jonquières map \mathfrak{F} is a plane de Jonquières map. More exactly, let \mathfrak{F} be defined by nonzero forms $\mathbf{f} = \{xq, yq, f\} \subset k[x, y, z]$ of degree d , where $q = g + zh$, with $g, h \in k[x, y]$ of degrees $d - 1, d - 2$, respectively, and $f = f_0 + zf_1$, with $f_0, f_1 \in k[x, y]$ of degrees $d, d - 1$, respectively. Then the linear system spanned by the Newton complementary dual $\hat{\mathbf{f}}$*

defines a plane de Jonquière's map of degree

$$\hat{d} := \max\{j_g + 1, j_h + 1, j_{f_0}, j_{f_1}\} - \min\{i_g, i_h + 1, i_{f_0}, i_{f_1} + 1\} + 1,$$

where

- $0 \leq i_g \leq d - 1$ is the first index such the term of g of order $(d - 1 - i_g, i_g)$ does not vanish
- $0 \leq j_g \leq d - 1$ is the last index such the term of g of order $(d - 1 - j_g, j_g)$ does not vanish;
- Similar definitions for $\{i_h, j_h\}$ (respectively, $\{i_{f_0}, j_{f_0}\}$ and $\{i_{f_1}, j_{f_1}\}$) regarding the form h (respectively, f_0 and f_1).

Proof. A straightforward inspection of the shape of the map gives the directrix vector

$$\underline{\alpha} = (\max\{d - i_g, d - 1 - i_h, d - i_{f_0}, d - i_{f_1}\}, \max\{j_g + 1, j_h + 1, j_{f_0}, j_{f_1}\}, 1)$$

as defined earlier. It remains to show that the resulting Cremona map is a de Jonquière's map.

For this, we display the Newton matrices of the forms $\mathbf{f} := \{xq, yq, f\}$, respecting the lexicographic order:

$$\begin{aligned} N(xq) &= \left(\begin{array}{cccc|cccc} d - i_g & > & \cdots & > & d - j_g & & d - 1 - i_h & > & \cdots & > & d - 1 - j_h \\ i_g & < & \cdots & < & j_g & & i_h & < & \cdots & < & j_h \\ 0 & & \cdots & & 0 & & 1 & & \cdots & & 1 \end{array} \right), \\ N(yq) &= \left(\begin{array}{cccc|cccc} d - 1 - i_g & > & \cdots & > & d - 1 - j_g & & d - 2 - i_h & > & \cdots & > & d - 2 - j_h \\ i_g + 1 & < & \cdots & < & j_g + 1 & & i_h + 1 & < & \cdots & < & j_h + 1 \\ 0 & & \cdots & & 0 & & 1 & & \cdots & & 1 \end{array} \right), \\ N(f) &= \left(\begin{array}{cccc|cccc} d - i_{f_0} & > & \cdots & > & d - j_{f_0} & & d - 1 - i_{f_1} & > & \cdots & > & d - 1 - j_{f_1} \\ i_{f_0} & < & \cdots & < & j_{f_0} & & i_{f_1} & < & \cdots & < & j_{f_1} \\ 0 & & \cdots & & 0 & & 1 & & \cdots & & 1 \end{array} \right). \end{aligned}$$

Here the symbol $>$ (respectively, $<$) is displayed to stress how the entries decrease (respectively, increase). Note that the sequences of entries represented by \cdots are typically lacunary as many terms have null coefficients. Actually, some blocks can collapse to one-column blocks. As a slight control, note that $d - j_g > 0$ since, by definition, $d - 1 - i_g \neq 0$; etc.

Clearly, \hat{f} is again a z -monoid. We next argue that two elements of $\hat{\mathbf{f}}$ are again of the form xQ, yQ for some z -monoid Q . For this, we show that the first coordinate of α satisfies $\alpha_1 - N(yq)_{1i} \geq 1 \forall i$. Namely, one has

$$\begin{aligned} \alpha_1 - N(yq)_{1i} &\geq \alpha_1 - \max\{d - 1 - i_g, d - 2 - i_h\} \geq \max\{d - i_g, d - 1 - i_h\} \\ &\quad - \max\{d - 1 - i_g, d - 2 - i_h\} = 1. \end{aligned}$$

One similarly shows that $\alpha_2 - N(yq)_{2i} \geq 1 \forall i$. This implies that there are forms Q_1, Q_2 such that $\hat{y}q = xQ_1$ and $\hat{x}q = yQ_2$. We claim that not only are Q_1, Q_2 z -monoids, but also $Q_1 = Q_2$. But this is immediate from looking at the respective Newton matrices

$$N(Q_1) = [\alpha - N(yq)] - \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad N(Q_2) = [\alpha - N(xq)] - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

To conclude, we know that since $\widehat{\mathbf{f}}$ defines a Cremona map without fixed part, according to the criterion of Enriques ([1, Theorem 5.1.1]) the k -vector space spanned by the elements of $\widehat{\mathbf{f}}$ contains an irreducible form F . Since the spanning elements are z -monoids, so must be F . Clearly then this vector space is spanned by $\{xQ, yQ, F\}$, thus defining a de Jonquières map. \square

3. Cremona maps galore with prescribed invariants

We retain the terminology and notation of Section 2.1. Thus, let $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ denote a Cremona map defined by a linear system spanned by forms f_0, \dots, f_n of the same degree satisfying the usual canonical restrictions. The common degree of these forms is the degree of \mathfrak{F} . The ideal generated by these forms in the homogeneous coordinate ring $R = k[x_0, \dots, x_n]$ of \mathbb{P}^n is the base ideal of \mathfrak{F} .

Definition 3.1. We will say that a Cremona map has a certain property if its base ideal has this property. Thus, a Cremona map whose base ideal is Cohen–Macaulay, of linear type, of codimension r , etc. is a Cohen–Macaulay Cremona map, a Cremona map of linear type, of codimension r , and so forth.

The following elementary result will subsequently be used.

Lemma 3.2. *Let $I \subset R$ be any ideal of a ring R , let $g \in R$ and let X denote an indeterminate over R . Then Xg is a zerodivisor on $RX/IR[X]$ (if and) only if g is a zerodivisor on R/I .*

Proof. The “if” assertion is obvious. To get the other direction, assume that g is not a zerodivisor on R/I and let $h \in R[X]$ be such that $h \cdot Xg \in IR[X]$. Then $hg \in IR[X] : (X) = IR[X]$ since X is an indeterminate over R . Therefore $h \in IR[X] : g = (I : g)R[X]$, i.e., every coefficient of h belongs to $I : g$. Since we are assuming that g is not a zero divisors on R/I , it follows that $h \in IR[X]$, as was to be shown. \square

Remark 3.3. The following souped-up version of Lemma 3.2 will be used below: Xg belongs to a minimal prime of $RX/IR[X]$ of minimal codimension if and only if g belongs to a minimal prime of R/I of minimal codimension.

3.1. Birational maps by recurrence. The following recurrence proposition clarifies and extends the result proved in [3, Proposition 3.2.2].

Proposition 3.4. *Let $\mathfrak{F} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{m-1}$ stand for a rational map defined by forms $f_0, \dots, f_{m-1} \in R = k[x_0, \dots, x_{n-1}]$ of degree d . Let*

$$g := g_\ell x_n^{d-\ell} + g_{\ell+1} x_n^{d-\ell-1} + \dots + g_{d-1} x_n + g_d \in R[x_n]$$

be a form of degree d , where $g_j \in R$ are forms, and g_ℓ is nonzero of degree $\ell \leq d-1$. If \mathfrak{F} is birational onto its image then the rational map $\mathfrak{G} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ defined by the forms $\{f_0, \dots, f_{m-1}, g\} \subset R[x_n]$ has topological, (i.e., field) degree at most $d-\ell$.

In particular, if $\ell = d-1$ then \mathfrak{G} is birational onto its image.

SUPPLEMENT. *Suppose that $\deg_{x_n}(g) = 1$ with $g_d \in I := (f_0, \dots, f_{m-1})$ (possibly vanishing). If g_{d-1} avoids all minimal primes of R/I of minimal codimension (respectively, is a nonzerodivisor on R/I and the latter is Cohen–Macaulay) then the codimension of $R[x_n]/(I, g)$ is one larger than that of R/I (respectively, $R[x_n]/(I, g)$ is Cohen–Macaulay).*

Proof. By assumption, the field extension $k(f_1, \dots, f_{m-1}) \subset k((x_0, \dots, x_{n-1})_d)$ is an equality, where $(x_0, \dots, x_{n-1})_d$ denotes the set of monomials of degree d in $k[x_0, \dots, x_{n-1}]$. On the other hand, it is easy to see that $k((x_0, \dots, x_{n-1})_d) = k(x_0^d, x_1/x_0, \dots, x_{n-1}/x_0)$. Therefore, and by the same token, it suffices to prove that x_n/x_0 satisfies an equation of degree $d - \ell$ with coefficients in $k(f_0, \dots, f_{m-1}, g)$.

Using the condensed monomial notation $\mathbf{x}^{\mathbf{a}} = x_0^{a_0} \cdots x_{n-1}^{a_{n-1}}$, let $\sum_{|\mathbf{a}|=j} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ be the expression of a form $f \in R$ of degree j . Then

$$\begin{aligned} f &= \sum_{|\mathbf{a}|=j} c_{\mathbf{a}} x_0^{\overbrace{a_0 + \cdots + a_{n-1}}^j} \underbrace{(x_1/x_0)^{a_1} \cdots (x_{n-1}/x_0)^{a_{n-1}}}_{h_{\mathbf{a}} \in k(f_0, \dots, f_{m-1})} \\ &= x_0^j \sum_{|\mathbf{a}|=j} c_{\mathbf{a}} h_{\mathbf{a}} = x_0^j h_j, \end{aligned}$$

for suitable $h_j \in k(f_0, \dots, f_{m-1})$, where we have used the equality $k(f_0, \dots, f_{m-1}) = k(x_0^d, x_1/x_0, \dots, x_{n-1}/x_0)$ given in the hypothesis.

Therefore

$$\begin{aligned} g &= x_0^\ell x_n^{d-\ell} h_\ell + x_0^{\ell+1} x_n^{d-\ell-1} h_{\ell+1} + \cdots + x_0^{d-1} x_n h_{d-1} + x_0^d h_d \\ &= x_0^\ell (x_n^{d-\ell} h_\ell + x_0 x_n^{d-\ell-1} h_{\ell+1} + \cdots + x_0^{d-\ell-1} x_n h_{d-1} + x_0^{d-\ell} h_d) \\ &= x_0^d \left(h_\ell \left(\frac{x_n}{x_0} \right)^{d-\ell} + h_{\ell+1} \left(\frac{x_n}{x_0} \right)^{d-\ell-1} + \cdots + h_{d-1} \left(\frac{x_n}{x_0} \right) + h_d \right). \end{aligned}$$

Now, $x_0^{-d} g \in k(x_0^d, x_1/x_0, \dots, x_{n-1}/x_0, g) = k(f_0, \dots, f_{m-1}, g)$. But, for every j one has $h_j \in k(f_0, \dots, f_{m-1})$, hence we are through.

The assertions in the supplement follow immediately from a slight modification of Lemma 3.2 and the remark after it, using the assumption that $g_d \in I$. \square

Forms of the shape $g = g_{d-1}x_n + g_d$, with $g_{d-1}, g_d \in R$ are called *monoids* — see [10] for an overview of the concept. It has been considered in Section 2.4 in the case $n = 2$ in connection to a de Jonquières map. We will often refer to a monoid as an x_n -monoid in order to stress the privileged variable x_n .

Remark 3.5. It is conceivable that the degree of \mathfrak{G} in Proposition 3.4 be exactly $d - \ell$ provided the coefficients g_j are sufficiently general forms. Drawing upon the characteristic-free methods developed in [5], where a certain Jacobian matrix is introduced, it would amount to be able to express the rank of that matrix in a formula such as $\max\{0, n + 1 - (d - \ell)\}$. Computation however can get pretty heavy. Thus, even at the far end of the spectrum, when $d - \ell = 1$ (the birational case) the degree of the inverse map to \mathfrak{G} is not easily predictable in terms of the degree of the inverse to \mathfrak{F} . This tells us that the corresponding two graphs have far different structure — alternatively, the equations defining the Rees algebra of the ideal $(I, g) \subset R[x_n]$ are not totally predicted from those defining the Rees algebra of the ideal $I \subset R$.

A slightly different type of construction yields Cremona maps which are Cohen–Macaulay with base ideal of codimension 2. As above, we first state isolate a general result which is possibly available in the literature in a disguised form.

Proposition 3.6. *Let R be a Cohen–Macaulay Noetherian ring and let $I \subsetneq R$ be an ideal such that R/I is perfect of codimension 2. Let $R \rightarrow T$ be a flat R -algebra such that $IT \neq T$ and $\text{ht } IT \geq \text{ht } I$. Then, for arbitrary elements $g \in IT$ and $f \in R$ such that the ideal $(f, g)T$ has codimension 2, the T -module $T/(If, g)T$ is perfect of codimension 2.*

Proof. Tensor a free R -resolution $0 \rightarrow F_1 \xrightarrow{\varphi} F_0 \rightarrow R$ of R/I with T to get a similar free T -resolution of T/IT . For simplicity, we write φ for $T \otimes_R \varphi$ and F_i for $T \otimes_R F_i$, $i = 0, 1$.

As one readily verifies, multiplication by g induces an injective T -module homomorphism $T/(IT : (g))f = T/IfT \hookrightarrow T/IfT$ with image $(If, g)T/IfT$. This homomorphism lifts to a map of complexes (free resolutions over T)

$$\begin{array}{ccccccccc} 0 & \rightarrow & F_1 & \xrightarrow{\varphi} & F_0 & \xrightarrow{f\mathbf{g}} & T & \rightarrow & T/IfT & \rightarrow & 0 \\ & & \uparrow & & c(g)\uparrow & & \cdot g\uparrow & & \cdot g\uparrow & & \\ 0 & \rightarrow & T & \xrightarrow{f} & T & \rightarrow & T/IfT & \rightarrow & 0 \end{array},$$

where \mathbf{g} is the map induced by a set of generators of I and $c(g) : T \rightarrow F_0$ is the content map that writes g as an element of IT , once a basis of F_0 is fixed.

Then the corresponding mapping cone is a T -free resolution (see [6, Exercise A3.30]):

$$(3.1) \quad 0 \rightarrow F_1 \oplus T \xrightarrow{\psi} F_0 \oplus T \xrightarrow{(f\mathbf{g}, g)} T \rightarrow T/(If, g)T \rightarrow 0,$$

where

$$\psi = \begin{pmatrix} \varphi & c(g) \\ \mathbf{0} & -f \end{pmatrix}.$$

To conclude, the ideal $(If, g) \subset T$ has codimension 2 since, by assumption, $(f, g) \subset T$ has codimension 2 and $(I, g)T = IT$ has codimension ≥ 2 . \square

The proposition has the following application to birational theory.

Theorem 3.7. *Let $f_0, \dots, f_{m-1} \in R = k[x_0, \dots, x_{n-1}]$ be forms of degree d satisfying the canonical restrictions and let $\mathfrak{F} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{m-1}$ denote the corresponding rational map. Set $I := (f_0, \dots, f_{m-1}) \subset R$. Let $f \in R$ and $g \in IR[x_n]$, be nonzero forms such that $\deg(g) = \deg(f) + d$ and $\gcd(f, g) = 1$ in $R[x_n]$, where x_n is a new variable. Suppose that:*

- *The base ideal I is a codimension 2 perfect ideal.*
- *g is an x_n -monoid.*
- *\mathfrak{F} is birational onto its image (hence, $m \geq n$).*

Then the rational map $\mathfrak{G} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ defined by the forms $\{f_0f, \dots, f_{m-1}f, g\} \subset R[x_n]$ is a birational map (onto its image) whose base ideal is perfect of codimension 2. In particular, if \mathfrak{F} is a perfect codimension 2 Cremona map of \mathbb{P}^{n-1} then \mathfrak{G} is a perfect codimension 2 Cremona map of \mathbb{P}^n .

Proof. Note that the assertion can be broken up into two separate assertions. The assertion about the perfectness of the base ideal (If, g) follows immediately from Proposition 3.6 by taking $T := R[x_n]$ and applying it to the ideal $I = (f_0, \dots, f_{m-1})$.

To prove the birationality claim one argues that $f_0f, \dots, f_{m-1}f$ also defines a birational map, hence one can apply Proposition 3.4 to this map (*not* to the rational map defined by f_0, \dots, f_{m-1} , and observe we do not use the supplement there). \square

Remark 3.8. The content of the above theorem is similar to the situation of a *de Jonquières parametrization* as in [9] — actually, to a special case called “inclusion case”. However, the present setup is slightly different as one is feeding in a new variable as well (a flat extension). This has the effect that the proof in the present situation is a bit less automatic. Moreover, as easy examples show, the hypothesis of g being a monoid is in general essential both in Theorem 3.7 and Proposition 3.4.

The following is the main result of the section.

Theorem 3.9. *Let $n \geq 2$, $d \geq 2$ be integers. Then, for every r in the integer interval $[2, n]$, there exist Cremona transformations of \mathbb{P}^n of codimension r and degree d , and there exist Cohen–Macaulay Cremona transformations of \mathbb{P}^n of codimension r and degree $d \geq n + 1 - r$.*

Proof. Induct on n . If $n = 2$ there is nothing to prove since any Cremona transformation of \mathbb{P}^2 has codimension 2 and, for any $d \geq 1$ the forms $x_0^d, x_0^{d-1}x_1, x_1^{d-1}x_2$ define a plane Cohen–Macaulay Cremona map of degree d .

So, suppose that $n \geq 3$. By the inductive hypothesis, for any r in the integer range $[2, n-1]$, \mathbb{P}^{n-1} admits Cremona maps of codimension r and degree d , and also Cohen–Macaulay ones of codimension r and degree $d \geq n-1+1-r = n+1-(r+1)$. By Proposition 3.4, for any integer s in the range $[2, n]$, we obtain Cremona maps of codimension s and degree d , and also Cohen–Macaulay Cremona transformations of \mathbb{P}^n of any codimension in the range $[3, n]$ and degree $d \geq n+1-s$. To show the existence of the latter also in codimension 2 we apply Theorem 3.7 by initiating with any Cohen–Macaulay plane Cremona (see the remark below and also Theorem 3.12 in the next section which gives a structured example of a perfect codimension 2 Cremona map of \mathbb{P}^{n-1} of any degree $\geq n-2+q$ with $q \geq 1$). \square

Remark 3.10. For $n = 2$ and arbitrary $d \geq 1$, the class of de Jonquières maps gives infinitely many distinct Cohen–Macaulay examples (see, e.g., [8, Proposition 4.2 and Corollary 4.5]), of which the monomial Cremona map in the proof is an instance. Thus, the proof of the theorem also shows that, for any $n \geq 2$, there are infinitely many such Cohen–Macaulay Cremona maps for any choices of $d \geq 1$ and of a prescribed codimension in the integer range $[2, n]$.

The lower bound $d \geq n + 1 - r$, with r the codimension of a Cohen–Macaulay Cremona map, is quite natural. Actually, it always holds for $r = 2$ since the lowest (standard) degree of a syzygy of the base ideal is 1, while it adds no condition for $r = n$. The natural question left is:

Question 3.11. Is the lower bound $d \geq n + 1 - r$, in the range $r \in [3, n-1]$, for Cohen–Macaulay Cremona maps, best possible for all $n \geq 3$?

3.2. A determinantal model. The following degeneration of the ordinary generic Hankel matrix has been introduced in [2, Section 4.1.1], where it has been called a *sub-Hankel* matrix.

$$(3.2) \quad \begin{pmatrix} x_0 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \\ x_2 & x_3 & x_4 & \dots & x_n & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ x_{n-2} & x_{n-1} & x_n & \dots & 0 & 0 \\ x_{n-1} & x_n & 0 & \dots & 0 & 0 \end{pmatrix}$$

It has been shown in [2] that the determinant $F \in k[x_0, \dots, x_n]$ of the sub-Hankel matrix is a homaloidal polynomial (with degree equal to the dimension of the ambient projective space). As a bonus one also obtained Cremona maps with Cohen–Macaulay codimension 2 base ideal. However, the degree of any of these maps is bounded above by the number of variables. Here we wish to use part of the coordinates of the corresponding polar map — i.e., a subset of the partial derivatives of F — in order to obtain a determinantal like model for the Cremona maps in the previous subsection which also have Cohen–Macaulay codimension 2 base ideal and, moreover, of degree arbitrarily larger than the number of variables.

Theorem 3.12. *Let $F \in k[x_0, \dots, x_n]$ denote the determinant of the matrix (3.2), with $n \geq 3$. Let the partial derivatives of F be denoted $F_{x_0}, \dots, F_{x_{n-1}}, F_{x_n}$. Then x_n divides F_{x_i} , for $0 \leq i \leq n-2$ and for any integer $q \geq 1$, the forms*

$$\frac{F_{x_0}}{x_n} x_n^q, \dots, \frac{F_{x_{n-2}}}{x_n} x_n^q, F_{x_{n-1}} x_n^{q-1} + \frac{F_{x_{n-2}}}{x_n} x_{n-1}^q$$

belong to $k[x_1, \dots, x_n]$ and define a codimension 2 Cohen–Macaulay Cremona map of \mathbb{P}^{n-1} .

Proof. First, one has $\gcd(F_{x_0}, \dots, F_{x_{n-2}}) = x_n$ and x_n does not divide $F_{x_{n-2}}/x_n$ (second claim of [2, Lemma 4.2 (ii)]).

Moreover, The first $n-2$ partial derivatives of F divided by their gcd x_n define a Cremona map with perfect codimension 2 base ideal ([2, Theorem 4.4 (i)]).

Next, $F_{x_0}, \dots, F_{x_{n-2}} \in k[x_2, \dots, x_n]$ ([2, Lemma 4.2 (ii)]). Also, by direct inspection of the derivatives one sees that $F_{x_{n-i}}$ is an x_i -monoid. In particular, $F_{x_{n-1}}$ is an x_1 -monoid, hence $g := F_{x_{n-1}} x_n^{q-1} + \frac{F_{x_{n-2}}}{x_n} x_{n-1}^q$ is also an x_1 -monoid since x_1 does not appear in $F_{x_{n-2}}$. It is clear that g is not divisible by x_n .

Finally, g belongs to the ideal generated by the first $n-2$ partial derivatives of F divided by their gcd x_n as a case of the relation (4.4) in [2, Lemma 4.2 (iii)].

Assembling the information, the result is now an application of Theorem 3.7, with $f := x_n^q$ and $f_i := F_{x_i}/x_n$, $i = 0, \dots, n-2$ and x_1 as the additional variable (in the role of x_n in that theorem). \square

Remark 3.13. Of course, one could trade the last coordinate g in the theorem by any x_1 -monoid of the right degree not divisible by x_n and belonging to the ideal generated by the first $n-2$ partial derivatives of f divided by their gcd x_n . However, we wished to keep the map as close as possible to the original polar map. Moreover, there is some computational evidence for conjecturing that its inverse map has the same shape, though of a different degree.

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