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ON p-ADIC PERIODS FOR MIXED TATE MOTIVES OVER A NUMBER FIELD

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Abstract. For a number field, we have a Tannaka category of mixed Tate motives at our disposal. We construct p-adic points of the associated Tannaka group by using p-adic Hodge theory. Extensions of two Tate objects yield functions on the Tannaka group, and we show that evaluation at our p-adic points is essentially given by the inverse of the Bloch–Kato exponential map.

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Introduction

For a number field E, one has an abelian category of mixed Tate motives $MT(E)$ [3]. A mixed Tate motive comes equipped with a weight filtration W, and the associated graded pieces are sums of Tate objects. There is a natural fibre functor ω defined by

$$\omega(M) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(\mathbb{Q}(n), \mathrm{gr}^W_{-2n}(M));$$

we denote by G_ω the corresponding Tannaka group.

If $\mathcal{O}$ denotes the ring of integers of E and $x \in \mathrm{Spec}(\mathcal{O})$ is a closed point, then Deligne and Goncharov construct a Tannaka subcategory $MT(\mathcal{O}_x)$ of $MT(E)$ consisting of motives which are unramified at x [3, 1.6]. We will denote its group of tensor automorphisms by G_x.

To a mixed Tate motive M we can attach its p-adic realization M_p which is a representation of the Galois group of E with coefficients in $\mathbb{Q}_p$. Suppose that the

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point x lies over the prime p, and $\iota : \overline{E} \to \overline{E}_x$ is an imbedding of the algebraic closure of E to that of the completion of E at x. This induces a homomorphism between the Galois groups, and restricting the Galois representation M_p via this homomorphism gives us a representation of the Galois group of E_x, which we denote by $M_{\iota,p}$. We will show that $M_{\iota,p}$ is always semistable. Furthermore, $M_{\iota,p}$ is crystalline if and only if M is unramified at x, i.e., $M \in MT(\mathcal{O}_x)$ (Theorem 2.2.3). In fact, p-adic representations attached to mixed Tate motives are contained in an abelian subcategory which admits a fibre functor τ similar to ω. Denoting by H_τ the corresponding Tannaka group over $\mathbb{Q}_p$, p-adic realization yields a group homomorphism

$$H_\tau \to G_\omega \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

The main purpose of this paper is to construct an $E_{x,st}$-valued point η_{st} of H_τ, where $\mathrm{Spec}(E_{x,st})$ is a one-dimensional affine space over the field E_x. The E_x-valued points of $\mathrm{Spec}(E_{x,st})$ correspond naturally to the extensions of the canonical logarithm $\log : \mathcal{O}_{E_x}^\times \to E$ to $E^\times$. Therefore, any choice of such an extension induces via η_{st} an E_x-valued point of H_τ and G_ω. For the Tannaka subcategory of crystalline representations the picture is simpler: if $H_{\tau,\mathrm{cris}}$ denotes their Tannaka group and $\pi : H_\tau \to H_{\tau,\mathrm{cris}}$ is the projection, then $\pi \circ \eta_{st}$ factors through $\mathrm{Spec}(E_x)$ and we obtain an E_x-valued point η of $H_{\tau,\mathrm{cris}}$. We denote by η_x^{ur} the image of η in G_x.

To state our main theorem, we need to recall how extensions M of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ in $MT(\mathcal{O}_x)$ give rise to functions on G_x for $n \geq 1$. The natural isomorphisms $\alpha : \mathbb{Q} \to \mathrm{Hom}(\mathbb{Q}(n), \mathrm{gr}^W_{-2n} M)$ and $\beta : \mathrm{Hom}(\mathbb{Q}(0), \mathrm{gr}^W_0 M) \to \mathbb{Q}$ induce elements $\alpha^{-1} \in \omega(M)^\vee$ and $\beta^{-1} \in \omega(M)$; we set $M(\eta_x^{ur}) = \alpha^{-1}(\eta_x^{ur} \cdot \beta^{-1}(1))$.

Theorem (Theorem 2.3.3)**.** *For all $n \geq 1$, the map*

$$\mathrm{Ext}^1_{MT(\mathcal{O}_x)}(\mathbb{Q}(0), \mathbb{Q}(n)) \to E_x, \quad M \mapsto M(\eta_x^{ur}),$$

is the composition of the p-adic realization

$$\mathrm{Ext}^1_{MT(\mathcal{O}_x)}(\mathbb{Q}(0), \mathbb{Q}(n)) \to \mathrm{Ext}^1_{\mathrm{crys}}(\mathbb{Q}_p(0), \mathbb{Q}_p(n))$$

and the inverse of the Bloch–Kato exponential map (2.3.1).

1. Filtered ϕ-modules and mixed Tate filtered ϕ-modules

1.1. Mixed Tate filtered ϕ-modules.

1.1.1. Let K be a p-adic field with residue field k, i.e., $\mathrm{char}(K) = 0$, K is complete with respect to a fixed discrete valuation and the residue field k is perfect of characteristic p. Let $W(k)$ be the ring of Witt vectors of k, $\sigma : W(k) \to W(k)$ the Frobenius lift and K_0 the field of fractions of $W(k)$.

1.1.2. We denote by MF_K^ϕ the category of filtered ϕ-modules, i.e., the objects are triples (M, ϕ, F), where (M, ϕ) is an isocrystal over K_0 and F is a descending, exhaustive and separated filtration on $M_K = M \otimes_{K_0} K$. We denote by $MF_K^{\phi,N}$ the category of filtered (ϕ, N)-modules, i.e., objects are tuples (M, ϕ, N, F) with $(M, \phi, F) \in MF_K^\phi$ and $N : M \to M$ is a K_0-linear endomorphism such that $N\phi = p\phi N$. We consider MF_K^ϕ as full subcategory of $MF_K^{\phi,N}$ via the functor $(M, \phi, F) \mapsto (M, \phi, 0, F)$.

The Dieudonné–Manin classification [4, II, Section 4.1] implies, by descent, that every isocrystal (M, ϕ) over K_0 admits a slope decomposition

$$M = \bigoplus_{\lambda \in \mathbb{Q}} M_\lambda,$$

with $\phi(M_\lambda) = M_\lambda$ and $(M_\lambda, \phi\mid_{M_\lambda})$ is isoclynic of slope λ. From the relation $N\phi = p\phi N$, it follows that $N(M_\lambda) \subseteq M_{\lambda-1}$. In the following, we will use the notation:

$$M_{\leq\lambda} := \bigoplus_{\substack{\lambda' \in \mathbb{Q} \\ \lambda' \leq \lambda}} M_{\lambda'}, \quad M_{\geq\lambda} := \bigoplus_{\substack{\lambda' \in \mathbb{Q} \\ \lambda' \geq \lambda}} M_{\lambda'}.$$

Definition 1.1.3. We say that an object $(M, \phi, F) \in MF_K^\phi$ is a *mixed Tate filtered ϕ-module* if the following properties are satisfied:

(1) There is an isomorphism of ϕ-modules

$$(M, \phi) \cong \bigoplus_{i \in I} (K_0, p^{n_i}\sigma),$$

for some index set I, and $n_i \in \mathbb{Z}$.

(2) For all $i \in \mathbb{Z}$ the natural map

$$F^i M_K \to M_{\geq i} \otimes_{K_0} K$$

is an isomorphism.

We say that $(M, \phi, N, F) \in MF_K^{\phi,N}$ is a *mixed Tate filtered (ϕ, N)-module* if (M, ϕ, F) is a mixed Tate filtered ϕ-module.

We denote by MT_K^ϕ (resp. $MT_K^{\phi,N}$) the full subcategory of MF_K^ϕ (resp. $MF_K^{\phi,N}$) with mixed Tate filtered ϕ-modules (resp. (ϕ, N)-modules) as objects. The categories MT_K^ϕ and $MT_K^{\phi,N}$ are additive. Again, we consider MT_K^ϕ as full subcategory of $MT_K^{\phi,N}$.

For $(M, \phi, N, F) \in MT_K^{\phi,N}$, it follows from Property (1) that all the slopes of (M, ϕ) are integers. From Property (2) we conclude that the Hodge polygon of (M_K, F) equals the Newton polygon of (M, ϕ).

Definition 1.1.4 (Tate objects)**.** Let $n \in \mathbb{Z}$ be an integer. We define the *Tate object* $K(n) \in MT_K^\phi$ by

$$K(n) := (K_0, p^{-n}\sigma, F),$$

with F defined by

$$F^j = \begin{cases} K, & \text{if } j \leq -n, \\ 0, & \text{if } j > -n. \end{cases}$$

Definition 1.1.5 (Weight filtration)**.** Let $(M, \phi, N, F) \in MT_K^{\phi,N}$. Let $i \in \mathbb{Z}$ be an integer. We define an object $W_{2i}(M, \phi, N, F)$ in MF_K^ϕ by

$$W_{2i}(M, \phi, N, F) := (M_{\leq i}, \phi\mid_{M_{\leq i}}, N\mid_{M_{\leq i}}, F \cap M_{\leq i}).$$

We define an object $\mathrm{gr}_{2i}^W(M, \phi, N, F)$ in MT_K^ϕ by

$$\mathrm{gr}_{2i}^W(M, \phi, N, F) := (M_i, \phi\mid_{M_i}, \tilde{F}),$$

where $\tilde{F}$ is defined as follows:

$$\tilde{F}^i M_i = M_i, \quad \tilde{F}^{i+1} M_i = 0.$$

Note that $N(M_{\leq i}) \subset M_{\leq i-1}$ and $N\,|_{M_{\leq i}}$ is well-defined.

Proposition 1.1.6. *Let $(M, \phi, N, F) \in MT_K^{\phi,N}$ and $i \in \mathbb{Z}$. The following statements hold.*

(1) *The object $W_{2i}(M, \phi, N, F)$ is contained in $MT_K^{\phi,N}$.*
(2) *There is an exact sequence*

$$0 \to W_{2(i-1)}(M, \phi, N, F) \to W_{2i}(M, \phi, N, F) \to \mathrm{gr}^W_{2i}(M, \phi, N, F) \to 0. \tag{1.1.1}$$

Proof. It is sufficient to prove the statement for $(M, \phi, 0, F)$, i.e., for objects in MT_K^{ϕ}.

For (1). It is obvious that

$$W_{2i}(W_{2(i+1)}(M, \phi, F)) = W_{2i}(M, \phi, F),$$

for all (M, ϕ, F). Therefore, we may reduce to the case

$$W_{2(i+1)}(M, \phi, F) = (M, \phi, F).$$

In this case $M = M_{\leq i} \oplus M_{i+1}$, and we have to prove that for all $j \in \mathbb{Z}$ the map

$$F^j \cap (M_{\leq i} \otimes_{K_0} K) \to (M_{\leq i})_{\geq j} \otimes_{K_0} K$$

is an isomorphism. Since (M, ϕ, F) is an object in MT_K^{ϕ}, the map is injective. In particular, the map is an isomorphism for all $j \geq i + 1$.

We need to show the surjectivity for $j \leq i$. By assumption, for every $m \in (M_{\leq i})_{\geq j} \otimes_{K_0} K$ there exists a preimage $m' \in F^j M_K$. By definition, the projection of m' to $M_{i+1} \otimes_{K_0} K$ vanishes, thus $m' \in F^j \cap (M_{\leq i} \otimes_{K_0} K)$.

For (2). There is an obvious morphism $W_{2(i-1)}(M, \phi, F) \to W_{2i}(M, \phi, F)$ in MT_K^{ϕ}. The morphism $W_{2i}(M, \phi, F) \to \mathrm{gr}^W_{2i}(M, \phi, F)$ is defined by the projection $M_{\leq i} \to M_i$. Since $F^{i+1} \cap (M_{\leq i} \otimes_{K_0} K) = 0$, the projection is compatible with the filtrations. Therefore the sequence (1.1.1) is well-defined.

In order to prove that the sequence is exact we need to show that it is an exact sequence of ϕ-modules and an exact sequence of filtered K-vector spaces. The first statement is obvious. For the second statement we note that all members in the sequence (1.1.1) are objects in MT_K^{ϕ}, thus the Hodge polygons equal the Newton polygons. In particular,

$$\dim(F^j \cap M_{\leq i}) = \dim(F^j \cap M_{\leq i-1}) + \dim \tilde{F}^j,$$

for all $j \in \mathbb{Z}$. This immediately implies the claim. □

Corollary 1.1.7. *The category $MT_K^{\phi,N}$ is contained in the category of weakly admissible filtered (ϕ, N)-modules.*

Proof. We use the fact that weakly admissible filtered (ϕ, N)-modules are stable under extensions. Therefore the claim follows from Proposition 1.1.6 provided we prove that $\mathrm{gr}^W_{2i}(M, \phi, N, F)$ is weakly admissible for all $(M, \phi, N, F) \in MT_K^{\phi,N}$ and all $i \in \mathbb{Z}$. By Definition 1.1.5, $\mathrm{gr}^W_{2i}(M, \phi, N, F)$ is isomorphic to a direct sum of Tate objects $K(-i)$. Since Tate objects are (weakly) admissible, we are done. □

In contrast to the category $MF_K^{\phi,N}$, the category of weakly admissible filtered (ϕ, N)-modules $MF_K^{\phi,N,wa}$ is an abelian category.

Proposition 1.1.8. *Let* $f : (M, \phi_M, N_M, F_M) \to (M', \phi_{M'}, N_{M'}, F_{M'})$ *be a morphism in* $MT_K^{\phi,N}$. *We denote by* $\ker(f)$ *and* $\operatorname{coker}(f)$ *the kernel of* f *and the cokernel of* f *in* $MF_K^{\phi,N,wa}$, *respectively. Then* $\ker(f)$ *and* $\operatorname{coker}(f)$ *are contained in* $MT_K^{\phi,N}$. *In particular,* $MT_K^{\phi,N}$ *is an abelian category.*

Proof. First, consider the full subcategory $\mathcal{C}$ of isocrystals over K_0 with objects (M, ϕ) such that there exists an isomorphism

$$(M, \phi) \cong \bigoplus_{i \in I} (K_0, p^{n_i}\sigma).$$

It is easy to see that $\mathcal{C}$, as subcategory of the category of isocrystals, contains all the kernels and cokernels of morphisms in $\mathcal{C}$.

We denote by f_0 the induced morphism $(M, \phi_M) \to (M', \phi_{M'})$. Then

$$\ker(f) = (\ker(f_0), \phi\,|_{\ker(f_0)}, N\,|_{\ker(f_0)}, F \cap (\ker(f_0) \otimes_{K_0} K)).$$

We know that $\ker(f_0) \in \mathcal{C}$ and thus satisfies Property (1) of Definition 1.1.3. It remains to show that

$$F^i_M \cap (\ker(f_0) \otimes_{K_0} K) \to \ker(f_0)_{\geq i} \otimes_{K_0} K$$

is an isomorphism. We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^i_M \cap (\ker(f_0) \otimes_{K_0} K) & \longrightarrow & F^i_M & \longrightarrow & F^i_{M'} \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \ker(f_0)_{\geq i} \otimes_{K_0} K & \longrightarrow & M_{\geq i} \otimes_{K_0} K & \longrightarrow & M'_{\geq i} \otimes_{K_0} K. \end{array}$$

Moreover, both rows are exact, which implies Property (2) of Definition 1.1.3.

The claim for the cokernel follows dually. □

1.1.9. The categories $MT_K^{\phi,N}$ and MT_K^{ϕ} are $\mathbb{Q}_p$-linear rigid $\otimes$-categories.

Lemma 1.1.10. *The functor*

$$\tilde{\omega} : MT_K^{\phi,N} \to (\mathbb{Q}_p\text{-vector spaces}), \quad (M, \phi, N, F) \mapsto \bigoplus_{n \in \mathbb{Z}} \tilde{\omega}_n(M, \phi, F), \tag{1.1.2}$$

with

$$\tilde{\omega}_n(M, \phi, F) = \operatorname{Hom}_{MT_K^{\phi}}(K(n), \operatorname{gr}^W_{-2n}(M, \phi, F)),$$

is a fibre functor. In particular, $(MT_K^{\phi,N}, \tilde{\omega})$ *and* $(MT_K^{\phi}, \tilde{\omega})$ *are Tannaka categories.*

Proof. It is easy to see that $\tilde{\omega}$ is a $\otimes$-functor. In order to see that $\tilde{\omega}$ is exact and faithful we will prove the existence of an isomorphism

$$\tilde{\omega}_{K_0} \xrightarrow{\cong} (\gamma : (M, \phi, N, F) \mapsto M), \tag{1.1.3}$$

where $\tilde{\omega}_{K_0}(M, \phi, N, F) = \tilde{\omega}(M, \phi, N, F) \otimes_{\mathbb{Q}_p} K_0$ and γ forgets about ϕ, N and F. Since γ is exact and faithful, this will imply the claim.

In order to construct (1.1.3), we observe that there is a functorial isomorphism

$$\text{(1.1.4)} \qquad \mathrm{Hom}_{MT_K^\phi}(K(n), \mathrm{gr}^W_{-2n}(M,\phi,N,F)) \otimes_{\mathbb{Q}_p} K_0 \to M_{-n}, \quad \phi \otimes a \mapsto a \cdot \phi(1).$$

□

Proposition 1.1.11. *An object $(M,\phi,N,F) \in MF_K^{\phi,N,wa}$ belongs to $MT_K^{\phi,N}$ if and only if there exists an increasing exhaustive separated filtration W by subobjects of (M,ϕ,N,F) in $MF_K^{\phi,N,wa}$ such that W_i/W_{i-1} vanishes if i is odd, and is a sum of Tate objects $K(-\frac{i}{2})$ if i is even.*

Proof. For $(M,\phi,N,F) \in MT_K^{\phi,N}$, such a filtration exists by Definition 1.1.5, Proposition 1.1.6, and the fact that $\mathrm{gr}^W_{2i}(M,\phi,N,F)$ is a sum of Tate objects $K(-i)$.

Suppose now that $(M,\phi,N,F) \in MF_K^{\phi,N,wa}$ admits a filtration W satisfying the assumptions. It is easy to see that (M,ϕ) satisfies Property (1) of Definition 1.1.3.

In general, if

$$0 \to M_1 \to M \to M_2 \to 0$$

is an exact sequence in $MF_K^{\phi,N,wa}$, and M_1, M_2 satisfy Property (2), then M satisfies Property (2). By induction on i we conclude that $W_i \in MT_K^{\phi,N}$ for all i. □

It is clear that any filtration as in Proposition 1.1.11 has to coincide with the weight filtration, and that any morphism between two objects in $MT_K^{\phi,N}$ has to be strict with respect to the weight filtrations on these objects.

1.2. The crystalline logarithmic point.

1.2.1. Recall from (1.1.2) that we have a fibre functor $\tilde{\omega}$ equipping MT_K^ϕ and $MT_K^{\phi,N}$ with the structure of Tannaka categories (Lemma 1.1.10). Let $G_{\tilde{\omega}}$ and $G_{\tilde{\omega}}^{st}$ denote the pro-algebraic groups which represent tensor automorphisms of $\tilde{\omega}$ on MT_K^ϕ and $MT_K^{\phi,N}$, respectively. In other words, we have $G_{\tilde{\omega}} = \underline{\mathrm{Aut}}^{\otimes}_{MT_K^\phi}\tilde{\omega}$ and $G_{\tilde{\omega}}^{st} = \underline{\mathrm{Aut}}^{\otimes}_{MT_K^{\phi,N}}\tilde{\omega}$. The goal of this section is to construct a non-trivial K-valued point η of $G_{\tilde{\omega}}$.

Definition 1.2.2. For $(M,\phi,F) \in MT_K^\phi$ we define

$$\eta(M,\phi,F) : M_K \to M_K$$

to be the unique endomorphism rendering the following diagram commutative:

$$\text{(1.2.1)} \qquad \begin{array}{ccc} M_K = \bigoplus_{i\in\mathbb{Z}} M_i \otimes_{K_0} K & \xrightarrow{\bigoplus_i \iota_i \otimes_{K_0} K} & \bigoplus_{i\in\mathbb{Z}} M_{\geq i} \otimes_{K_0} K \\ \Big\downarrow{\scriptstyle \eta(M,\phi,F)} & & \Big\downarrow{\scriptstyle (\bigoplus_{i\in\mathbb{Z}} \pi_i)^{-1}} \\ M_K & \xleftarrow[\sum_{i\in\mathbb{Z}}]{} & \bigoplus_{i\in\mathbb{Z}} F^i M_K, \end{array}$$

where $\iota_i : M_i \to M_{\geq i}$ is the obvious inclusion, $\pi_i : F^iM_K \to M_{\geq i} \otimes_{K_0} K$ is the projection and therefore by definition an isomorphism (Definition 1.1.3(2)), and $\sum_{i\in\mathbb{Z}}$ is the sum over the obvious inclusions.

Lemma 1.2.3. *The morphisms η from Definition 1.2.2 define a tensor automorphism of the fibre functor $\tilde{\omega}_K = \tilde{\omega} \otimes_{\mathbb{Q}_p} K$.*

Proof. Via the $\otimes$-isomorphism (1.1.3) we may identify $\tilde{\omega} \otimes_{\mathbb{Q}_p} K_0$ with the forgetful functor $(M, \phi, F) \mapsto M$. After tensoring with K we obtain $\tilde{\omega}_K(M, \phi, F) = M_K$.

First, let us prove that $\eta(M, \phi, F)$ is an automorphism. We denote by

$$\eta(M, \phi, F)[i, j] : M_j \otimes_{K_0} K \to M_i \otimes_{K_0} K$$

the composition with the inclusion $M_j \otimes_{K_0} K \to M_K$ and the projection $M_K \to M_i \otimes_{K_0} K$. The following claim implies that $\eta(M, \phi, F)$ is an automorphism.

Claim 1.2.4.

$$\eta(M, \phi, F)[i, j] = \begin{cases} 0, & \text{if } i > j, \\ id_{M_i \otimes_{K_0} K}, & \text{if } i = j. \end{cases} \tag{1.2.2}$$

Proof of Claim 1.2.4. For $a \in M_j \otimes_{K_0} K$ we have $\eta(M, \phi, F)(a) = \pi_j^{-1}(a)$, where $\pi_j : F^j M_K \to M_{\geq j} \otimes_{K_0} K$ is the projection. For $i \geq j$ the projection $M_K \to M_i \otimes_{K_0} K$ factors as

$$M_K \xrightarrow{\pi_j} M_{\geq j} \otimes_{K_0} K = \bigoplus_{l \geq j} M_l \otimes_{K_0} K \xrightarrow{\text{projection}} M_i \otimes_{K_0} K.$$

Since $\pi_j \circ \eta(M, \phi, F)(a) = a$ is concentrated in the j-th component, the claim follows. □

Since the diagram (1.2.1) is functorial, η defines a natural transformation. The compatibility with the tensor product is obvious. □

1.2.5. By Lemma 1.2.3, we obtain a K-valued point $\eta \in G_{\tilde{\omega}}(K)$; we call this point the *logarithmic point*. Let us check that η is not the identity.

Proposition 1.2.6. *Let $n \in \mathbb{Z}$ be an integer. We have*

$$\mathrm{Ext}^1_{MT_K^{\phi}}(K(0), K(n)) \cong \begin{cases} K, & \text{if } n > 0, \\ 0, & \text{if } n \leq 0. \end{cases} \tag{1.2.3}$$

Let

$$0 \to K(n) \xrightarrow{\iota} (E, \phi, F) \xrightarrow{\pi} K(0) \to 0 \tag{1.2.4}$$

be an extension. For $n \neq 0$, there are unique sections $f : E \to K_0$ and $v : K_0 \to E$ of the underlying maps of K_0-isocrystals of ι and π, respectively. The isomorphism (1.2.3), for $n \neq 0$, is given by the formula

$$E \mapsto f(\eta(E, \phi, F)(v(1))).$$

Proof. First, we consider the case $n = 0$. Let (E, ϕ, F) be as in (1.2.4). We have $F^0(E_K) = E_K$ and $F^1(E_K) = 0$ by Definition 1.1.3(2). In view of Definition 1.1.3(2) there is an isomorphism $(E, \phi) \cong (K_0, \sigma) \oplus (K_0, \sigma)$, thus there is a section of π in MT_K^{ϕ}.

For $n \neq 0$: From the slope decomposition we obtain natural sections f, v as ϕ-modules. If $n < 0$ then $F^1 E_K = \iota(K)$ which means $(E, \phi, F) = K(0) \oplus K(n)$.

For $n > 0$, we can uniquely write $F^{-n+1} E_K = K\langle a \cdot \iota(1) + v(1) \rangle$ with $a \in K$. Let us compute $f(\eta(E, \phi, F)(v(1)))$. Since $v(1) \in E_0$, we have $\eta(E, \phi, F)(v(1)) = \pi_0^{-1}(v(1))$. On the other hand, $a \cdot \iota(1) + v(1) \in F^{-n+1} E_K = F^0 E_K$, and maps to $v(1)$ under π_0. Therefore $\eta(E, \phi, F)(v(1)) = \pi_0^{-1}(v(1)) = a \cdot \iota(1) + v(1)$ and hence $f(\eta(E,\phi,F)(v(1))) = a$. It is clear that $F^{-n+1}E_K$ is the only invariant for extensions. □

1.2.7. Recall that we have a fibre functor $\tilde{\omega}$ (1.1.2) to the category of $\mathbb{Q}_p$-vector spaces. In the obvious way $\tilde{\omega}$ factors through the category of graded $\mathbb{Q}_p$-vector spaces. Furthermore, we have an automorphism η of $\tilde{\omega}_K$ (Lemma 1.2.3).

Definition 1.2.8. We define $\mathcal{C}_\eta$ to be the category of pairs (V,η), where V is a finite-dimensional graded $\mathbb{Q}_p$-vector space and $\eta : V\otimes K \to V\otimes K$ is a K-linear map such that for all $n\in\mathbb{Z}$:

$$(\eta - id)(V_n\otimes K) \subset \bigoplus_{i>n} V_i \otimes K. \tag{1.2.5}$$

Morphisms $(V_1,\eta_1)\to(V_2,\eta_2)$ are $\mathbb{Q}_p$-linear morphisms $\tau : V_1\to V_2$ which respect the grading and commute with the endomorphisms η_i, i.e., $\eta_2\circ(\tau\otimes id_K) = (\tau\otimes id_K)\circ\eta_1$.

The category $\mathcal{C}_\eta$ is a $\otimes$-category with

$$(V_1,\eta_1)\otimes(V_2,\eta_2) = (V_1\otimes V_2, \eta_1\otimes\eta_2).$$

Proposition 1.2.9. *The functor*

$$\Psi : MT_K^\phi \to \mathcal{C}_\eta$$
$$(M,\phi,F)\mapsto \left(\bigoplus_{n\in\mathbb{Z}}\tilde{\omega}_n(M,\phi,F), \eta(M,\phi,F)\right)$$

is an equivalence of $\otimes$-categories.

Proof. By Lemma 1.2.3, η is functorial and Ψ is a $\otimes$-functor. It follows from (1.2.2) that

$$(\eta - id)(\tilde{\omega}_n\otimes K)\subset\bigoplus_{i>n}\tilde{\omega}_i\otimes K.$$

We define a functor

$$\Phi : \mathcal{C}_\eta \to MT_K^\phi$$
$$(\oplus_{n\in\mathbb{Z}}V_n,\eta)\mapsto\left(\oplus_{n\in\mathbb{Z}}(V_{-n}\otimes_{\mathbb{Q}_p}K_0, p^{-n}\otimes\sigma), F\right),$$

with the following filtration:

$$F^i := \eta\left(\bigoplus_{j\ge i} V_{-j}\otimes_{\mathbb{Q}_p} K\right),$$

for all i. Property (1.2.5) implies that Φ is well-defined. From Definition 1.2.2 it easily follows that $\Psi\circ\Phi = id_{\mathcal{C}_\eta}$.

On the other hand, we have $\Phi\circ\Psi \xrightarrow{\cong} id_{MT_K^\phi}$ via

$$\Phi\circ\Psi(M,\phi,F)\to(M,\phi,F)$$
$$\bigoplus_{n\in\mathbb{Z}}\tilde{\omega}_{-n}(M,\phi,F)\otimes_{\mathbb{Q}_p}K_0 \xrightarrow{(1.1.4)} M.$$

□

1.3. The semistable logarithmic point.

1.3.1. Let K be as in Section 1.1.1 with residue field k. We denote by ν_K the valuation of K.

1.3.2. Recall that we have a homomorphism

$$[.] : k^\times \to \mathcal{O}_K^\times, \quad x \mapsto [x],$$

by taking the Teichmüller lift. Denoting by $U_K := \{x \in \mathcal{O}_K^\times; x \in 1 + m_K\}$ the 1-units, we obtain a decomposition

$$\mathcal{O}_K^\times = k^\times \times U_K.$$

The logarithm

$$\log : \mathcal{O}_K^\times \to \mathcal{O}_K \tag{1.3.1}$$

is by definition trivial on the factor $k^\times$ and is given by

$$\log(u) = \sum_{n\geq 1} (-1)^{n+1} \frac{(u-1)^n}{n}, \qquad \text{for all } u \in U_K.$$

1.3.3. We consider $\mathcal{O}_{K,\mathbb{Q}}^\times := \mathcal{O}_K^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ and $K_{\mathbb{Q}}^\times := K^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}$-vector spaces, therefore we may form the symmetric algebras $\mathrm{Sym}_{\mathbb{Q}}(\mathcal{O}_{K,\mathbb{Q}}^\times)$ and $\mathrm{Sym}_{\mathbb{Q}}(K_{\mathbb{Q}}^\times)$. The exact sequence

$$0 \to \mathcal{O}_{K,\mathbb{Q}}^\times \to K_{\mathbb{Q}}^\times \xrightarrow{\nu_K} \mathbb{Q} \to 0$$

implies that $\mathrm{Spec}(\mathrm{Sym}_{\mathbb{Q}}(K_{\mathbb{Q}}^\times))$ is a one-dimensional affine space over Spec $(\mathrm{Sym}_{\mathbb{Q}}(\mathcal{O}_{K,\mathbb{Q}}^\times))$. In other words, for $x \in K^\times$ with $\nu_K(x) \neq 0$, the map

$$\mathrm{Sym}_{\mathbb{Q}}(\mathcal{O}_{K,\mathbb{Q}}^\times)[X] \to \mathrm{Sym}_{\mathbb{Q}}(K_{\mathbb{Q}}^\times), \quad X \mapsto x,$$

is an isomorphism.

The logarithm (1.3.1) induces a ring homomorphism

$$\mathrm{Sym}_{\mathbb{Q}}(\mathcal{O}_{K,\mathbb{Q}}^\times) \to K. \tag{1.3.2}$$

Definition 1.3.4. We define the K-algebra K_{st} by

$$K_{st} := \mathrm{Sym}_{\mathbb{Q}}(K_{\mathbb{Q}}^\times) \otimes_{\mathrm{Sym}_{\mathbb{Q}}(\mathcal{O}_{K,\mathbb{Q}}^\times)} K.$$

By base change, $\mathrm{Spec}(K_{st})$ is a one-dimensional affine space over K. We have a natural logarithm

$$\log_{st} : K^\times \to K_{st}, \quad x \mapsto x \otimes 1. \tag{1.3.3}$$

The K-valued points of $\mathrm{Spec}(K_{st})$ admit the following description:

$$\begin{aligned} \mathrm{Spec}(K_{st})(K) &= \{\text{extensions } \log : K^\times \to K \text{ of } (1.3.1)\} \\ f &\mapsto f^* \circ \log_{st}. \end{aligned} \tag{1.3.4}$$

By an extension $\log : K^\times \to K$ we mean a homomorphism such that the restriction to $\mathcal{O}_K^\times$ equals (1.3.1).

1.3.5. The p-adic Hodge theory of K (and fixed valuation ν_K) depends for semistable representations on the choice of a logarithm

$$\log : K^\times \to K.$$

It will be important for us that our constructions do not depend on a particular choice, and for this we have to recall the basic constructions of p-adic Hodge theory.

We denote by R the ring

$$R := \varprojlim \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}},$$

where the maps are given by rising to the p-th power $x \mapsto x^p$. Denoting by $C_K = \widehat{\overline{K}}$ the p-adic completion of $\overline{K}$ we have a multiplicative bijection

$$\varprojlim \mathcal{O}_{C_K} \to R,$$

where the projective system is defined by rising to the p-th power again. In other words, we can represent every element x in R by $(x^{(0)}, x^{(1)}, \dots)$ with $x^{(n)} \in \mathcal{O}_{C_K}$ and $x^{(n-1)} = (x^{(n)})^p$.

Let $\nu_{\bar{K}}$ (resp. ν_{C_K}) be the extension of ν_K (resp. $\nu_{\bar{K}}$) to $\bar{K}$ (resp. C_K). The map

$$\nu_R : R\backslash\{0\} \to \mathbb{Q}, \quad x \mapsto \nu_{C_K}(x^{(0)})$$

can be extended to a valuation

$$\nu_R : \mathrm{Frac}(R)^\times \to \mathbb{Q}$$

with valuation ring R.

Let B_{cris} be the crystalline period ring; we define

$$B_{st} = \mathrm{Sym}_{\mathbb{Q}}(\mathrm{Frac}(R)^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathrm{Sym}_{\mathbb{Q}}(R^\times \otimes_{\mathbb{Z}} \mathbb{Q})} B_{\mathrm{cris}},$$

where $\mathrm{Sym}_{\mathbb{Q}}(R^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \to B_{\mathrm{cris}}$ is induced by the crystalline logarithm

$$\log_{\mathrm{cris}} : R^\times \to B_{\mathrm{cris}}.$$

Again, $R^\times = \bar{k}^\times \times (1 + m_R)$; $\log_{\mathrm{cris}}$ is trivial on $\bar{k}^\times$ and given by

$$\log_{\mathrm{cris}}(u) = \sum_{n \geq 1} (-1)^{n+1} \frac{([u]-1)^n}{n}$$

for $u \in 1 + m_R$, where $[u]$ denotes the Teichmüller lift of u in the Witt ring $W(R)$ of R.

By construction we have a natural logarithm

$$\log_{st} : \mathrm{Frac}(R)^\times \to B_{st}, \quad x \mapsto x \otimes 1.$$

The ring B_{st} has the following properties.

(1) We have a $\mathrm{Gal}(\bar{K}/K)$-action on B_{st} extending the action on B_{cris}.
(2) We have a Frobenius map $\phi : B_{st} \to B_{st}$ extending the Frobenius map on B_{cris}. Moreover,

$$\phi \circ \log_{st} = p \log_{st}.$$

(3) We have a B_{cris}-linear derivation $N : B_{st} \to B_{st}$ such that

$$N(\log_{st}(x)) = \nu_R(x), \qquad \text{for all } x \in \mathrm{Frac}(R)^\times.$$

After choosing a logarithm

$$\log : K^{\times} \to K,$$

which extends (1.3.1), we obtain a morphism of B_{cris}-algebras

$$\gamma_{\log} : B_{st} \to B_{dR}.$$

We denote by $\gamma_{\log,K} : B_{st} \otimes_{K_0} K \to B_{dR}$ the induced map of $B_{\mathrm{cris}} \otimes_{K_0} K$-algebras. The morphism depends on the choice of log, and the filtration on $B_{st} \otimes_{K_0} K$ induced by the filtration on B_{dR} via $\gamma_{\log,K}$ depends on log.

Proposition 1.3.6. *For* $\log, \log' \in \mathrm{Spec}(K_{st})(K)$ *there is a unique ring homomorphism*

$$\delta_{\log,\log'} : B_{st} \otimes_{K_0} K \to B_{st} \otimes_{K_0} K$$

such that $\gamma_{\log',K} \circ \delta_{\log,\log'} = \gamma_{\log,K}$. *The map* $\delta_{\log,\log'}$ *is given by*

$$\delta_{\log,\log'} = \exp\left(\frac{\log(x) - \log'(x)}{\nu_K(x)} N\right) \tag{1.3.5}$$

for every $x \in K \backslash \mathcal{O}_K^{\times}$.

Proof. Uniqueness follows from the fact that $\gamma_{\log,K}$ is injective.

Choose $\tilde{p} \in R$ with $\tilde{p}^{(0)} = p$. By definition we have

$$\gamma_{\log}(\log_{st}(\tilde{p})) = \log_{dR}([\tilde{p}]/p) + \log(p),$$

where $\log_{dR}$ is defined by the usual series since $[\tilde{p}]/p$ is a 1-unit in B_{dR}. Since $\mathrm{Spec}(B_{st})$ is a one-dimensional affine space over $\mathrm{Spec}(B_{\mathrm{cris}})$, there exists a unique morphism of $B_{\mathrm{cris}} \otimes_{K_0} K$-algebras $\delta_{\log,\log'}$ such that

$$\delta_{\log,\log'}(\log_{st}(\tilde{p})) = \log_{st}(\tilde{p}) + \log(p) - \log'(p).$$

Obviously, $\delta_{\log,\log'}$ satisfies $\gamma_{\log',K} \circ \delta_{\log,\log'} = \gamma_{\log,K}$ and the equality (1.3.5). ☐

By using $\gamma_{\log,K}$ we obtain a filtration on $B_{st} \otimes_{K_0} K$. The p-adic Hodge theory [2, Thm. A] asserts that the functor

$$\begin{aligned} D_{st,\log} :&(\text{semistable } \mathbb{Q}_p\text{-representations of } \mathrm{Gal}(\bar{K}/K)) \to MF_K^{\phi,N,w.a.} \\ &V \mapsto \left(B_{st} \otimes_{\mathbb{Q}_p} V\right)^{\mathrm{Gal}(\bar{K}/K)} \end{aligned} \tag{1.3.6}$$

is an equivalence of categories. We will use the subscript log in $D_{st,\log}$ to emphasize the dependence on log.

Denoting by forget_F the functor $\mathrm{forget}_F(M, \phi, N, F) = (M, \phi, N)$, we get

$$\mathrm{forget}_F \circ D_{st,\log} = \mathrm{forget}_F \circ D_{st,\log'},$$

because only the filtration depends on the embedding to B_{dR}. Proposition 1.3.6 implies that for the filtrations we have the following comparison:

$$F^i_{D_{st,\log'}(V)} = \exp\left(\frac{\log(x) - \log'(x)}{\nu_K(x)} N\right) F^i_{D_{st,\log}(V)}, \tag{1.3.7}$$

for all $i \in \mathbb{Z}$ and all $x \in K \backslash \mathcal{O}_K^{\times}$.

Definition 1.3.7. Let K be a p-adic field. We denote by MT_{G_K} the full subcategory of p-adic representations V of $\mathrm{Gal}(\bar{K}/K)$ which admit an increasing exhaustive separated filtration W by subrepresentations of V such that W_i/W_{i-1} vanishes if i is odd, and is a sum of Tate objects $\mathbb{Q}_p(-\frac{i}{2})$ if i is even. We call an object of MT_{G_K} a *mixed Tate representation* of $\mathrm{Gal}(\bar{K}/K)$.

Proposition 1.3.8. *Let* $\log \in \mathrm{Spec}(K_{st})(K)$*. Then*

$$MT_{G_K} = D_{st,\log}^{-1}(MT_K^{\phi,N}).$$

In particular, every mixed Tate representation is semistable.

Proof. From Proposition 1.1.11 it follows that every object in $D_{st,\log}^{-1}(MT_K^{\phi,N})$ admits a filtration W satisfying the properties of Definition 1.3.7.

Now, suppose that V is a p-adic representation of $\mathrm{Gal}(\bar{K}/K)$ which admits a filtration W as in Definition 1.3.7. If we know that V is semistable then clearly $D_{st,\log}(V) \in MT_K^{\phi,N}$ by Proposition 1.1.11, again. Therefore it suffices to prove that V is semistable.

We use induction on the length of the filtration W of V. If the filtration W has length ≤ 1, semistability of V follows from those of $\mathbb{Q}_p(n)$. In general, let n be the smallest integer such that $W_{2n}V = V$. Then we have an exact sequence

$$0 \to (W_{2n-2}V) \otimes \mathbb{Q}_p(n) \to V \otimes \mathbb{Q}_p(n) \to (V/W_{2n-2}V) \otimes \mathbb{Q}_p(n) \to 0.$$

By the induction hypothesis the terms on the left and right are semistable. Moreover, since the weights of the term on the left are ≤ -2 and the term on the right has weight 0, we have

$$F^0 D_{dR}((W_{2n-2}V) \otimes \mathbb{Q}_p(n)) = 0$$
$$F^0(D_{dR}((V/W_{2n-2}V) \otimes \mathbb{Q}_p(n))) = D_{dR}((V/W_{2n-2}V) \otimes \mathbb{Q}_p(n)).$$

Therefore [5, Proposition 1.28] shows that the middle term is also semistable. □

Obviously,

$$\tau = \tilde{\omega} \circ D_{st,\log} \tag{1.3.8}$$

is independent of log, and (MT_{G_K}, τ) is a Tannaka category (by Lemma 1.1.10).

1.3.9. Recall from Lemma 1.1.10 that $(MT_K^{\phi,N}, \tilde{\omega})$ is a Tannaka category. We will use the ring K_{st} (Definition 1.3.4) and $\log_{st}$ (1.3.3).

Definition 1.3.10. For a logarithm $\log \in \mathrm{Spec}(K_{st})(K)$ and $(M, \phi, N, F) \in MT_K^{\phi,N}$ we define $\eta_{st,\log}(M, \phi, N, F) \in \mathrm{End}_{K_{st}}(M \otimes_{K_0} K_{st})$ by

$$\eta_{st,\log}(M, \phi, N, F) := \exp\left(\frac{\log(x) - \log_{st}(x)}{\nu_K(x)} N\right) \eta(M, \phi, F),$$

for $x \in K^\times \backslash \mathcal{O}_K^\times$. For the definition of $\eta(M, \phi, F)$ we refer to Definition 1.2.2.

Obviously, $\eta_{st,\log}$ does not depend on the choice $x \in K^\times \backslash \mathcal{O}_K^\times$, but it depends on log.

Lemma 1.3.11. *Let* $\log \in \operatorname{Spec}(K_{st})(K)$. *The morphisms* $\eta_{st,\log}$ *from Definition 1.3.10 define a tensor automorphism of the fibre functor* $\tilde{\omega}_{K_{st}} = \tilde{\omega} \otimes_{\mathbb{Q}_p} K_{st}$. *In other words,* $\eta_{st,\log} \in G^{st}_{\tilde{\omega}}(K_{st})$ *with* $G^{st}_{\tilde{\omega}} = \underline{\operatorname{Aut}}^{\otimes}_{MT_K^{\phi,N}} \tilde{\omega}$.

Proof. Via the $\otimes$-isomorphism (1.1.3) we may identify $\tilde{\omega} \otimes_{\mathbb{Q}_p} K_0$ with the forgetful functor $(M, \phi, N, F) \mapsto M$. After tensoring with K_{st} we obtain $\tilde{\omega}_{K_{st}}(M, \phi, N, F) = M \otimes_{K_0} K_{st}$. Lemma 1.2.3 implies that $\eta(M, \phi, F)$ is a tensor automorphism, thus it suffices to prove the statement for $\exp\left(\frac{\log(x) - \log_{st}(x)}{\nu_K(x)} N\right)$. The functoriality follows immediately. The compatibility with the $\otimes$-structure follows from

$$N_{M_1 \otimes M_2} = N_{M_1} \otimes 1 + 1 \otimes N_{M_2}.$$

□

Lemma 1.3.12. *The* K_{st}*-valued point*

$$\eta_{st} = \eta_{st,\log} \circ D_{st,\log}$$

of $\underline{\operatorname{Aut}}^{\otimes}_{MT_{G_K}} \tau$ *is independent of the choice of* $\log \in \operatorname{Spec}(K_{st})(K)$.

Proof. Let $\log, \log' \in \operatorname{Spec}(K_{st})(K)$ and $V \in MT_{G_K}$. In view of (1.3.7) we get

$$\eta(\text{forget}_N D_{st,\log'}(V)) = \exp\left(\frac{\log(x) - \log'(x)}{\nu_K(x)} N\right) \eta(\text{forget}_N D_{st,\log}(V)), \tag{1.3.9}$$

for very $x \in K \backslash \mathcal{O}_K^{\times}$, and $\text{forget}_N(M, \phi, N, F) = (M, \phi, F)$. Thus,

$$\begin{aligned} \eta_{st,\log'} D_{st,\log'}(V) &= \exp\left(\frac{\log'(x) - \log_{st}(x)}{\nu_K(x)} N\right) \eta(\text{forget}_N D_{st,\log'}(V)) \\ &= \exp\left(\frac{\log(x) - \log_{st}(x)}{\nu_K(x)} N\right) \eta(\text{forget}_N D_{st,\log}(V)) \\ &= \eta_{st,\log} D_{st,\log}(V). && \text{by (1.3.9)}, \end{aligned}$$

□

Example 1.3.13. By Kummer theory any $q \in K^{\times}$ defines an extension V of the $\operatorname{Gal}(\bar{K}/K)$-representation $\mathbb{Q}_p(0)$ by $\mathbb{Q}_p(1)$. This in turn gives via $D_{st,\log}$ an extension of $K(0)$ by $K(1)$ in $MT_K^{\phi,N}$:

$$0 \to K(1) \to M \to K(0) \to 0,$$

which may be described as follows. The underlying K_0-space of M has a basis e_0, e_1 such that the following conditions are satisfied:

(1) the action of ϕ is given by $\phi(e_i) = p^{-i} e_i$, for $i = 0, 1$,
(2) e_1 is the image of $1 \in K(1)$,
(3) e_0 maps to $1 \in K(0)$.

The filtration is given by $F^{-1} M_K = M_K$, $F^0 M_K = K \cdot \langle \log(q) e_1 + e_0 \rangle$ and $F^1 M_K = 0$. Finally N is given by $N e_0 = -\nu_K(q) \cdot e_1$ and $N e_1 = 0$. Then we easily compute

$$\eta_{st}(V) = \begin{pmatrix} 1 & 0 \\ \frac{-\nu_K(q)(\log(x) - \log_{st}(x))}{\nu_K(x)} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \log(q) & 1 \end{pmatrix}$$

for the obvious basis of

$$\tau(V) = \mathrm{Hom}(\mathbb{Q}(0), \mathbb{Q}(0)) \oplus \mathrm{Hom}(\mathbb{Q}(1), \mathbb{Q}(1)),$$

and every $x \in K_0 \backslash \mathcal{O}_{K_0}^{\times}$, or, equivalently, for every $x \in K \backslash \mathcal{O}_K^{\times}$. If $\nu_K(q) \neq 0$, then we can take $q = x$ in order to see that

$$\eta_{st}(V) = \begin{pmatrix} 1 & 0 \\ \log_{st}(q) & 1 \end{pmatrix}$$

holds for all $q \in K^{\times}$.

2. Mixed Tate motives over a number field and logarithmic points

2.1. Mixed Tate motives.

2.1.1. Let E be a number field and S a set of finite places. Let $\mathcal{O}$ be the ring of integers of E, and $|\mathrm{Spec}(\mathcal{O})|$ the maximal spectrum of $\mathcal{O}$. We denote by

$$\mathcal{O}_S := \bigcap_{x \in \mathrm{Spec}(\mathcal{O}) \backslash S} \mathcal{O}_x$$

the ring of S-integers of E; the elements of $\mathcal{O}_S$ are integral outside of S. We will be mainly interested in two cases for S. In the first case, we have $S = |\mathrm{Spec}(\mathcal{O})|$ and $\mathcal{O}_S = E$. In the second case, we have $S = |\mathrm{Spec}(\mathcal{O})| \backslash \{x\}$, for a point $x \in |\mathrm{Spec}(\mathcal{O})|$, and $\mathcal{O}_S = \mathcal{O}_x$ is the local ring at x.

2.1.2. Deligne and Goncharov defined in [3, 1.6] an abelian category of mixed Tate motives $MT(\mathcal{O}_S)$. By definition it is the full subcategory of $MT(E)$ consisting of objects which are unramified outside S in the following sense. Let $x \in |\mathrm{Spec}(\mathcal{O})|$ be a point lying over a prime p; then we say that $M \in MT(E)$ is unramified at x if for all primes $\ell \neq p$ the corresponding Galois representation M_ℓ is unramified at x, i.e., the inertia subgroup I_x (which is only well-defined up to conjugation) acts trivially at M_ℓ [3, Proposition 1.8].

2.1.3. For extensions of Tate objects we know that:

$$\mathrm{Ext}^1_{MT(\mathcal{O}_S)}(\mathbb{Q}(0), \mathbb{Q}(1)) = \mathcal{O}_S^{\times} \otimes \mathbb{Q},$$
$$\mathrm{Ext}^1_{MT(\mathcal{O}_S)}(\mathbb{Q}(0), \mathbb{Q}(n)) = \begin{cases} 0, & \text{if } n \leq 0, \\ \mathrm{Ext}^1_{MT(E)}(\mathbb{Q}(0), \mathbb{Q}(n)), & \text{if } n \neq 1, \end{cases}$$
$$\mathrm{Ext}^2_{MT(\mathcal{O}_S)}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0, \quad \text{for all } n \in \mathbb{Z},$$

(see [3, Proposition 1.9]).

2.1.4. Every object of $MT(\mathcal{O}_S)$ comes equipped with a finite increasing functorial weight filtration, indexed by even integers. For all $n \in \mathbb{Z}$ the graded pieces $\mathrm{gr}^W_{2n}(M)$ are sums of copies of $\mathbb{Q}(-n)$.

In view of [3, 1.1] the $\otimes$-functor

$$\omega : MT(\mathcal{O}_S) \to (\mathbb{Q}\text{-vector spaces}), \tag{2.1.1}$$
$$\omega(M) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(\mathbb{Q}(n), \mathrm{gr}^W_{-2n}(M)),$$

is a fibre functor, therefore $MT(\mathcal{O}_S)$ is a Tannaka category. We denote by $G_{S,\omega}$ the group scheme of $\otimes$-automorphisms of ω. By [3, 2.1] we can write $G_{S,\omega}$ as a semi-direct product:

$$G_{S,\omega} = \mathbb{G}_m \ltimes U_{S,\omega},$$

where $U_{S,\omega}$ is a unipotent group and $G_{S,\omega} \to \mathbb{G}_m$ is induced by the obvious grading of ω. If $S = |\mathrm{Spec}(\mathcal{O})|$ then we simply write $G_\omega = G_{S,\omega}$.

2.1.5. *Functor to p-adic representations.* Let $x \in |\mathrm{Spec}(\mathcal{O})|$ be a point lying over a prime p. Let $K = E_x$ be the completion of E at the place x. Choose algebraic closures $\bar{E}$, $\bar{K}$, and an embedding $\iota : \bar{E} \to \bar{K}$.

To $M \in MT(E)$ we can attach a Galois representation M_p of $\mathrm{Gal}(\bar{E}/E)$ with coefficients in $\mathbb{Q}_p$, which is called the p-adic realization of M. By using ι, we get a continuous homomorphism

$$\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gal}(\bar{E}/E),$$

and we can restrict M_p in order to obtain a p-adic representation $M_{\iota,p}$ of $\mathrm{Gal}(\bar{K}/K)$.

Proposition 2.1.6. *The assignment $M \mapsto M_{\iota,p}$ defines a functor*

$$(.)_{\iota,p} : MT(E) \to MT_{G_K}.$$

See Definition 1.3.7 for MT_{G_K}.

Proof. The p-adic realization is functorial. Thus, we only need to show that $M_{\iota,p} \in MT_{G_K}$, which follows immediately from the existence of the weight filtration of M and Definition 1.3.7. □

The set $\{\iota : \bar{E} \to \bar{E}_x\}$ of embeddings over E is a torsor under the Galois group $\mathrm{Gal}(\bar{E}/E)$, and for every $g \in \mathrm{Gal}(\bar{E}/E)$ there is a natural transformation:

$$\alpha_g : (.)_{\iota,p} \xrightarrow{\cong} (.)_{\iota\circ g,p}. \tag{2.1.2}$$

Lemma 2.1.7. *For the fibre functor τ (defined in (1.3.8)) and the fibre functor ω defined in (2.1.1) we have a canonical isomorphism*

$$\tau \circ (.)_{\iota,p} \cong \omega \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

For every $g \in \mathrm{Gal}(\bar{E}/E)$, the diagram

$$\begin{array}{ccc} \tau \circ (.)_{\iota,p} & \xrightarrow{(2.1.2)} & \tau \circ (.)_{\iota\circ g,p} \\ & \searrow^{\cong} \quad \swarrow_{\cong} & \\ & \omega \otimes_{\mathbb{Q}} \mathbb{Q}_p & \end{array} \tag{2.1.3}$$

is commutative.

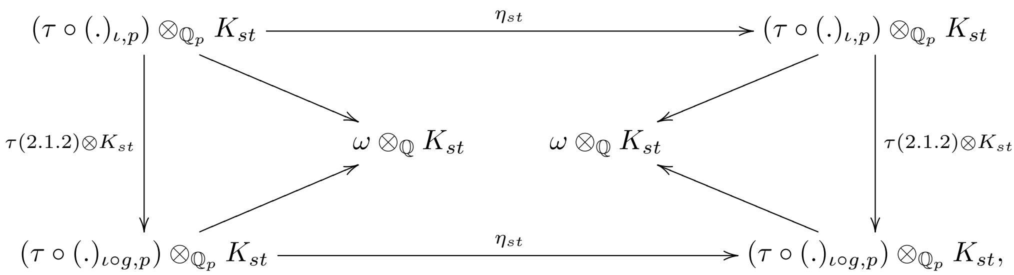
Proof. Straightforward. □

2.1.8. Recall that we have constructed a K_{st}-valued η_{st} of $\underline{\mathrm{Aut}}^{\otimes}\tau$ (Lemma 1.3.12).

Proposition 2.1.9. *For every embedding ι, $\eta_x := \eta_{st} \circ (.)_{\iota,p}$ defines a K_{st}-valued point of $\underline{\mathrm{Aut}}^{\otimes}_{MT(E)}\omega$ which is independent of the choice of ι.*

Proof. Since $\tau \circ (.)_{\iota,p} = \omega \otimes_{\mathbb{Q}} \mathbb{Q}_p$ by Lemma 2.1.7, $\eta_{st} \circ (.)_{\iota,p}$ is a K_{st}-valued point of $\underline{\mathrm{Aut}}^{\otimes}\omega$.

The independence of the choice of ι follows from the commutative diagram



where the triangles are commutative by Lemma 2.1.7, and the square is commutative because η_{st} is functorial. □

2.2. Crystalline characterization of unramified motives.

2.2.1. Let E be a number field, and let M be a mixed Tate motive over E, i.e., an object in $MT(E)$. Let ν be a finite place of E, M is unramified at ν [3, Definition 1.4, Section 1.7] if the coaction [3, (1.2.2)]

$$e_M : \omega(M) \to \mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) \otimes \omega(M)$$

of $\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) = E^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$ on $\omega(M)$ factors through a coaction of $\mathcal{O}_{\nu}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.2.2. Recall from Proposition 2.1.6 that $M_{\iota,p}$ is a mixed Tate Galois representation of $G_K = \mathrm{Gal}(\bar{K}/K)$ for the completion $K = E_{\nu}$ at ν. In particular, $M_{\iota,p}$ is semistable (Proposition 1.3.8). In the following we will simply write $M_p = M_{p,\iota}$. We call M_p *crystalline* if the monodromy operator N of $D_{st}(M_p)$ is trivial, or equivalently if

$$(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} M_p)^{G_K} \to (B_{st} \otimes_{\mathbb{Q}_p} M_p)^{G_K}$$

is an isomorphism.

Theorem 2.2.3. *Let M be a mixed Tate motive over E and ν a finite place of E. Then M is unramified at ν if and only if M_p is crystalline.*

Proof. First note that the statement that M is unramified at ν is equivalent to the statement that for every subquotient N of M which is of the form

$$0 \to \mathbb{Q}(n+1) \to N \to \mathbb{Q}(n) \to 0,$$

for some n, the extension class $\mathrm{Ext}^1(\mathbb{Q}(n), \mathbb{Q}(n+1)) = \mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) = E^{\times} \otimes \mathbb{Q}$ lies in $\mathcal{O}_{\nu}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$ [3, Section 1.4].

Also in the category of p-adic representations of a p-adic field K, a representation in $\mathrm{Ext}^1_{G_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$ that is associated to some $q \in K^{\times} \otimes \mathbb{Q} \subseteq \varprojlim_n (K^{\times}/(K^{\times})^{p^n}) \otimes \mathbb{Q} = \mathrm{Ext}^1_{G_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$ is crystalline if and only if $q \in \mathcal{O}_K^{\times} \otimes \mathbb{Q}$ [6, Example 2.3.2].

First, suppose that M_p is crystalline, then every subquotient of M_p is crystalline. So in order to prove that M is unramified at ν we may assume that $M = N$, where N is as above with $n = 0$ (after Tate twist). Therefore we have an extension in $\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))$, defined by some $q \in E^\times \otimes_{\mathbb{Z}} \mathbb{Q}$, whose p-adic realization is crystalline at ν. Then the above remark implies that the image of q in $E_\nu^\times \otimes \mathbb{Q}$ lies in $\hat{\mathcal{O}}_\nu^\times \otimes \mathbb{Q}$, hence $q \in \mathcal{O}_\nu^\times \otimes \mathbb{Q}$ and M is unramified at ν.

Suppose conversely that M is unramified at ν. We have to show that the monodromy operator N on $D_{st}(M_p) =: D(M)$ vanishes. Note that N maps the slope λ piece of $D(M)$ to the slope $\lambda - 1$ piece. Therefore, if N is non-zero on $D(M)$ then there exists an n such that N is non-zero on

$$D(W_{2n}M_p/W_{2n-4}M_p) = D((W_{2n}M/W_{2n-4}M)_p).$$

Replacing M by $(W_{2n}M/W_{2n-4}M)\otimes\mathbb{Q}(n)$ we may assume that M is defined by a class in $\mathrm{Ext}^1(\mathbb{Q}(0)^{\oplus r}, \mathbb{Q}(1)^{\oplus s}) = \mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))^{\oplus rs}$, M is unramified, and N is non-zero on $D(M)$. By passing to a subquotient we may further assume that $r = s = 1$. This gives an extension in $\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))$, which is unramified at ν (and hence defined by some $q \in \mathcal{O}_\nu^\times \otimes \mathbb{Q}$) and whose p-adic realization is not crystalline at ν. This is a contradiction. □

2.2.4. Recall the notation of Section 2.1.1. Let $x \in |\mathrm{Spec}(\mathcal{O})|$ be a point; in the following we will work with $S = |\mathrm{Spec}(\mathcal{O})| \backslash \{x\}$, thus $\mathcal{O}_S = \mathcal{O}_x$.

Let p be the prime lying under x. In view of Theorem 2.2.3, we know that $MT(\mathcal{O}_x)$ is the full subcategory of $MT(E)$ consisting of motives M such that the p-adic realization M_p is crystalline at x.

We denote by G_x the group scheme of $\otimes$-automorphisms of the fibre functor (see (2.1.1))

$$\omega : MT(\mathcal{O}_x) \to (\mathbb{Q}\text{-vector spaces}). \tag{2.2.1}$$

The group scheme G_x is a quotient of $G_\omega = \mathrm{Aut}^\otimes_{MT(E)}\omega$.

Lemma 2.2.5. *The morphism* $\mathrm{Spec}(E_{x,st}) \xrightarrow{\eta_x} G_\omega \to G_x$ *factors through the structure morphism* $\mathrm{Spec}(E_{x,st}) \to \mathrm{Spec}(E_x)$ *and thus defines a point* $\eta_x^{ur} \in G_x(E_x)$.

Proof. The point η_x was defined in Proposition 2.1.9. If $M \in MT(\mathcal{O}_x)$ then $D_{st}(M_{\iota,p})$ has vanishing monodromy operator N and $\eta_{st}(M_{\iota,p}) = \eta_{st,\log}D_{st}(M_{\iota,p})$ takes values in E_x by Definition 1.3.10. □

2.3. Main theorem.

2.3.1. Let $x \in |\mathrm{Spec}(\mathcal{O})|$ and let E_x be the completion of E at x. Bloch and Kato [1, Definition 3.10] define an exponential map

$$\exp : E_x \to \mathrm{Ext}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(n)), \qquad \text{for all } n \geq 1, \tag{2.3.1}$$

where Ext^1 is computed in the category of p-adic representation of $\mathrm{Gal}(\bar{E}_x/E_x)$. Note that, in fact, the image of the exponential map lies among the crystalline representations $\mathrm{Ext}^1_{\mathrm{crys}}(\mathbb{Q}_p(0), \mathbb{Q}_p(n))$ [1, Example 3.9]. Via p-adic Hodge theory, we obtain a map

$$E_x \to \mathrm{Ext}^1_{\mathrm{crys}}(\mathbb{Q}_p(0), \mathbb{Q}_p(n)) \cong \mathrm{Ext}^1_{MT^\phi_{E_x}}(E_x(0), E_x(n)), \tag{2.3.2}$$

which, by abuse of notation, will also be called the Bloch–Kato exponential map.

2.3.2. For an extension $M \in \operatorname{Ext}^1_{MT(\mathcal{O}_x)}(\mathbb{Q}(0), \mathbb{Q}(n))$ with $n \geq 1$, there are natural maps $v_0 : \mathbb{Q} \to \omega(M)$ and $f_n : \omega(M) \to \mathbb{Q}$ defined as follows. By definition, there are isomorphisms $\alpha : \mathbb{Q}(n) \to \operatorname{gr}^W_{-2n} M$ and $\beta : \operatorname{gr}^W_0 M \to \mathbb{Q}(0)$; we define

$$v_0 : \mathbb{Q} = \operatorname{Hom}(\mathbb{Q}(0), \mathbb{Q}(0)) \xrightarrow{\beta^{-1}} \omega_0(M) \to \omega(M),$$

$$f_n : \omega(M) \to \omega_n(M) \xrightarrow{\alpha^{-1}} \operatorname{Hom}(\mathbb{Q}(n), \mathbb{Q}(n)) = \mathbb{Q}.$$

Given a point $s : \operatorname{Spec}(L) \to G_x$, we get $s(M) : \omega(M) \otimes L \to \omega(M) \otimes L$. This gives an element of L as follows:

$$M(s) := f_n(s(M)(v_0(1))) \in L.$$

Theorem 2.3.3. *Let E be a number field and $\mathcal{O}$ be the ring of integers. Let $x \in |\operatorname{Spec}(\mathcal{O})|$ be a closed point over a prime p. For the Tannaka category $(MT(\mathcal{O}_x), \omega)$ of mixed Tate motives we denote by G_x the group scheme of $\otimes$-automorphisms of ω. For all $n \geq 1$, the map*

$$\operatorname{Ext}^1_{MT(\mathcal{O}_x)}(\mathbb{Q}(0), \mathbb{Q}(n)) \to E_x, \quad M \mapsto M(\eta^{ur}_x),$$

induced by $\eta^{ur}_x \in G_x(E_x)$ (see Lemma 2.2.5), is the composition of the p-adic realization

$$\operatorname{Ext}^1_{MT(\mathcal{O}_x)}(\mathbb{Q}(0), \mathbb{Q}(n)) \to \operatorname{Ext}^1_{\text{crys}}(\mathbb{Q}_p(0), \mathbb{Q}_p(n))$$

and the inverse of the Bloch–Kato exponential map (2.3.1).

Proof. Let us prove that evaluation at the point η^{ur}_x has the desired compatibility with the Bloch–Kato exponential map (see (2.3.2))

$$\exp : E_x \to \operatorname{Ext}^1_{\text{crys}}(\mathbb{Q}_p(0), \mathbb{Q}_p(n)) \xrightarrow{\cong} \operatorname{Ext}^1_{MT_{E_x}}(E_x(0), E_x(n)).$$

For this we need to recall the construction of the exponential map. For the rest of the proof let $K := E_x$. First there is an exact sequence [1, Proposition 1.17]:

$$0 \to \mathbb{Q}_p \to B^{\varphi=1}_{\text{crys}} \oplus B^+_{dR} \to B_{dR} \to 0, \tag{2.3.3}$$

where the first map sends x to (x, x) and the second one sends (x, y) to $x - y$.

For $n \geq 1$, the Bloch–Kato construction gives a map

$$K = (\mathbb{Q}_p(n) \otimes B_{dR})^{G_K} \to \operatorname{Ext}^1_{\text{crys}}(\mathbb{Q}_p(0), \mathbb{Q}_p(n)).$$

This map is obtained as follows. First tensor the above exact sequence with $\mathbb{Q}_p(n)$:

$$0 \to \mathbb{Q}_p(n) \to (\mathbb{Q}_p(n) \otimes B^{\varphi=1}_{\text{crys}}) \oplus (\mathbb{Q}_p(n) \otimes B^+_{dR}) \to \mathbb{Q}_p(n) \otimes B_{dR} \to 0.$$

Then an element a in $K(n) = (\mathbb{Q}_p(n) \otimes B_{dR})^{G_K}$ gives a map $\mathbb{Q}_p \to \mathbb{Q}_p(n) \otimes B_{dR}$, pulling back the above exact sequence via this map gives the extension we were looking for.

More explicitly, for $a \in K$ the extension constructed above is:

$$0 \to V_n \to V \to V_0 \to 0,$$

where $V_0 = \mathbb{Q}_p \cdot t^n \otimes at^{-n}$, $V_n = \mathbb{Q}_p \cdot t^n$, and V is a two-dimensional representation of G_K with basis which can be described as follows. By the exact sequence (2.3.3), there exists $x \in B^{\varphi=1}_{\text{crys}}$ and $y \in B^+_{dR}$ such that $at^{-n} = x - y$. Then V has basis $\{(t^n \otimes x, t^n \otimes y), (t^n \otimes 1, t^n \otimes 1)\}$. For $\sigma \in G_K$,

$$\sigma(t^n \otimes x, t^n \otimes y) = (t^n \otimes x, t^n \otimes y) + \gamma(\sigma)(t^n \otimes 1, t^n \otimes 1),$$

for some $\gamma(\sigma) \in \mathbb{Q}_p$. Therefore

$$\chi_{\mathrm{cyc}}(\sigma)^n \sigma(x) = x + \gamma(\sigma)$$

and

$$\chi_{\mathrm{cyc}}(\sigma)^n \sigma(y) = y + \gamma(\sigma).$$

Let us now try to find what this extension corresponds to after we apply the functor $(\cdot \otimes B_{\mathrm{crys}})^{G_K}$. First note that $(V \otimes B_{\mathrm{crys}})^{G_K}$ has basis

$$e_n := (t^n \otimes 1, t^n \otimes 1) \otimes t^{-n}$$

and

$$e_0 := (t^n \otimes x, t^n \otimes y) \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes x.$$

That e_n is invariant under the Galois action is clear. In order to see that e_0 is G_K invariant let $\sigma \in G_K$. Then

$$\begin{aligned}\sigma(e_0) = (t^n \otimes (x + \gamma(\sigma)), t^n \otimes (y + \gamma(\sigma)))\\ \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes (x + \gamma(\sigma)) = e_0.\end{aligned}$$

Now note that $\varphi(e_n) = p^{-n} e_n$ and $\varphi(e_0) = e_0$. Furthermore e_n is the image of $1 \in K(n)$ and e_0 maps to $1 \in K(0)$ in the exact sequence (note that $\mathbb{Q}_p(0)$ is identified with V_0 via the map that sends 1 to $t^n \otimes at^{-n}$):

$$0 \to K(n) \to (V \otimes B_{\mathrm{crys}})^{G_K} \to K(0) \to 0.$$

Therefore, in order to compare Bloch–Kato's construction we need only compute the filtration on $(V \otimes B_{\mathrm{crys}})^{G_K} \otimes_{K_0} K$. So we need to compute the 0th piece of the filtration on $(V \otimes B_{dR})^{G_K}$.

We claim that $ae_n + e_0 \in Fil^0(V \otimes B_{dR})^{G_K}$. This follows immediately from

$$ae_n + e_0 = (t^n \otimes x, t^n \otimes y) \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes y,$$

and the fact that $y \in B_{dR}^+$.

In order to complete the proof, let us start with M in $\mathrm{Ext}^1_{MT(\mathcal{O}_x)}(\mathbb{Q}(0), \mathbb{Q}(n))$. The p-adic realization of M gives an element M_x in $\mathrm{Ext}^1_{MT^{\phi}_{E_x}}(E_x(0), E_x(n))$. Then $M_x = \exp(a)$, for some $a \in E_x$. On the other hand, by the computation above $\exp(a)$ is *the* extension in $\mathrm{Ext}^1_{MT^{\phi}_{E_x}}(E_x(0), E_x(n))$, whose 0th filtration is given by $\langle a \cdot e_n + e_0 \rangle$, where $\phi(e_i) = p^{-i} e_i$ and e_i is the element 1 in $E_x(i)$, for $i = 0, n$ (cf. Proposition 1.2.6). In view of the proof of Proposition 1.2.6 we see that $f_n(\eta_x^{ur}(M) v_0(1)) = f(\eta(M_x, \phi, F)(v(1))) = a$. This completes the proof. □

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