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PROOF OF THE INDEX CONJECTURE IN HOFER GEOMETRY

Yasha Savelyev

Abstract. Let γ be an Ustilovsky geodesic and H its generating function. We give a simple proof of a generalization of the conjecture stated in [7], relating the Morse index of γ, as a critical point of the Hofer length functional, with the Conley Zehnder index of the extremizers of H, considered as periodic orbits.

1. Introduction

There has not been much study of the Morse index of geodesics for the Hofer length functional on path spaces of the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$. Maybe this is because the problem of Morse theory for the Hofer length functional seems completely hopeless. This is possibly true to a large extent, however in [7] we showed that doing Morse theory for the Hofer length functional "virtually" can give some interesting results in symplectic topology.

In the special case where γ is an S^1-subgroup in $\mathrm{Ham}(M, \omega)$ generated by a Morse Hamiltonian H, a key point in [7] was using a relationship of the Morse index of γ with the Conley–Zehnder index of the linearized flow at the extremizers of H, in some special cases.

Remark 1.1. We did not use the words Conley–Zehnder index in [7], but rather the index of a certain Cauchy–Riemann operator, but this could be directly related to the above CZ index.

Indeed as a byproduct we arrived at the conjecture that the two indexes must coincide. A lower bound for the Morse index in terms of the Conley–Zehnder index was proved by Karshon–Slimowitz in [2] by constructing a beautiful explicit local family of shortenings of γ. Here we give a simple proof of the conjecture for more general Ustilovsky geodesics, using calculus of variations, already worked out in [8] for the Hofer length functional.

In the future work, we will use this coincidence to extend the virtual Morse theory picture of [7] from very special flag manifolds to general monotone symplectic manifolds.

2. Statement and proof

2.1. The group of Hamiltonian symplectomorphisms and Hofer metric.

Given a smooth function $H : M^{2n} \times [0, 1] \to \mathbb{R}$, there is an associated time-dependent Hamiltonian vector field X_t, $0 \leq t \leq 1$, defined by

$$\omega(X_t, \cdot) = -dH_t(\cdot). \tag{2.1}$$

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The vector field X_t generates a path $\gamma : [0,1] \to \mathrm{Diff}(M)$, starting at *id*. Given such a path γ, its end point $\gamma(1)$ is called a Hamiltonian symplectomorphism. The space of Hamiltonian symplectomorphisms forms a group, denoted by $\mathrm{Ham}(M, \omega)$.

In particular, the path γ above lies in $\mathrm{Ham}(M, \omega)$. It is well-known that any smooth path γ in $\mathrm{Ham}(M, \omega)$ with $\gamma(0) = id$ arises in this way (is generated by $H : M \times [0,1] \to \mathbb{R}$ as above). Given a general smooth path γ, the *Hofer length*, $L(\gamma)$ is defined by

$$L(\gamma) := \int_0^1 \max_M H_t^\gamma - \min_M H_t^\gamma dt,$$

where H^γ is a generating function for the path $t \mapsto \gamma(0)^{-1}\gamma(t)$, $0 \leq t \leq 1$. The Hofer distance $\rho(\phi, \psi)$ is defined by taking the infinum of the Hofer length of paths from ϕ to ψ. We only mention it, to emphasize that it is a deep and interesting theorem that the resulting metric is non-degenerate (cf. [1, 3]). This gives $\mathrm{Ham}(M, \omega)$ the structure of a Finsler manifold.

We now consider L as a functional on the space of paths in $\mathrm{Ham}(M, \omega)$ starting at *id* and ending at some fixed end points, denote this by $\Omega\mathrm{Ham}(M, \omega)$. It is shown by Ustilovsky that γ is a smooth critical point of

$$L : \Omega\mathrm{Ham}(M, \omega) \to \mathbb{R},$$

if there is a unique pair of points $x_{\max}, x_{\min} \in M$ maximizing, respectively minimizing the generating function H_t^γ at each moment t, and such that H_t^γ is Morse at $x_{\max}$, $x_{\min}$. We shall call such a γ *Ustilovsky geodesic*.

Consequently, it makes sense to ask for the Morse index of Ustilovsky geodesics, (which might a priori be infinite.) Moreover, it is easy to see that $\mathrm{index}_L(\gamma) = \mathrm{index}_{L_+}(\gamma) + \mathrm{index}_{L_-}(\gamma)$, where:

$$L_+(\gamma) := \int_0^1 \max(H_t^\gamma) dt, \tag{2.2}$$

for H_t^γ in addition normalized by the condition:

$$\int_M H_t^\gamma \cdot \omega^n = 0. \tag{2.3}$$

The functional L_- is defined similarly as above. It will be the Morse index of γ with respect to L_+ that we compute.

Fix a small ϵ, $0 < \epsilon < 1$, s.t. the linearized flow (isotopy) at $x_{\max}$ of $\gamma|_{[0,\epsilon]}$ has no non-trivial periodic orbits with positive period. Let us denote the periodic orbit of the isotopy $\gamma|_{[0,\epsilon]}$ associated to $x_{\max}$ by $x_{\max,0}$, and likewise the periodic orbit of the isotopy $\gamma|_{[0,1]}$ associated to $x_{\max}$ by $x_{\max,1}$. We will say that γ is *non-degenerate* if $x_{\max,1}$ is non-degenerate in the sense of Floer theory, in other words the time 1 linearized flow at $x_{\max}$ has no non-trivial time 1 periodic orbits.

Theorem 2.1. *For γ a non-degenerate Ustilovsky geodesics as above, the Morse index of γ with respect to L_+ is*

$$|CZ(x_{\max,1}) - CZ(x_{\max,0})|. \tag{2.4}$$

Remark 2.2. The above expression is independent of any choices of normalization of CZ index appearing in literature. Moreover, it is precisely the index of the real linear

Cauchy Riemann operator on which the conjecture is based in [7]. A better way to understand this coincidence is outlined in Section 1.3 of that paper.

Proof. The Morse index theorem [4] cannot be directly applied to

$$L_+ = \int_0^1 L(\dot\gamma(t), \gamma(t))dt,$$

$L(\dot\gamma(t), \gamma(t)) = \max_M H_t$, for $H_t = \dot\gamma(t) \in T_{\gamma(t)}\mathrm{Ham}(M, \omega) \equiv C^\infty_{\mathrm{norm}}(M)$, with the latter being smooth functions normalized to have zero mean, (2.3). This is because it clearly does not satisfy the Legendre condition that $\frac{d^2}{d\xi^2} L(\dot\gamma(t), \gamma(t)) > 0$, for every variation ξ of $\dot\gamma$, (for every t). However, Ustilovsky shows that there is a related functional (actually a quadratic form) $\mathcal{L}_+$ on the vector space $\Omega_0 T_{x_{\max}} M$ (based loop space at 0 on the tangent space). With the Hessian of L_+ at γ coinciding with the Hessian of $\mathcal{L}_+$ at 0, and to which the Morse theorem does apply. This is beautiful, but we refer the reader to [8] and [5, Section 12.4] for further details.

The Morse theorem gives us the following procedure for the calculation of the Morse index of γ with respect to $\mathcal{L}_+$. Denote by γ_τ the restriction of γ to $[0, \tau] \subset [0, 1]$. Then $index(\gamma_\tau)$ is a locally constant, lower semi-continuous function in τ, and jumps at a discrete set of $\tau_i \in (0, 1)$ called conjugate times. The value of the jump $mult(\tau_i)$ is the dimension of the solution space of the associated Jacobi equation. Informally speaking this is dimension of the space of infinitesimal variations of γ_τ through extremals with the same endpoints. And a point $\tau \in (0, 1]$ is defined to be a conjugate time if this dimension is non-zero.

In the case of the functional $\mathcal{L}_+$, it is shown in [8] that $\tau_0 \in (0, 1]$ is a conjugate time if and only if the time τ_0 linearized flow of H at the extremizer $x_{\max}$ of H has periodic orbits, and the multiplicity $mult(\tau_0)$ is the dimension of the space of these periodic orbits.

To keep notation simple, let us denote by $\gamma_{\max}$ the restriction of the linearization of γ at $x_{\max}$ to $[\epsilon, 1]$. We will use the construction of Maslov and Conley–Zehnder index given in [6]. For the normalizations used in [6] we show that the absolute value of the Conley–Zehnder index for the path $\gamma_{\max}$ is exactly the Morse index of γ for L_+, from which the statement of the theorem immediately follows by additivity of the Conley–Zehnder/Maslov index with respect to concatenations (and with respect to those normalizations).

Note first that $\gamma_{\max}$ has a crossing at $\tau_0 \in (0, 1)$ with the Maslov cycle if and only if $\gamma_{\max}(\tau_0)$ has 1-eigenvectors, i.e., if and only if τ_0 is a conjugate time. Moreover, the dimension of the intersection I_{τ_0} of the diagonal $\Lambda \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with the graph $Gr(\gamma_{\max}(\tau_0)) = \{(z, \gamma_{\max}(\tau_0)z) | z \in \mathbb{R}^{2n}\}$, is exactly the multiplicity of τ_0. The crossing form Q at τ_0 can then be identified with the Hessian of $H^\gamma_{\tau_0}$ at $x_{\max}$, which follows by [6, Remark 5.4]. Since this is non-degenerate by assumption, all the crossings are regular. Our conventions are

$$\omega(X_H, \cdot) = -dH(\cdot),$$
$$\omega(\cdot, J\cdot) > 0.$$

Consequently the crossing form is negative definite, and so is negative definite on I_{τ_0}. So the signature of Q on I_{τ_0} (number of positive minus number of negative eigenvalues) is just the $-mult(\tau_0)$. The Conley–Zehnder index of $\gamma_{\max}$ is then the sum over conjugate times τ_i of $-mult(\tau_i)$. Consequently Morse index of γ, is $|CZ(\gamma_{\max})|$. □

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Centre de Recherches Mathématiques, Université de Montréal, C.P. 6128, Succ. Centre-ville Montréal H3C 3J7, Québec, Canada

E-mail address: yasha.savelyev@gmail.com