

THE TRANSCENDENTAL LATTICE OF THE SEXTIC FERMAT SURFACE

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ABSTRACT. We prove that the integral polarized Hodge structure on the transcendental lattice of a sextic Fermat surface is decomposable. This disproves a conjecture of Kulikov related to a Hodge theoretic approach to proving the irrationality of the very general cubic fourfold.

1. Introduction

Proving the irrationality of a very general cubic fourfold $X \subset \mathbb{P}^5$ over the complex numbers is a well-known problem in algebraic geometry. At present, not a single cubic fourfold is provably irrational. However, families of rational cubic fourfolds are described by Fano [14], Tregub [32, 33], and Beauville–Donagi [6]. Hassett [18, 19] identifies, via lattice theory, a countably infinite number of subvarieties, of codimension 2 in the moduli space, consisting of rational cubic fourfolds. So far, all known rational cubic fourfolds lie on two divisors of the moduli space, corresponding to the existence of a plane or a quartic scroll marking. Even the construction of additional classes of rational cubic fourfolds is an open problem.

Recently, Kulikov [21] initiated a conjectural approach to the irrationality problem for cubic fourfolds. The strategy is modeled on that of Clemens and Griffiths [11] for cubic threefolds, with the role of the intermediate Jacobian played by the integral polarized Hodge structure T_X on the transcendental part of the middle cohomology $H^4(X, \mathbb{Z})$. Assuming the existence of a birational map $r : \mathbb{P}^4 \dashrightarrow X$, by Hironaka’s resolution of singularities, we can resolve r to a birational morphism $f : X' \rightarrow X$ by a sequence

$$X' = X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 = \mathbb{P}^4 \dashrightarrow X$$

of blow-ups $X_i \rightarrow X_{i-1}$ along points, smooth curves, or smooth surfaces. Blow-ups along points, curves, and surfaces of $p_g = 0$ do not contribute to the transcendental lattice, hence there is a decomposition of polarized Hodge structures

$$T_{X'} = \bigoplus_j T_{S_j}(-1)$$

where the sum is taken over all surfaces S_j of $p_g \geq 1$ that are the centers of blow-ups in the resolution. Here, for S a smooth projective surface, T_S denotes the transcendental part of the middle cohomology $H^2(S, \mathbb{Z})$. On the other hand, there is a decomposition

$$T_{X'} = f^*T_X \oplus (f^*T_X)^\perp.$$

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Kulikov proves [21, Lemmas 2,3] that for X very general, comparing these two decompositions yields an index j_0 such that $T_{S_{j_0}} = f^*T_X(1) \oplus T'$ for some *nontrivial* polarized Hodge substructure $T' \subset (f^*T_X)^\perp$. The nontriviality of T' follows from standard estimates on the 2nd Betti number of a minimal model of S_{j_0} . Then the irrationality of the very general cubic fourfold would follow from the following conjecture.

Conjecture 1.1 (Nondecomposability Conjecture [21, p. 59]). *Let S be a smooth projective surface over the complex numbers. Then the integral polarized Hodge structure T_S on the transcendental part of $H^2(S, \mathbb{Z})$ is indecomposable.*

For curves, the integral polarized Hodge structure on H^1 is indeed indecomposable because of Riemann's theorem describing the theta divisor of a Jacobian in geometric terms and showing that it is irreducible. Hence, the Jacobian is indecomposable as a polarized abelian variety and this is what is used substantially in the proof of the irrationality of cubic threefolds by Clemens and Griffiths [11]. An essential point in Conjecture 1.1 is the indecomposability over \mathbb{Z} . In fact, counterexamples over \mathbb{Q} abound; note also that Jacobians of curves are decomposable if one considers them within the category of abelian varieties up to isogeny.

In this paper we prove that Conjecture 1.1 cannot be true for an arbitrary surface.

Theorem 1.2. *Let $S \subset \mathbb{P}^3$ be the sextic Fermat surface. Then the integral polarized Hodge structure T_S on the transcendental part of $H^2(S, \mathbb{Z})$ is decomposable.*

After recalling some general theory in Section 2, we prove this in Section 3. Admittedly, the sextic Fermat surface S , defined in \mathbb{P}^3 by

$$x_0^6 + x_1^6 + x_2^6 + x_3^6 = 0$$

is quite special. For one, it has maximal Picard rank $\rho(S) = h^{1,1}(S) = 86$, a fact known to Beauville, cf. [31, Remark 3.3(ii)] or [29, Ex. 3]. The rank of T_S is 20. See also Remark 4.7 for a discussion of why having maximal Picard rank might be significant in this context.

In our analysis, we make use of the description of the integral Hodge structure of Fermat varieties as a module over the group ring of the automorphism group in the formulation of Looijenga [23, Section 2], which in turn draws on many previous sources [17, 26, 28, 30].

After a discussion of the results of this paper with the second author at an Oberwolfach workshop in May 2013, V. Kulikov suggested that his conjecture could be modified to the effect that surfaces with decomposable integral polarized Hodge structure on the transcendental lattice enjoy some Hodge theoretic rigidity property, and that this would still imply irrationality of the very general cubic fourfold. In Section 4, we work out the details of Kulikov's suggestion.

One may also wonder if surfaces S for which Conjecture 1.1 fails are always defined over $\overline{\mathbb{Q}}$.

Finally, we mention a few other conjectural approaches to prove the irrationality of the very general cubic fourfold. A derived categorical approach due to Kuznetsov [22] has seen recent activity [1, 2, 5, 24]. Using the theory of semiorthogonal decompositions Kuznetsov constructs a triangulated category $\mathbf{A}_X \subset \mathbf{D}^b(X)$ and conjectures

that it encodes all the information concerning the rationality of X . The irrationality of the very general cubic fourfold would be a consequence. This approach runs into some difficulties due to certain pathologies that semiorthogonal decompositions of derived categories may exhibit, see [7, 8, 9], but which can possibly be overcome if complemented by new ideas.

There is also a cohomological invariant approach due to Colliot-Thélène. Many unramified cohomology groups of X vanish as a consequence of the integral Hodge conjecture [13, Theorem 1.1] proved by Voisin [35, Theorem 18] and the triviality of the Brauer group [27, Theorem A.1]. Nevertheless, by a result of Merkurjev [25, Theorem 2.11], the vanishing of *all* unramified cohomology groups arising from cycle modules is controlled by the vanishing of the Chow group $A_0(X_F)$ of 0-cycles of degree 0 on $X_F = X \times_{\mathbb{C}} F$ over all field extensions F/\mathbb{C} . Hence the detection of such a nontrivial 0-cycle on a cubic fourfold over a sufficiently complicated field F would imply irrationality. Results in this direction are obtained in [4, 12, 36].

2. Integral polarized Hodge structures on Fermat surfaces

In this section, we describe the integral polarized Hodge structure on the cohomology of Fermat varieties, especially Fermat surfaces, building on and developing further the description in [23]. For general background on Hodge structures, we follow [34, I Section 7.1].

Definition 2.1.

- (1) An integral polarized Hodge structure (IPHS) of weight $n \in \mathbb{N}$ is a triple $(H_{\mathbb{Z}}, H^{p,q}, Q)$ where

- $H_{\mathbb{Z}}$ is a free \mathbb{Z} -module of finite rank and $Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \rightarrow \mathbb{Z}$ is a nondegenerate (i.e., nondegenerate over \mathbb{Q}) bilinear form with symmetry property $Q(x, y) = (-1)^n Q(y, x)$ for all $x, y \in H_{\mathbb{Z}}$.
- The $H^{p,q}$, for $0 \leq p, q \leq n$, are complex linear subspaces in the complexification $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ with the property that

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$$

and such that $H^{p,q} = \overline{H^{q,p}}$, with the conjugation on $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ being induced by the conjugation on \mathbb{C} .

- Extend Q to $H_{\mathbb{C}}$ by linearity. Then we require the orthogonality condition

$$Q(x, y) = 0 \text{ if } x \in H^{p,q}, y \in H^{p',q'} \text{ with } p \neq q',$$

which is called the first Hodge–Riemann bilinear relation. We also require the positivity condition

$$(\sqrt{-1})^{p-q} Q(x, \bar{x}) > 0 \text{ for } 0 \neq x \in H^{p,q},$$

which is called the second Hodge–Riemann bilinear relation, although it will not actually play any role in this paper.

- (2) The notion of morphism of integral polarized Hodge structures $(H_{\mathbb{Z},1}, H_1^{p,q}, Q_1)$ and $(H_{\mathbb{Z},2}, H_2^{p,q}, Q_2)$ is the natural one: it is a \mathbb{Z} -linear homomorphism $f : H_{\mathbb{Z},1} \rightarrow H_{\mathbb{Z},2}$ which is an isometry, i.e., $Q_2(f(x), f(y)) = Q_1(x, y)$, and for

$f_{\mathbb{C}} = f \otimes \text{id}$ we have $f_{\mathbb{C}}(H_1^{p,q}) \subset H_2^{p,q}$. Such an f is necessarily an embedding since Q_1 and Q_2 are nondegenerate.

- (3) There is a natural notion of direct sum of two integral polarized Hodge structures $(H_{\mathbb{Z},1}, H_1^{p,q}, Q_1)$ and $(H_{\mathbb{Z},2}, H_2^{p,q}, Q_2)$; it is simply given by

$$(H_{\mathbb{Z},1} \oplus H_{\mathbb{Z},2}, H_1^{p,q} \oplus H_2^{p,q}, Q_1 \oplus Q_2).$$

An integral polarized Hodge structure is indecomposable if it is not a direct sum of two nontrivial integral polarized Hodge structures.

Let X be an n -dimensional smooth projective variety. We assume that $n = 2m$ is even. Consider the middle cohomology $H_{\mathbb{Z}} := H^n(X, \mathbb{Z})/(\text{torsion})$ and its Hodge decomposition $H_{\mathbb{C}} = H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$ into the spaces of harmonic (p, q) -forms. Consider the bilinear form Q defined as the restriction to $H_{\mathbb{Z}}$ of

$$Q(x, y) = \int_X x \wedge y, \quad \text{for } x, y \in H_{\mathbb{C}}.$$

This triple almost defines an integral polarized Hodge structure; the second Hodge–Riemann bilinear relation in Definition 2.1(1) is not satisfied, but it holds if we pass to primitive cohomology and replace Q by $(-1)^{n+1}Q$.

Definition 2.2. Let $h \in H^{1,1} \cap H^2(X, \mathbb{Z})$ be a polarization class on X .

- (a) The IPHS on the primitive middle cohomology of X , denoted by

$$(H_0^n(X, \mathbb{Z}), H_0^{p,q}, Q_0),$$

is defined as follows: $H_0^n(X, \mathbb{Z}) \subset H^n(X, \mathbb{Z})/(\text{torsion})$ is the kernel of the map $H^n(X, \mathbb{Z})/(\text{tors}) \rightarrow H^{n+2}(X, \mathbb{Z})/(\text{tors})$ defined by cup product with h . Then $H_0^{p,q} = H^{p,q} \cap (H_0^n(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})$ and Q_0 is the restriction of Q to $H_0^n(X, \mathbb{Z})$.

- (b) We call $A_X = H^{m,m} \cap (H^n(X, \mathbb{Z})/(\text{tors}))$ the algebraic lattice, $A_X^0 = H_0^{m,m} \cap H_0^n(X, \mathbb{Z})$ the primitive algebraic lattice and $T_X = A_X^{\perp}$ the transcendental lattice of X . The transcendental IPHS is

$$(T_X, H_T^{p,q}, Q_T),$$

where $H_T^{p,q} = H^{p,q} \cap (T_X \otimes \mathbb{C})$ and Q_T is the restriction of Q to T_X . It is an integral polarized Hodge substructure of the primitive middle cohomology.

We will now assume $n = 2$ and describe this structure for the Fermat surface $X_d = \{x_0^d + x_1^d + x_2^d + x_3^d = 0\}$ of degree d in \mathbb{P}^3 , taking our point of departure from [23], which we would like to simplify and amplify in several respects.

Looijenga’s computation starts by considering homology. Poincaré duality gives an isomorphism

$$P : H_2(X_d, \mathbb{Z}) \simeq H^2(X_d, \mathbb{Z}).$$

Lemma 2.3. *If we endow $H_2(X_d, \mathbb{Z})$ with the intersection product and $H^2(X_d, \mathbb{Z})$ with Q , then P is an isomorphism of integral lattices.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc}
 H_2(X_d, \mathbb{Z}) \otimes H_2(X_d, \mathbb{Z}) & \xrightarrow{Q'} & \mathbb{Z} \\
 \downarrow P \otimes \text{id} & & \parallel \\
 H^2(X_d, \mathbb{Z}) \otimes H_2(X_d, \mathbb{Z}) & \xrightarrow{\cap} & \mathbb{Z} \\
 \downarrow \text{id} \otimes P & & \parallel \\
 H^2(X_d, \mathbb{Z}) \otimes H^2(X_d, \mathbb{Z}) & \xrightarrow{Q} & \mathbb{Z}
 \end{array}$$

where \cap is the topological cap product, Q' the intersection product, and Q the bilinear form on cohomology as defined above. The assertion follows. By abuse of notation, we will also write Q (and not Q') for the pairing on homology in the following. \square

Looijenga now works with the primitive homology $H_2^0(X_d, \mathbb{Z})$ defined as the orthogonal to h (the embedding hyperplane class from \mathbb{P}^3), viewed as an element of $H_2(X_d, \mathbb{Z})$ (so this is $P^{-1}(h)$, to be precise). Hence, Poincaré duality induces an isomorphism of lattices

$$P : H_2^0(X_d, \mathbb{Z}) \rightarrow H_0^2(X_d, \mathbb{Z}).$$

Remark 2.4. The Fermat hypersurface X_d is invariant under the action of the group $\mathbb{P}_d^4/\mathbb{P}_d$ where $\mathbb{P}_d^4/\mathbb{P}_d$ acts on X_d via rescaling the coordinates. Therefore, $H_2^0(X_d, \mathbb{Z})$ and $H_0^2(X_d, \mathbb{Z})$ are naturally modules over the group algebra $\mathbb{Z}[\mathbb{P}_d^4/\mathbb{P}_d]$.

The following is a consequence of [23], Corollary 2.2 and the computation following Remark 2.3 on p. 6.

Proposition 2.5. *The lattice $H_2^0(X_d, \mathbb{Z})$ is isomorphic, as a \mathbb{Z} -module, to the quotient ring*

$$H_2^0(X_d, \mathbb{Z}) \simeq \mathbb{Z}[u_0, u_1, u_2, u_3]/I_d$$

where I_d is the ideal

$$I_d = \left(u_0 u_1 u_2 u_3 - 1, \frac{u_0^d - 1}{u_0 - 1}, \dots, \frac{u_3^d - 1}{u_3 - 1} \right).$$

The intersection form is given as follows: abbreviating

$$u^K := u_0^{k_0} \cdots u_3^{k_3} \text{ for } K = (k_0, \dots, k_3)$$

then $u^K \cdot u^L$ is the coefficient of 1 in

$$-u^{K-L}(1-u_0)(1-u_1)(1-u_2)(1-u_3),$$

where we calculate in the group ring

$$\mathbb{Z}[\mathbb{P}_d^4/\mathbb{P}_d] = \mathbb{Z}[u_0, u_1, u_2, u_3]/(u_0 u_1 u_2 u_3 - 1, u_0^d - 1, \dots, u_3^d - 1).$$

Moreover, the $\mathbb{Z}[\mathbb{P}_d^4/\mathbb{P}_d]$ -module structure on $H_2^0(X_d, \mathbb{Z})$ is induced by letting the i th standard generator of \mathbb{P}_d^4 act via multiplication by u_i .

Remark 2.6. Let $G = \mathbb{P}_d^4/\mathbb{P}_d$. Fix a primitive d th root of unity ζ_d . The characters $\chi : G \rightarrow \mathbb{C}^*$ of G are given by

$$\chi(u_i) = \zeta_d^{k_i}, \quad k_i \in \{0, \dots, d-1\} \text{ and } \sum_{0 \leq i \leq 3} k_i \equiv 0 \pmod{d}.$$

For a tuple $K = (k_0, k_1, k_2, k_3) \in \{0, \dots, d-1\}^4$ with zero sum modulo d , we denote its associated character by χ_K . Characters of G correspond bijectively with characters of $\mathbb{Z}[G]$.

The characters of $\mathbb{Z}[G]$ that descend to characters of $H_2^0(X_d, \mathbb{Z})$, via the isomorphism of Proposition 2.5, correspond exactly to those tuples $K = (k_0, \dots, k_3)$ with zero sum mod d and with $k_i \neq 0$ for all $0 \leq i \leq 3$. Note that the complex zeros Z of the ideal I_d are precisely the points

$$P_K := (\zeta_d^{k_0}, \dots, \zeta_d^{k_3})$$

with K as in the preceding sentence.

We now have to describe how $H_2^0(X_d, \mathbb{Z}) \simeq H_0^2(X_d, \mathbb{Z})$, viewed as a sublattice of $H_0^2(X_d, \mathbb{Z}) \otimes \mathbb{C} = H_0^2(X_d, \mathbb{C})$, is positioned relative to the Hodge subspaces of $H_0^2(X_d, \mathbb{C})$. Note that this will allow us to compute everything: the algebraic part, the transcendental part, and the induced integral polarized Hodge structure.

The Poincaré duality isomorphism P is equivariant for the natural actions of G on homology and cohomology (it is given by cap product with the fundamental class, which is invariant). Via P , we identify $H_0^{p,q} \subset H_0^2(X_d, \mathbb{C})$ with its image in $H_2^0(X_d, \mathbb{C})$, which we denote by the same letter. We will write $L = H_0^2(X_d, \mathbb{Z})$.

Proposition 2.7. *The Hodge subspace $H_0^{p,q} \subset L \otimes \mathbb{C}$ is*

$$H_0^{p,q} = \bigoplus_{\chi_K} (L \otimes \mathbb{C})_{\chi_K}$$

where $(L \otimes \mathbb{C})_{\chi_K}$ is the eigenspace of the character χ_K and the sum runs over all characters with

$$|K| := \sum_i k_i = (q+1)d.$$

In other words, for $Z^{p,q} = \{P_K \mid \sum_i k_i = (q+1)d\} \subset Z$,

$$H_0^{p,q} = \{\varphi \in \mathbb{C}[u_0, \dots, u_3]/I_d \mid \text{supp}(\varphi) \subset Z^{p,q}\}$$

Proof. We can identify the corresponding character spaces in $L \otimes_{\mathbb{Z}} \mathbb{C} \simeq H_2^0(X_d, \mathbb{C})$ and $H_0^2(X_d, \mathbb{C})$ via the diagram

$$\begin{array}{ccc} H_2^0(X_d, \mathbb{C}) = L_{\mathbb{C}} & \xrightarrow[\simeq]{P} & H_0^2(X_d, \mathbb{C}) \\ \uparrow & & \uparrow \\ H_2^0(X_d, \mathbb{Z}) = L & \xrightarrow[\simeq]{P} & H_0^2(X_d, \mathbb{Z}) \end{array}$$

We then apply [23], section after Corollary 2.4 on p. 8. □

One can compare the character basis of Proposition 2.7 with the character basis for Griffith's presentation of the primitive Hodge subspaces in terms of residues

$$R^{pd-4} \xrightarrow{\simeq} H_0^{3-p,p-1}(X_d)$$

where $R = \bigoplus_{j \geq 0} R^j$ is the Jacobian ring of X_d , see [34, II Theorem 6.10]. This comparison is made in [23], section after Corollary 2.4.

Lemma 2.8. *The quadratic form Q on L can be written as*

$$Q(\varphi, \psi) = \sum_{P \in Z} \alpha_P \varphi(P) \psi(\bar{P})$$

for some $\alpha_P \in \mathbb{Q}(\zeta_d)^*$ for each $P \in Z$.

Proof. First, note that Q is invariant under G and also satisfies

$$Q(v, w) = Q(gv, gw) = \chi(g)\chi'(g)Q(v, w),$$

for all $v \in (L \otimes \mathbb{C})_\chi$, $w \in (L \otimes \mathbb{C})_{\chi'}$ and all $g \in G$. Hence, $Q(v, w) \neq 0$ only if $\chi = (\chi')^{-1} = \overline{\chi'}$. \square

Remark 2.9. The whole construction up to now is also invariant under the symmetric group \mathfrak{S}_4 acting by permutations on the u_i . Therefore, α_P is constant on the orbits of the action of \mathfrak{S}_4 on Z .

Proposition 2.10. *Consider the action of the Galois group $\Gamma = \text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ on Z . Let*

$$Z_A := \{P_K \mid \Gamma \cdot P_K \subset Z^{1,1}\}$$

and $Z_T := Z \setminus Z_A$. Then

$$A_X^0 = \{\varphi \in \mathbb{Z}[u_0, \dots, u_3]/I_d \mid \text{supp}(\varphi) \subset Z_A\}$$

and

$$T_X = \{\varphi \in \mathbb{Z}[u_0, \dots, u_3]/I_d \mid \text{supp}(\varphi) \subset Z_T\}.$$

Proof. By [30], Theorem I(iii), we get the assertion about A_X^0 . The assertion about T_X then follows from Lemma 2.8. \square

3. The Fermat sextic

Let now X_6 be the sextic Fermat surface in \mathbb{P}^3 and ζ a primitive 6th root of unity. Here, $\Gamma = \text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ is generated by complex conjugation. In this case, $Z_A = Z^{1,1}$ and therefore by Proposition 2.10, we have that

$$A_X^0 \otimes \mathbb{C} = H_0^{1,1}, \quad \text{and} \quad A_X \otimes \mathbb{C} = H^{1,1},$$

i.e., X_6 is a surface of maximal Picard rank. Hence

$$T_X \otimes \mathbb{C} = H^{2,0} \oplus H^{0,2}.$$

Notation 3.1. *Let*

$$\begin{aligned} Z_{(1,1,1,3)} &:= \{P_{(1,1,1,3)}, P_{(1,1,3,1)}, P_{(1,3,1,1)}, P_{(3,1,1,1)}, \\ &\quad P_{(5,5,5,3)}, P_{(5,5,3,5)}, P_{(5,3,5,5)}, P_{(3,5,5,5)}\}, \\ Z_{(1,1,2,2)} &:= \{P_{(1,1,2,2)}, P_{(2,2,1,1)}, P_{(5,5,4,4)}, P_{(4,4,5,5)}\}, \\ Z_{(1,2,1,2)} &:= \{P_{(1,2,1,2)}, P_{(2,1,2,1)}, P_{(5,4,5,4)}, P_{(4,5,4,5)}\}, \\ Z_{(1,2,2,1)} &:= \{P_{(1,2,2,1)}, P_{(2,1,1,2)}, P_{(5,4,4,5)}, P_{(4,5,5,4)}\} \end{aligned}$$

and

$$L_\beta := \{\varphi \in L \mid \text{supp}(\varphi) \subset Z_\beta\}.$$

We have

$$Z_T = Z_{(1,1,1,3)} \cup Z_{(1,1,2,2)} \cup Z_{(1,2,1,2)} \cup Z_{(1,2,2,1)}$$

and

$$T_X \otimes_{\mathbb{Z}} \mathbb{Q} = (L_{(1,1,1,3)} \oplus L_{(1,1,2,2)} \oplus L_{(1,2,1,2)} \oplus L_{(1,2,2,1)}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In the rest of this section we show that this decomposition holds even over \mathbb{Z} . The necessary computations were checked using a Macaulay2 script [3, 15].

Proposition 3.2. *There is a sublattice $L'_{(1,1,1,3)}$ of $L_{(1,1,1,3)}$ with a basis such that the intersection form is given by*

$$Q_{(1,1,1,3)} = \begin{pmatrix} 32 & 8 & 8 & 8 & 4 & 16 & 16 & 16 \\ 8 & 32 & 8 & 8 & 16 & 4 & 16 & 16 \\ 8 & 8 & 32 & 8 & 16 & 16 & 4 & 16 \\ 8 & 8 & 8 & 32 & 16 & 16 & 16 & 4 \\ 4 & 16 & 16 & 16 & 32 & 8 & 8 & 8 \\ 16 & 4 & 16 & 16 & 8 & 32 & 8 & 8 \\ 16 & 16 & 4 & 16 & 8 & 8 & 32 & 8 \\ 16 & 16 & 16 & 4 & 8 & 8 & 8 & 32 \end{pmatrix}.$$

We have $\det Q_{(1,1,1,3)} = 2^{16} 3^{12}$.

Proof. Consider the matrix

$$M_{(1,1,1,3)} = 12(\zeta + 1) \begin{pmatrix} \zeta^4 & \zeta^2 & \zeta^2 & \zeta^2 & \zeta^1 & \zeta^3 & \zeta^3 & \zeta^3 \\ \zeta^2 & \zeta^4 & \zeta^2 & \zeta^2 & \zeta^3 & \zeta^1 & \zeta^3 & \zeta^3 \\ \zeta^2 & \zeta^2 & \zeta^4 & \zeta^2 & \zeta^3 & \zeta^3 & \zeta^1 & \zeta^3 \\ \zeta^2 & \zeta^2 & \zeta^2 & \zeta^4 & \zeta^3 & \zeta^3 & \zeta^3 & \zeta^1 \\ \zeta^1 & \zeta^3 & \zeta^3 & \zeta^3 & \zeta^4 & \zeta^2 & \zeta^2 & \zeta^2 \\ \zeta^3 & \zeta^1 & \zeta^3 & \zeta^3 & \zeta^2 & \zeta^4 & \zeta^2 & \zeta^2 \\ \zeta^3 & \zeta^3 & \zeta^1 & \zeta^3 & \zeta^2 & \zeta^2 & \zeta^4 & \zeta^2 \\ \zeta^3 & \zeta^3 & \zeta^3 & \zeta^1 & \zeta^2 & \zeta^2 & \zeta^2 & \zeta^4 \end{pmatrix}.$$

Denote by P_i the i th point of $Z_{(1,1,1,3)}$. By interpolation we find polynomials φ_j in $\mathbb{Z}[u_0, \dots, u_3]/I_6$ with $\varphi_j(P_i) = (M_{(1,1,1,3)})_{ij}$ and zero on all other points in Z . We can choose

$$\varphi_1 = (u_3^4, u_3^3, u_3^2, u_3, 1) \begin{pmatrix} 1 \\ s_1^2 - 2s_2 + s_1 + 2 \\ s_1^2 - 3s_2 + 1 \\ -s_1^2 s_2 + 2s_2^2 - s_3 + 2s_1^2 - 5s_2 + s_1 + 1 \\ -s_1^2 s_3 + 3s_2 s_3 - s_1^2 s_2 + 3s_2^2 - s_1 s_3 - 3s_3 - 2s_2 - 1 \end{pmatrix}$$

with $s_1 = u_0 + u_1 + u_2$, $s_2 = u_0 u_1 + u_0 u_2 + u_1 u_2$ and $s_3 = u_0 u_1 u_2$. By applying appropriate permutations of the variables we obtain $\varphi_2, \varphi_3, \varphi_4$. The remaining polynomials are obtained from these by applying the substitution $u_i \mapsto u_i^{-1} = u_i^5$. This induces complex conjugation on the points.

Let $L'_{(1,1,1,3)}$ be the sublattice of $L_{(1,1,1,3)}$ spanned by the φ_j . By Lemma 2.8 and Remark 2.9 we have that the intersection form on $L'_{(1,1,1,3)}$ is

$$Q_{(1,1,1,3)} = \alpha_{P_{(1,1,1,3)}} M_{(1,1,1,3)}^t \overline{M_{(1,1,1,3)}}$$

where $M_{\overline{(1,1,1,3)}}$ is obtained from $M_{(1,1,1,3)}$ by interchanging the first four rows with the last four rows (since complex conjugation interchanges the first four points in $Z_{(1,1,1,3)}$ with the last four points). We compute $\alpha_{P_{(1,1,1,3)}}$ by evaluating $Q(\varphi_1, \varphi_1)$ in two different ways: firstly, by using Looijenga's formula in Proposition 2.5, and secondly, by Lemma 2.8. One finds

$$\alpha_{P_{(1,1,1,3)}} = \frac{1}{108}.$$

Direct computation gives the above matrix for $Q_{(1,1,1,3)}$ and its determinant. \square

Similarly, we have the following.

Proposition 3.3. *There is a sublattice $L'_{(1,1,2,2)}$ of $L_{(1,1,2,2)}$ with a basis such that the intersection form is given by*

$$Q_{(1,1,2,2)} = \begin{pmatrix} 24 & 12 & 0 & 0 \\ 12 & 24 & 0 & 0 \\ 0 & 0 & 24 & 12 \\ 0 & 0 & 12 & 24 \end{pmatrix}.$$

We have $\det Q_{(1,1,2,2)} = 2^8 3^6$. The same is true for the lattices $L_{(1,2,1,2)}$ and $L_{(1,2,2,1)}$.

Proof. Consider

$$M_{(1,1,2,2)} = 12(\zeta + 1) \begin{pmatrix} \zeta^0 & \zeta^5 & \zeta^1 & \zeta^0 \\ \zeta^2 & \zeta^3 & \zeta^4 & \zeta^5 \\ \zeta^5 & \zeta^0 & \zeta^4 & \zeta^5 \\ \zeta^3 & \zeta^2 & \zeta^1 & \zeta^0 \end{pmatrix}.$$

Denote by P_i the i th point of $Z_{(1,1,2,2)}$. By interpolation we find polynomials ψ_j in $\mathbb{Z}[u_0, \dots, u_3]/I_6$ with $\psi_j(P_i) = (M_{(1,1,2,2)})_{ij}$ and zero on all other points in Z . We can choose

$$\begin{aligned} \psi_1 &= q_1 q_2 r_1^2 - q_1 r_1^2 r_2 + q_1 r_2^2 - q_1^3 r_1 + 3q_1 q_2 r_1 + q_2 r_1^2 - q_1^2 r_2 - q_1 r_1 r_2 \\ &\quad - q_1^3 + 3q_1 q_2 - 2q_1^2 r_1 + 3q_2 r_1 - q_1 r_2 + r_1 r_2 - q_1^2 \\ &\quad + 2q_2 - q_1 r_1 - r_1^2 + 2r_2 - 2q_1 - 2r_1 - 2 \end{aligned}$$

with $q_1 = u_0 + u_1$, $q_2 = u_0 u_1$, $r_1 = u_2 + u_3$ and $r_2 = u_2 u_3$. Replacing u_i by u_i^5 gives ψ_2 and the values of the second column. The third and forth columns are realized by $\psi_3 = u_1 \psi_1$ and $\psi_4 = u_1 \psi_2$. By Lemma 2.8 and Remark 2.9, we have that the intersection form on $L'_{(1,1,2,2)}$ is

$$Q_{(1,1,2,2)} = \alpha_{P_{(1,1,2,2)}} M_{(1,1,2,2)}^t M_{\overline{(1,1,2,2)}}$$

where $M_{\overline{(1,1,2,2)}}$ is obtained from $M_{(1,1,2,2)}$ by interchanging the first two rows with the last two rows.

We compute $\alpha_{P_{(1,1,2,2)}}$ by evaluating $Q(\psi_1, \psi_1)$ in two different ways: firstly, by using Looijenga's formula in Proposition 2.5, and secondly, by Lemma 2.8. One finds

$$\alpha_{P_{(1,1,2,2)}} = \frac{1}{72}.$$

Direct computation gives the above matrix for $Q_{(1,1,2,2)}$ and its determinant. The existence of $L'_{(1,2,1,2)}$ and $L'_{(1,2,2,1)}$ with the analogous bases follows by symmetry. \square

Proposition 3.4. *Let*

$$T'_X = L'_{(1,1,3,3)} \oplus L'_{(1,1,2,2)} \oplus L'_{(1,2,1,2)} \oplus L'_{(1,2,2,1)}.$$

Then we have an equality of lattices $T'_X = T_X$. In particular, always $L'_\beta = L_\beta$.

Proof. It is clear that T'_X is a sublattice of T_X of finite index. Consider the basis of T'_X consisting of the union of the basis vectors of the L'_β constructed above. One can check that the reductions of the vectors of this basis modulo 2 and 3 are still linearly independent. Since the discriminant of T'_X is only divisible by primes 2 and 3, this proves that there is no sublattice of L which contains T'_X as a proper sublattice of finite index. In particular, $T_X = T'_X$. \square

This completes the proof of Theorem 1.2.

Remark 3.5. Proposition 3.4 implies also that the discriminant of the transcendental lattice T_X is $2^{40}3^{30}$ and consequently the discriminant of the Picard lattice is $-2^{40}3^{30}$ (the sign is negative since the signature of Q on $H^{1,1}$ is $(1, 85)$). We could not find this number in the literature.

Remark 3.6. We found the matrices $M_{(1,1,1,3)}$ and $M_{(1,1,2,2)}$ as follows: using Proposition 2.7 we find a \mathbb{Q} -basis of $L_{(1,1,1,3)}$ and of $L_{(1,1,2,2)} \oplus L_{(1,2,1,2)} \oplus L_{(1,2,2,1)}$. Clearing denominators, we find vectors in the lattice L that form a basis over \mathbb{Q} of $L_{(1,1,1,3)}$ and of $L_{(1,1,2,2)} \oplus L_{(1,2,1,2)} \oplus L_{(1,2,2,1)}$, respectively. These vectors generate lattices M and N , which are not saturated, however. For each prime p dividing the discriminant of M , for example, we reduce a set of basis vectors mod p in the ambient L , and if the reductions happen to become linearly dependent, we lift the dependency relation to \mathbb{Z} and find a vector divisible by p . Continuing in this way we arrive at a saturated sublattice M' spanning the same \mathbb{Q} -subspace as M . Using the LLL-algorithm we find vectors in M' with small coefficients. Among these we choose one with small length; evaluating this on $Z_{(1,1,1,3)}$ gives the first column of $M_{(1,1,1,3)}$. The remaining columns are obtained using the \mathfrak{S}_4 -symmetry and conjugation.

4. Rigidity and transcendental lattice decompositions

By a (projective) family of surfaces $\pi : \mathcal{S} \rightarrow B$, we mean a flat surjective morphism of schemes or analytic spaces, all of whose fibers are projective surfaces. All families considered will actually have smooth fibers, so that π is even a smooth map. If B is a scheme or analytic space, then by a “very general” point, we mean any point outside of a suitable countable union of proper algebraic or analytic subsets.

After a discussion about Theorem 1.2, V. Kulikov suggested the following question as a possible starting point in modifying his conjecture.

Question 4.1. *Let $\pi : \mathcal{S} \rightarrow B$ be a family of surfaces over an irreducible analytic space B such that a very general fiber \mathcal{S}_b has decomposable integral polarized Hodge structure on the transcendental lattice. Then must the image of the period map associated to the family necessarily be a single point?*

Even while we might wonder whether Question 4.1 has a negative answer, the main result of this section is the following.

Theorem 4.2. *A positive answer to Question 4.1 would imply that the very general cubic fourfold X is irrational.*

The integral polarized Hodge structures on the transcendental part T_X of the middle cohomology of a cubic fourfold X are uncountably many. Moreover, in [21] it is proved that, if such X were rational, their T_X must occur as *proper* summands of T_S , for S a smooth projective surface. Therefore, it suffices to prove the following.

Theorem 4.3. *A positive answer to Question 4.1 would imply that there are only countably many (weight 2) integral polarized Hodge structures arising as a proper direct summand of the integral polarized Hodge structure on T_S for S a surface.*

Remark that surfaces with $p_g \leq 1$ have indecomposable T_S , so one can restrict to surfaces with $p_g \geq 2$.

Since a given IPHS can have only countably many different decompositions, it suffices to prove that there are only countably many IPHS arising from the transcendental lattice of a surface which are decomposable.

Proof of Theorem 4.3. We divide the proof into a series of technical steps.

Step 1. There are countably many families $\mathcal{S}_i \rightarrow B_i$ of projective surfaces over irreducible base varieties B_i , such that every isomorphism class of a projective surface is represented by a fiber of some such family. One can take for example the universal families over the Hilbert schemes of two-dimensional subschemes of \mathbb{P}^n , $n \in \mathbb{N}$, since there are only countably many Hilbert polynomials.

Hence, it suffices to prove the following.

Lemma 4.4. *Let $\pi : \mathcal{S} \rightarrow B$ be a family of surfaces over an analytic space B . Then the integral polarized Hodge structures arising from $T_{\mathcal{S}_b}$, where \mathcal{S}_b , for $b \in B(\mathbb{C})$, is a fiber and the IPHS on $T_{\mathcal{S}_b}$ is decomposable, are countably many.*

Step 2. We first prove a linear algebra result.

Lemma 4.5. *Let $D = \mathrm{SO}(2p, q)/\mathrm{SO}(2p) \times \mathrm{U}(q)$ be the period domain classifying polarized weight two Hodge structures on $V := L \otimes \mathbb{C}$, where L is a fixed lattice. Let \check{D} be the compact dual of D and $\check{D} = \mathrm{IGrass}(p, V)$, so that $D \subset \check{D}$ is a (classically) open subset.*

Let $M \subset L$ be a sublattice of L . Let $Z \subset \check{D}$ be the subset such that M is contained in the F^1 -part of the Hodge filtration corresponding to the points in Z (this is a closed algebraic subset). Then for each $z \in Z \cap D$, we get an induced integral polarized Hodge structure on $M^\perp \subset L$, by putting $H_{M^\perp}^{p,q} := H^{p,q} \cap (M^\perp \otimes \mathbb{C})$. Let $Z_M \subset Z \cap D$ be the locus of points z where this induced integral polarized Hodge structure is decomposable. Then Z_M is an intersection with D of countably many locally closed algebraic subvarieties of \check{D} .

To prove this, note first that a given lattice, M^\perp in our case, can decompose in at most a countable number of ways into two nontrivial summands. We fix such a decomposition of M^\perp . We then have to prove that the points $z \in \check{D}$ such that the corresponding Hodge filtration decomposes in a way compatible with the fixed decomposition of M^\perp , form a finite union of locally closed algebraic subsets of \check{D} . This follows from the following.

Lemma 4.6. *Let $F = \mathrm{Fl}(k_1, \dots, k_r; V)$ be the flag variety of flags (F_1, \dots, F_r) of type (k_1, \dots, k_r) in a complex vector space V . Let $W \subset V$ be a fixed subspace with a given direct sum decomposition $W = W_1 \oplus W_2$ into subspaces $W_1 \subset W$ and $W_2 \subset W$. Then the subset Z_{W_1, W_2} of flags such that*

$$F_i \cap W = (F_i \cap W_1) \oplus (F_i \cap W_2), \quad \forall i,$$

is a finite union of locally closed algebraic subsets.

The proof of this Lemma is straightforward: it suffices to prove it in case $F = \mathrm{Fl}(k; V)$ is a Grassmannian. The set of k -subspaces of V intersecting W in a subspace of fixed dimension k' is a locally closed subset of the Grassmannian. We now fix $k' \leq k$ and also positive integers a and b with $a + b = k'$. We have a morphism

$$\mathrm{Gr}(a, W_1) \times \mathrm{Gr}(b, W_2) \rightarrow \mathrm{Gr}(k', W)$$

(direct sum of subspaces), whose image is a closed subset $G \subset \mathrm{Gr}(k', W)$. Over $\mathrm{Gr}(k', W)$ we have the tautological bundle \mathcal{E} whose fiber over a point is the given subspace of dimension k' of W . We consider the relative Grassmannian

$$\psi : \mathrm{Gr}(k - k', V \otimes \mathcal{O}/\mathcal{E}) \rightarrow \mathrm{Gr}(k', W).$$

Now $\mathrm{Gr}(k - k', V \otimes \mathcal{O}/\mathcal{E})$ is proper and has a natural morphism f to $\mathrm{Gr}(k, V)$. Its image is closed (but consists also of subspaces whose dimension of intersection with W is strictly larger than k' of course). In any case, $f(\psi^{-1}(G))$, intersected with the locus of subspaces L of dimension k in V whose intersection with W has dimension exactly k' , is exactly the locus of such subspaces L such that $L \cap W$ decomposes into a direct sum $L \cap W_1$ of dimension a and $L \cap W_2$ of dimension b . Since there are only finitely many choices for a and b , the result follows.

Step 3. Now we continue with the proof of Lemma 4.4. Look at the period map

$$p_{\mathcal{S}} : B \rightarrow D$$

of the family $\pi : \mathcal{S} \rightarrow B$. It is holomorphic by [16]. (We may assume without loss of generality that B is so small that there are no monodromy phenomena, i.e., that the local system $R^2\pi_*\mathbb{Z}_{\mathcal{S}}$ is trivialized; certainly B is covered by countably many such open subspaces). Since the Hodge filtration varies holomorphically, the locus where a given rational cohomology class of type $2p$ remains of type (p, p) is a complex analytic subspace of B (even more is true, see [10], but we do not need this). It follows that the Picard rank and algebraic lattice $A_{\mathcal{S}_b}$ are constant (equal to A) for b outside a countable union of analytic subsets of B , say $\{B_i\}_{i \in \mathbb{N}}$. Let $U = B - \bigcup_i B_i$. Look at $p^{-1}(Z_A) \subset B$, where $Z_A \subset D$ is the subset of the respective period domain from Lemma 4.5. The restriction of \mathcal{S} to each irreducible component of $p^{-1}(Z_A)$ which meets U nontrivially fulfills the hypotheses of Question 4.1. Hence, each such component gives only one isomorphism class of an integral polarized Hodge structure, assuming the a positive answer to Question 4.1. Thus the isomorphism classes of integral polarized Hodge structures coming from decomposable $T_{\mathcal{S}}$ of fibers \mathcal{S}_b such that $b \in U$ form a countable set.

Step 4. We repeat the argument of Step 3 for each of the countably many families $\mathcal{S}|_{B_i} \rightarrow B_i$. We get new analytic subsets $\{B_{ij}\}$ in each B_i in this way (countably many) and repeat the argument for the $\mathcal{S}|_{B_{ij}} \rightarrow B_{ij}$, and so forth. Each time, the dimension of the base decreases, and after finitely many steps, we reach zero-dimensional bases (countably many). This concludes the proof of Lemma 4.4, and with it, the proof of Theorem 4.3. \square

Remark 4.7. We will argue that Question 4.1 has a positive answer for families of surfaces with maximal Picard rank $\pi : \mathcal{S} \rightarrow B$. Examples of such families were given by Remke Kloosterman in [20] in the form of elliptic surfaces over \mathbb{P}^1 with $p_g > 1$ and constant j -invariant (these families are not isotrivial, i.e., the surfaces have moduli). This reference was kindly pointed out to us by Matthias Schütt.

To argue, we can assume that B is a small, simply connected open subset of \mathbb{C}^n . If a very general fiber is a surface of maximal Picard rank, then all fibers will have maximal Picard rank. The result then follows from Proposition 4.4 in [20], or rather its proof: if $\Theta_{\mathcal{S}_b}$ the tangent bundle of \mathcal{S}_b and p the period map, then

$$B \rightarrow H^1(\mathcal{S}_b, \Theta_{\mathcal{S}_b}) \xrightarrow{d_p} \mathrm{Hom}(H^{2,0}(\mathcal{S}_b), H^{1,1}(\mathcal{S}_b)) \oplus \mathrm{Hom}(H^{1,1}(\mathcal{S}_b), H^{0,2}(\mathcal{S}_b))$$

factors over $\mathrm{Hom}(T^{2,0}(\mathcal{S}_b), T^{1,1}(\mathcal{S}_b)) \oplus \mathrm{Hom}(T^{1,1}(\mathcal{S}_b), T^{0,2}(\mathcal{S}_b))$, which is zero because $T^{1,1}(\mathcal{S}_b) = 0$. Therefore, p is constant.

Note that the sextic Fermat surface is the only Fermat surface of maximal Picard rank; for other Fermat surfaces up to degree 8 we have not been able to decompose the IPHS on the transcendental lattices. One might wonder, therefore, whether decomposability of T_S with its IPHS forces S to have maximal Picard rank. This would certainly imply a positive answer to Question 4.1 and with it irrationality of very general cubic fourfolds.

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