Math. Res. Lett. **21** (2014), *no.* 1, 127–131

A REMARK ON THE MODIFIED ZAKHAROV–KUZNETSOV EQUATION IN THREE SPACE DIMENSIONS

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Abstract. The Cauchy Problem for the modified Zakharov–Kuznetsov equation in three space dimensions is shown to be locally well-posed in $H^s(\mathbb{R}^3)$ for $s > \frac{1}{2}$. Combined with the conservation of mass and energy this result implies global well-posedness for small data in $H^1(\mathbb{R}^3)$.

1. Introduction and main result

The Cauchy Problem for the modified Zakharov–Kuznetsov (mZK) equation in two space dimensions

$$u_t + \partial_x^3 u + \partial_x \partial_y^2 u + \partial_x(u^3) = 0, \quad u(0,x,y) = u_0(x,y), \quad (x,y) \in \mathbb{R}^2 \tag{1.1}$$

has been studied extensively in recent years. Local well-posedness in $H^1(\mathbb{R}^2)$ was obtained in 2003 by Biagioni and Linares; see [1]. Combined with the conservation of mass and energy their local result implies global well-posedness, provided the data are sufficiently small in $L^2(\mathbb{R}^2)$. The local result was generalized to data in $H^s(\mathbb{R}^2)$, $s > \frac{3}{4}$ by Linares and Pastor in [6]; the same authors showed global well-posedness in $H^s(\mathbb{R}^2)$, $s > \frac{53}{63}$, under an additional smallness assumption on the L^2-norm of the data [7]. Further progress on the local problem was reached by Ribaud and Vento in [10], who established well-posedness in $H^s(\mathbb{R}^2)$ for $s > \frac{1}{4}$. The critical space obtained by scaling considerations for mZK in two dimensions is $L^2(\mathbb{R}^2)$. So the local well-posedness of (1.1) in $H^s(\mathbb{R}^2)$ with $0 \leq s \leq \frac{1}{4}$ is still an open problem.

In contrast to the two-dimensional case, no results concerning the Cauchy Problem for the mZK-equation in three space dimensions are known, yet. In this short note, we shall apply mostly well-known linear and a new bilinear estimate for solutions of the corresponding linear equation to establish the following result.

Theorem 1. *The Cauchy Problem for the mZK equation*

$$u_t + \partial_x \Delta u + \partial_x(u^3) = 0, \quad u(0,x,y) = u_0(x,y) \tag{1.2}$$

with $x \in \mathbb{R}$ and $y \in \mathbb{R}^2$ is locally well-posed for data $u_0 \in H^s(\mathbb{R}^3)$, provided that $s > \frac{1}{2}$.

Received by the editors August 9, 2013.

2000 *Mathematics Subject Classification.* Primary: 35Q53. Secondary: 37K40.

Key words and phrases. modified Zakharov–Kuznetsov equation — local and global well-posedness — Fourier restriction norm method.

We remark that from the scaling point of view this theorem covers the whole subcritical range. Combining the above local result with the conservation of mass and energy as in [2, p. 3], we obtain global well-posedness for small data in $H^1(\mathbb{R}^3)$.

Corollary 1. *Let $u_0 \in H^s(\mathbb{R}^3)$ with $s \geq 1$. Then there exists $\varepsilon > 0$, such that for $\|u_0\|_{H^1} < \varepsilon$, the local solution of the Cauchy Problem* (1.2) *guaranteed by Theorem 1 extends globally in time.*

2. Estimates for free solutions of the linear equation

Let $U_\phi(t)u_0$ denote the solution of the Cauchy Problem for the linear equation

$$u_t + \partial_x \Delta u = 0, \quad u(0,x,y) = u_0(x,y), \tag{2.1}$$

where $t \in \mathbb{R}$ is the time variable, $x \in \mathbb{R}$, $y \in \mathbb{R}^2$ are the space variables and $\Delta = \partial_x^2 + \Delta_y$, $\Delta_y = \partial_{y_1}^2 + \partial_{y_2}^2$. The dual variables of $(t,x,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$ are denoted by $(\tau, \xi, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$ and the phase function is given here by $\phi(\xi,\eta) = \xi(\xi^2 + |\eta|^2)$. The Fourier transform in space and time is denoted by $\mathcal{F}$, and fractional derivatives with respect to x, y and t are specified by subscripts, e.g., $D_x^s := \mathcal{F}^{-1}|\xi|^s\mathcal{F}$, $D_t^b := \mathcal{F}^{-1}|\tau|^b\mathcal{F}$ and $|\nabla_{xy}|^s := \mathcal{F}^{-1}|(\xi,\eta)|^s\mathcal{F}$. The corresponding Bessel potential operators are denoted by Japanese brackets, e.g., $\langle D_y\rangle^s := \mathcal{F}^{-1}(1+|\eta|^2)^{\frac{s}{2}}\mathcal{F}$.

Concerning the solutions of (2.1) Linares and Saut [8, Proposition 3.1] obtained the Strichartz type estimate,

$$\|D_x^{\theta\varepsilon/2} U_\phi u_0\|_{L_t^p L_{xy}^q} \lesssim \|u_0\|_{L^2_{xy}}, \tag{2.2}$$

provided $0 < \varepsilon < 1$, $0 < \theta < (1 + \frac{\varepsilon}{3})^{-1}$, $\frac{2}{p} = \theta(1 + \frac{\varepsilon}{3})$ and $\frac{1}{q} = \frac{1-\theta}{2}$. The L^4-estimate corresponding to $\theta = \frac{1}{2}$ and $\varepsilon = 0$ is excluded in [8], but nonetheless true. In fact, modifying the proof of Theorem 2 in [4] appropriately we obtain the bilinear estimate

$$\|U_\phi u_0 U_\phi v_0\|_{L^2_{txy}} \lesssim \|D_x^{-\frac{1}{2}} u_0\|_{L^2_{xy}} \|\langle D_x\rangle^s v_0\|_{L^2_{xy}}, \tag{2.3}$$

provided $s > \frac{1}{2}$. Especially for $u_0 = v_0$, we get with $P_{\Delta_k} = \mathcal{F}^{-1}\chi_{\{\xi \sim 2^k\}}\mathcal{F}$ that

$$\|P_{\Delta_1} U_\phi u_0\|_{L^4_{txy}} \lesssim \|P_{\Delta_1} u_0\|_{L^2_{xy}}.$$

Now let u_k be defined by $\mathcal{F}u_k(\xi,\eta) = \mathcal{F}u_0(2^k\xi, 2^k\eta)$. Then

$$P_{\Delta_k} U_\phi(t) u_0(x,y) = 2^{3k} P_{\Delta_1} U_\phi(2^{3k}t) u_k(2^k x, 2^k y)$$

and hence

$$\begin{aligned}\|P_{\Delta_k} U_\phi u_0\|_{L^4_{txy}} &= 2^{3k} \|P_{\Delta_1} U_\phi(2^{3k}\cdot) u_k(2^k\cdot, 2^k\cdot)\|_{L^4_{txy}} \\ &= 2^{\frac{3k}{2}} \|P_{\Delta_1} U_\phi u_k\|_{L^4_{txy}} \lesssim 2^{\frac{3k}{2}} \|P_{\Delta_1} u_k\|_{L^2_{xy}} = \|P_{\Delta_k} u_0\|_{L^2_{xy}}.\end{aligned}$$

Applying the Littlewood–Paley theorem we see that

$$\|U_\phi u_0\|_{L^4_{txy}} \lesssim \|u_0\|_{L^2_{xy}}. \tag{2.4}$$

Apart from the Strichartz type estimates and their bilinear refinement, we can rely on a local smoothing effect of Kato type in order to deal with the derivative in the nonlinearity. As was shown by Ribaud and Vento, in the case of the linear ZK equation (2.1) it reads

$$\|\nabla_{xy} U_\phi u_0\|_{L_x^\infty L^2_{yt}} \lesssim \|u_0\|_{L^2_{xy}}, \tag{2.5}$$

see Proposition 3.1 of [9]. To complement the use of the local smoothing effect, we combine a Sobolev embedding in the time variable with the L^4-Strichartz estimate in order to obtain the following maximal function inequality. The argument given below was taken from [5, Proof of Theorem 2.4].

$$\begin{aligned} \|U_\phi u_0\|_{L^4_{xy}L^\infty_t} &\lesssim \|\langle D_t\rangle^{\frac{1}{4}+}U_\phi u_0\|_{L^4_{xyt}} \\ &= \|\langle \partial_x \Delta\rangle^{\frac{1}{4}+}U_\phi u_0\|_{L^4_{xyt}} \lesssim \|u_0\|_{H^s}, \quad s > \frac{3}{4}. \end{aligned} \tag{2.6}$$

3. Proof of the local result for modified ZK

Now let $X_{s,b}$ denote the Bourgain space associated with the phase function $\phi(\xi,\eta) = \xi(\xi^2+|\eta|^2)$, more precisely let

$$X_{s,b} = \{u \in \mathcal{S}'(\mathbb{R}^4) : \|u\|_{X_{s,b}} < \infty\},$$

where, with $(\xi,\eta,\tau) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}$,

$$\|u\|_{X_{s,b}} = \|\langle(\xi,\eta)\rangle^s\langle\tau-\phi(\xi,\eta)\widehat{u}\rangle^b\|_{L^2_{\xi\eta\tau}}.$$

Then by the transfer principle the estimates for free solutions discussed in Section 2 imply corresponding estimates in $X_{s,b}$ — norms for $b > \frac{1}{2}$. For example, we have

$$\|u\|_{L^4_{txy}} \lesssim \|u\|_{X_{0,b}}, \tag{3.1}$$

$$\|\nabla_{xy}u\|_{L^\infty_x L^2_{yt}} \lesssim \|u\|_{X_{0,b}}, \tag{3.2}$$

and, for $s > \frac{3}{4}$,

$$\|u\|_{L^4_{xy}L^\infty_t} \lesssim \|u\|_{X_{s,b}}. \tag{3.3}$$

The bilinear estimate (2.3) is converted into

$$\|uv\|_{L^2_{xyt}} \lesssim \|D_x^{-\frac{1}{2}}u\|_{X_{0,b}}\|\langle D_x\rangle^s v\|_{X_{0,b}}, \tag{3.4}$$

where $s > \frac{1}{2}$ and again $b > \frac{1}{2}$ are assumed. In the sequel, we proceed similar as in [3, Proof of Theorem 2] and combine these estimates with duality and interpolation arguments to obtain the following Proposition, which in turn implies Theorem 1.

Proposition 1. *For any $s > \frac{1}{2}$ there exists a $b' > -\frac{1}{2}$, such that for all $b > \frac{1}{2}$ the estimate*

$$\|\partial_x(uvw)\|_{X_{s,b'}} \lesssim \|u\|_{X_{s,b}}\|v\|_{X_{s,b}}\|w\|_{X_{s,b}}$$

holds true.

Proof. Dualizing the bilinear estimate (3.4), we obtain for $s, b > \frac{1}{2}$

$$\|D_x^{\frac{1}{2}}(uv)\|_{X_{0,-b}} \lesssim \|u\|_{L^2_{xyt}}\|\langle D_x\rangle^s v\|_{X_{0,b}}. \tag{3.5}$$

Here and in (3.4), we may clearly replace the $\langle D_x\rangle^s$ on the right by $\langle\nabla_{xy}\rangle^s$. Now pointwise estimates on Fourier side show that for u, v and w with nonnegative Fourier transforms we have

$$\mathcal{F}D_x^{\frac{1}{2}}(uvw) \lesssim \mathcal{F}(D_x^{\frac{1}{2}}u)vw + \mathcal{F}u(D_x^{\frac{1}{2}}v)w + \mathcal{F}uvD_x^{\frac{1}{2}}w,$$

and that a similar inequality holds with $\langle\nabla_{xy}\rangle^s$ instead of $D_x^{\frac{1}{2}}$. This gives

$$\|\partial_x(uvw)\|_{X_{s,-b}} \lesssim \|D_x^{\frac{1}{2}}((D_x^{\frac{1}{2}}\langle\nabla_{xy}\rangle^s u)vw)\|_{X_{0,-b}} + \|D_x^{\frac{1}{2}}((D_x^{\frac{1}{2}}u)(\langle\nabla_{xy}\rangle^s v)w)\|_{X_{0,-b}} + \cdots,$$

where the dots indicate those terms, which arise by permuting u, v and w. For the first contribution we use first (3.5) and then (3.4) to obtain the upper bound

$$\|(D_x^{\frac{1}{2}}\langle\nabla_{xy}\rangle^s u)v\|_{L^2_{xyt}}\|\langle D_x\rangle^s w\|_{X_{0,b}} \lesssim \|u\|_{X_{s,b}}\|v\|_{X_{s,b}}\|w\|_{X_{s,b}}.$$

For the second contribution we start again with (3.5) and continue with Hölder's inequality and two applications of (3.1) to see that it is bounded by

$$\|(D_x^{\frac{1}{2}}u)(\langle\nabla_{xy}\rangle^s v)\|_{L^2_{xyt}}\|\langle D_x\rangle^s w\|_{X_{0,b}} \lesssim \|u\|_{X_{s,b}}\|v\|_{X_{s,b}}\|w\|_{X_{s,b}}.$$

Thus we have achieved

$$\|\partial_x(uvw)\|_{X_{s,-b}} \lesssim \|u\|_{X_{s,b}}\|v\|_{X_{s,b}}\|w\|_{X_{s,b}}, \tag{3.6}$$

where $s, b > \frac{1}{2}$. It remains to replace the $-b < -\frac{1}{2}$ on the left by a $b' > -\frac{1}{2}$. For that purpose we estimate for $\sigma > \frac{5}{4}$

$$\begin{aligned}\|\langle\nabla_{xy}\rangle^\sigma\partial_x(uvw)\|_{L^2_{xyt}} &\lesssim \|((\langle\nabla_{xy}\rangle^\sigma\partial_x u)vw\|_{L^2_{xyt}} + \|(\partial_x u)((\langle\nabla_{xy}\rangle^\sigma v)w\|_{L^2_{xyt}} + \cdots \\ &\lesssim \|\langle\nabla_{xy}\rangle^\sigma\partial_x u\|_{L^\infty_x L^2_{yt}}\|v\|_{L^4_x L^\infty_{yt}}\|w\|_{L^4_x L^\infty_{yt}} \\ &+ \|(\partial_x u)((\langle\nabla_{xy}\rangle^\sigma v)\|_{L^4_x L^2_{yt}}\|w\|_{L^4_x L^\infty_{yt}} + \cdots\end{aligned}$$

For the first contribution we get the upper bound $\|u\|_{X_{\sigma,b}}\|v\|_{X_{\sigma,b}}\|w\|_{X_{\sigma,b}}$ by Kato smoothing for the first and by the maximal function estimate for the second and third factor. The second contribution is estimated by

$$\|\partial_x u\|_{L^\infty_x L^4_{yt}}\|\langle\nabla_{xy}\rangle^\sigma v\|_{L^4_{xyt}}\|w\|_{L^4_x L^\infty_{yt}} \lesssim \|u\|_{X_{\sigma,b}}\|v\|_{X_{\sigma,b}}\|w\|_{X_{\sigma,b}},$$

where we have used a Sobolev embedding in x and the L^4-Strichartz estimate for the first, the same Strichartz estimate for the second and the maximal function estimate for the third factor. This shows that for $b > \frac{1}{2}$ and $\sigma > \frac{5}{4}$

$$\|\langle\nabla_{xy}\rangle^\sigma\partial_x(uvw)\|_{L^2_{xyt}} \lesssim \|u\|_{X_{\sigma,b}}\|v\|_{X_{\sigma,b}}\|w\|_{X_{\sigma,b}}. \tag{3.7}$$

Finally interpolation among (3.6) and (3.7) gives the claimed estimate. □

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