Math. Res. Lett. **21** (2014), *no.* 1, 133–139

ON THE QUOTIENT OF $\mathbb{C}^4$ BY A FINITE PRIMITIVE GROUP OF TYPE (I)

Ilya Karzhemanov

Abstract. We study rationality problem for the quotient of $\mathbb{C}^4$ by a finite primitive group G of Type (I). We prove that this quotient is a rational variety for any such G.

1. Introduction

Given a complex affine space $\mathbb{C}^n = \mathrm{Spec}(\mathbb{C}[x_1,\ldots,x_n])$ and a finite group G acting linearly on $\mathbb{C}^n$, one of the fundamental questions to ask is whether the field of G-invariant rational functions on $\mathbb{C}^n$ is also a purely transcendental extension of $\mathbb{C}$, or, in other words, whether variety $\mathbb{C}^n/G$ is rational (see [4] (and references therein) for an extensive overview of the current state of the problem). By a simple argument (see [4, Proposition 1.2]), one can show that $\mathbb{C}^n/G$ is birationally isomorphic to $(\mathbb{P}(\mathbb{C}^n)/G) \times \mathbb{P}^1$, and hence $n = 4$ is the first non-trivial issue, since the Lüroth problem has a positive solution for $n \leqslant 3$. The case of $n = 4$ has been treated in detail in [4]. However, for some of the groups G (non-)rationality of $\mathbb{C}^4/G$ was not established.

Namely, let $\mathbb{O}, \mathbb{I} \subset SL_2(\mathbb{C})$ be the octahedron and icosahedron subgroups, respectively. Identify $U_0 := \mathbb{C}^4$ with the space of (2×2)-matrices $A := \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, $X_i \in \mathbb{C}$, and consider the action of the group $G := \mathbb{O} \times \mathbb{I}$ on U_0 such that $\mathbb{O}$ and $\mathbb{I}$ act by multiplying A from the left and right, respectively. Furthermore, by the above argument in order to establish rationality of U_0/G, one may assume that $G := (\mathbb{O} \times \mathbb{I}) \cdot \mathbb{C}^*$ for the standard diagonal action of $\mathbb{C}^*$ on U_0. Then for such group action, we prove the following:

Theorem 1.1. *The 3-fold U_0/G is rational.*

Theorem 1.1 settles the remaining case in [4] of quotients of $\mathbb{P}^3$ (or, equivalently, $\mathbb{C}^4$) by *finite primitive groups of Type (I)* (see [4, Section 2] for the description of these).

Let us outline the proof of Theorem 1.1. Recall that in [4], after taking the $\mathbb{C}^*$-quotient of U_0 and passing to the projectivized G-action on $\mathbb{P}^3$, with G now equal $\mathbb{O}\times\mathbb{I}$, one can notice that $\mathbb{P}^3/G$ is birationally isomorphic to $SL_2(\mathbb{C})/G$ for the induced G-action on $SL_2(\mathbb{C}) \subset U_0$. Further, compactifying $SL_2(\mathbb{C})$ by a smooth Fano 3-fold W with either $\mathbb{O}$- or $\mathbb{I}$-action, one might try to prove that the corresponding quotient of W is rational by finding an equivariant birational map of W onto a product of positive-dimensional varieties (see [4, Section 2], where this idea worked perfectly well for all finite primitive groups of Type (I), except for the given G).

Received by the editors June 6, 2012.

Our approach is more direct (and simpler in a sense). Namely, let the group $\mathbb{Z}/2\mathbb{Z}$ act on U_0 by multiplying every X_i by -1, so that the G-action descends to $U_0/(\mathbb{Z}/2\mathbb{Z})$. A natural generalization of the construction of $\mathbb{P}^1$ leads to a projective compactification V' of $U_0/(\mathbb{Z}/2\mathbb{Z})$ (see Section 2 below).[1] This V' turns out to be a Fano 4-fold with isolated terminal singularities, of Picard number 1 and Fano index 4, i.e., V' is a quadratic cone in $\mathbb{P}^5$ by a result of T. Fujita (see Lemma 2.15). Furthermore, the G-action on $U_0/(\mathbb{Z}/2\mathbb{Z})$ extends to a regular action on V', and $V' \subset \mathbb{P}^5$ happens to have three linearly independent G-invariant hyperplane sections (see Lemma 3.3). Then, considering the corresponding G-equivariant linear projection $V \dashrightarrow \mathbb{P}^2$, we split the threefold V'/G birationally into a product of positive-dimensional varieties, thus proving rationality of V'/G (see Lemma 3.4). It is now easy to see that U_0/G is also rational (see Lemma 3.5).

Remark 1.2. Instead of $\mathbb{O}\times\mathbb{I}$ one may take any other finite primitive group G of Type (I) and prove that the corresponding quotient $\mathbb{C}^4/G$ is rational, repeating literally the arguments in Sections 2 and 3 below. This gives another proof of Theorem 2.1 in [4].

Notation. We use standard notions and facts from [3]. Also throughout the paper we use the following notation:

(†) Given two varieties X and Y, $X \approx Y$ denotes birational equivalence between them. For an algebraic group G acting regularly on both X and Y, we write $X \approx_G Y$ if there exists a G-equivariant birational map $X \dashrightarrow Y$.

2. One explicit compactification

2.1. Take another copy U_1 of $\mathbb{C}^4$. Identify U_1 with the space of (2×2)-matrices, as U_0 above. Let $\varphi_1 : U_0 \dashrightarrow U_1$ be birational map induced by the morphism $GL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C})$ which sends every invertible matrix $A \in U_0$ to $A^{-1} \in U_1$. Set $X_i^{(1)} := \varphi_1^{-1*}(X_i)$, $1 \leqslant i \leqslant 4$. These extend to affine coordinates on U_1. Put also $\Delta_0 := \det A$ and $\Delta_1 := \varphi_1^{-1*}(\Delta_0)$.

Further, let $l_{\alpha,\beta}$ be the linear automorphism of U_0 which permutes X_α and X_β in A with $\alpha+\beta \neq 5$. Take another copy $U_{\alpha,\beta}$ of $\mathbb{C}^4$, as U_0 and U_1 above, and consider birational map $\varphi_{\alpha,\beta} := \varphi_1 \circ l_{\alpha,\beta} : U_0 \dashrightarrow U_{\alpha,\beta}$. Set $X_i^{(\alpha,\beta)} := \varphi_{\alpha,\beta}^{-1*}(X_i)$. These extend to affine coordinates on $U_{\alpha,\beta}$. Put also $\Delta_{\alpha,\beta} := \varphi_{\alpha,\beta}^{-1*}(\Delta_0)$.

Now glue U_0, U_1, $U_{\alpha,\beta}$ together via the maps φ_1, $\varphi_{\alpha,\beta}$ for various α, β. We get a smooth complex 4-fold V so that U_0, U_1, $U_{\alpha,\beta}$ are analytic domains covering V. Note that $\Delta_1 = \Delta_0^{-1}$ on $U_0 \cap U_1$ and $\Delta_{\alpha,\beta} = l_{\alpha,\beta}^*(\Delta_0)$ on $U_0 \cap U_{\alpha,\beta}$.

Lemma 2.2. *$V = G(2,4)$, the Grassmanian of 2-planes in $\mathbb{C}^4$.*

Proof. Evident (by definition of the complex structure on $G(2,4)$). □

2.3. Let us now replace each of U_i and $U_{\alpha,\beta}$ in **2.1** by $\mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ acts via $X_i \mapsto -X_i$, $1 \leqslant i \leqslant 4$. Note that the gluing maps φ_1 and $\varphi_{\alpha,\beta}$ are $(\mathbb{Z}/2\mathbb{Z})$-equivariant, hence we can glue the six copies of $\mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z})$ together via φ_1, $\varphi_{\alpha,\beta}$ as above. We get an algebraic space V' (with $\{U_0, U_1, U_{\alpha,\beta}\}_{\alpha,\beta}$ being an open cover of V' in the orbifold topology).

[1]By "V' compactifies $U_0/(\mathbb{Z}/2\mathbb{Z})$" we mean that $\mathbb{C}(V') = \mathbb{C}(U_0/(\mathbb{Z}/2\mathbb{Z}))$ for the fields of meromorphic functions.

Remark 2.4. Note that the gluing maps $\varphi_1, \varphi_{1,2}, \ldots$ on V' are rather *algebraic* (see [1, Chapter 1]) than analytic. Indeed, $\varphi_1, \varphi_{1,2}$, etc., when lifted to the universal covers of the charts $U_0 := \mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z}), \ldots$, are only $\mathbb{Z}/2\mathbb{Z}$-equivariant, but not $\mathbb{Z}/2\mathbb{Z}$-invariant. It is easy to see, however, that the complex (scheme) structure on V' is provided by the charts $U_0 \cup U_1, U_0 \cup U_{1,2}, \ldots$ (but *not* by $\{U_0, U_1, U_{\alpha,\beta}\}_{\alpha,\beta}$), glued from $U_0, U_1, U_{1,2}$, etc., via $\varphi_1, \varphi_{1,2}, \ldots$.

Lemma 2.5. *V' is compact.*

Proof. Let $\Delta \subset \mathbb{C}$ be a small disk around 0. We have to prove that any (analytic) family of points $O_t \in V'$, parameterized by $\Delta \setminus \{0\} \ni t$, extends to a family at $t = 0$. This follows from Lemma 2.2 and the fact that the gluing maps $\varphi_1, \varphi_{1,2}, \ldots$ are $\mathbb{Z}/2\mathbb{Z}$-equivariant. □

The next lemma is straightforward from the construction of V' (cf. Remark 2.4):

Lemma 2.6. *$\mathbb{C}(V') = \mathbb{C}(U_0)$.*

Remark 2.7. One can easily see that the quotient map $\mathbb{C}^4 \longrightarrow U_0 := \mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z})$ does not induce a *regular* map $V = G(2,4) \longrightarrow V'$. Thus, in view of Lemma 2.6, V' is only *birationally* a quotient $V/(\mathbb{Z}/2\mathbb{Z})$.

2.8. Let D_0 be a divisor on V' with local equations $\Delta_0 = 0$ on U_0 and $\Delta_{\alpha,\beta} = 0$ on $U_{\alpha,\beta}$ for all α, β (cf. **2.1**). Note that the defining equations of D, when lifted to the universal covers of $U_0, U_1, \ldots$, are $(\mathbb{Z}/2\mathbb{Z})$-*invariant* (cf. Remark 2.4). Then the sheaf property (see [1, Chapter 2]) implies that D_0 is a Cartier divisor on V'. Let $\mathcal{L} := \mathcal{O}_{V'}(D_0)$ be the corresponding line bundle.

Lemma 2.9. *D_0 is irreducible and $\mathcal{L}$ carries a Hermitian metric $|\cdot|$ such that $1 = |\Delta_0| = |\Delta_{\alpha,\beta}|$ on $U_0 \cap U_1$ and $U_0 \cap U_{\alpha,\beta}$ for all α, β.*

Proof. Evident. □

Proposition 2.10. *D_0 is ample.*

Proof. Let $\theta \in H^0(V', \mathcal{L})$ be the global section such that $(\theta)_0 = D_0$. Put $\theta_0 := \theta\big|_{U_0}$, $\theta_1 := \theta\big|_{U_1}$, $\theta_{\alpha,\beta} := \theta\big|_{U_{\alpha,\beta}}$.

Restrict $\mathcal{L}$ to U_0 and define a Hermitian metric h_0 on $\mathcal{L}\big|_{U_0}$ as follows:

$$h_0 := (1 + |X_1|^2)|\theta_0|.$$

Then on $U_0 \cap U_1$, we have

$$|\theta_1| = |\theta_0| \frac{1}{|\Delta_0|} = |\theta_0|$$

and hence

$$h_0 = |\theta_1| + \frac{|X_1|^2}{|\Delta_0|^2}|\theta_1| = \left(1 + |X_1^{(1)}|^2\right)|\theta_1|.$$

This extends h_0 to a metric on $\mathcal{L}$ over $U_0 \cup U_1$. Repeating the same construction, with U_1 replaced by $U_{\alpha,\beta}$, we obtain a global metric on $\mathcal{L}$, equal

$$(1 + |X_1^{(\alpha,\beta)}|^2)|\theta_{\alpha,\beta}|$$

on each $U_{\alpha,\beta}$. Moreover, starting with the metric

$$h := |\theta_0| \prod_{i=1}^{4} (1 + |X_i|^2)^{1/4}$$

on $\mathcal{L}$ over U_0, the same argument yields to a metric[2] on $\mathcal{L}$ over X which extends h. Let us again denote this new metric by h and consider the $(1,1)$-form $\Theta := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h \in c_1(\mathcal{L})$. Then from the Nakai–Moishezon criterion (see [2, Theorem 5.1]), we get the following:

Lemma 2.11. *If $\sqrt{-1}\Theta > 0$, then D_0 is ample.*

Further, the condition $\sqrt{-1}\Theta > 0$ is local, so we restrict ourselves to the chart U_0 (the argument is the same for U_1 and $U_{\alpha,\beta}$), and on U_0 we have

$$\sqrt{-1}\Theta = \frac{1}{8\pi}\sum_{i=1}^{4} \frac{dX_i \wedge d\bar{X}_i}{(1 + |X_i|^2)^2} > 0.$$

□

2.12. There is a unique (prime) Cartier divisor $D_\infty \sim D_0$ on V' with equation $\Delta_1 = 0$ on U_1. Indeed, one can define D_∞ by taking the closure of the locus $(\Delta_1 = 0) \subset U_1$ in V', and $D_\infty \sim D_0$ because of the rational map $V' \dashrightarrow \mathbb{P}^1$ which extends the map $A \mapsto \det A$ on U_0. Equivalently, one can notice that the divisors D_∞ and $D_0 + (f)$ determine the same valuations on the function field $\mathbb{C}(V')$, where f is a rational function on V', equal Δ_0^{-1} on U_0 (cf. Remark 2.14 below). Note also that $D_0 \neq D_\infty$ (cf. the similar construction of $\mathbb{P}^1$ and of the divisors $0, \infty \in \mathbb{P}^1$).

Lemma 2.13. *$K_{V'} \sim -4D_0$.*

Proof. Let us start with the form $\omega := dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4$ on U_0. We have

$$\hat{X}_j := d\left(\frac{X_j}{X_1X_4 - X_2X_3}\right) = \frac{dX_j}{X_1X_4 - X_2X_3} - \frac{X_j d\big(X_1X_4 - X_2X_3\big)}{\big(X_1X_4 - X_2X_3\big)^2}$$

for all j, and it is easy to see that

$$\begin{aligned}\hat{X}_1 \wedge \hat{X}_2 \wedge \hat{X}_3 \wedge \hat{X}_4 &= \frac{dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4}{\big(X_1X_4 - X_2X_3\big)^4} \\ &\quad - \frac{\sum_{1\leqslant j\leqslant 4} X_j d\big(X_1X_4 - X_2X_3\big) dX_1 \wedge \cdots \wedge d\hat{X}_j \wedge \cdots \wedge dX_4}{\big(X_1X_4 - X_2X_3\big)^5} \\ &= \frac{dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4}{\big(X_1X_4 - X_2X_3\big)^4}.\end{aligned}$$

Then we get

$$dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4 = \frac{1}{\Delta_1^4}\, dX_1^{(1)} \wedge dX_2^{(1)} \wedge dX_3^{(1)} \wedge dX_4^{(1)}$$

[2]Equal $|\theta_{\alpha,\beta}| \prod_{i=1}^{4}(1 + |X_i^{(\alpha,\beta)}|^2)^{1/4}$ on $U_{\alpha,\beta}$.

on $U_0 \cap U_1$. This extends ω to a meromorphic form on $U_0 \cup U_1$. Note that $K_{V'} = -4D_\infty \sim -4D_0$ on $U_0 \cup U_1$.

Repeating the same construction, with U_1 replaced by $U_{\alpha,\beta}$, we obtain a global meromorphic section of the line bundle $\mathcal{O}_{V'}(K_{V'})$, equal

$$\frac{1}{l^*_{\alpha,\beta}(\Delta_{\alpha,\beta})^4}\, dX_1^{(\alpha,\beta)} \wedge dX_2^{(\alpha,\beta)} \wedge dX_3^{(\alpha,\beta)} \wedge dX_4^{(\alpha,\beta)}$$

on $U_0 \cap U_{\alpha,\beta}$ for all α, β. Hence $K_{V'} = -4D_\infty \sim -4D_0$ on V'. □

Remark 2.14. It follows from the proof of Lemma 2.13 that the equation of the divisor D_∞ on $U_{\alpha,\beta}$ is $l^*_{\alpha,\beta}(\Delta_{\alpha,\beta}) = 0$ for all α, β.

Lemma 2.15. *V' is a quadratic cone with a unique singular point.*

Proof. Firstly, V' has only isolated terminal singularities (by definition of the latter and construction of V'). Now the assertion follows from Lemma 2.13, Proposition 2.10 and [3, Theorem 3.1.14]. □

3. Proof of Theorem 1.1

3.1. Consider V' as in Section 2. Let us show that the G-action extends from $U_0 = \mathbb{C}^4/(\mathbb{Z}/2\mathbb{Z})$ to a regular action on V' (note G is obviously defined on U_0).

By construction of V', every $g \in G$ determines a birational automorphism $g : V' \dashrightarrow V'$, regular and bijective on $U_0 \cup U_1$. Furthermore, we have $V' \setminus (U_0 \cup U_1) \subseteq D_0 \cup D_\infty$, since

$$U_0 \cup U_1 \supseteq V' \setminus (D_0 \cup D_\infty) = U_0 \cap U_1 \cap \bigcap_{\alpha,\beta} U_{\alpha,\beta}$$

(cf. **2.1** and the equations of $D_0,\ D_\infty$). Then, since $g(D \cap U_0) = D \cap U_0$, $g(D_\infty \cap U_1) = D_\infty \cap U_1$ and $D_0,\ D_\infty$ are irreducible, we obtain that g is an isomorphism in codimension 2 on V', and hence $g_*(D) = D$, $g_*(D_\infty) = D_\infty$ in $\mathrm{Pic}(V')$. This implies that g is induced by an automorphism of $\mathbb{P}^5 \supset V'$. Thus, we get $g \in \mathrm{Aut}(V')$ and $U_0/G \approx V'/G$ (cf. Lemma 2.6).

Remark 3.2. Note that given the embedding $U_0 := \mathbb{C}^4 \subset G(2,4) =: V$, the G-action extends from U_0 to V by similar arguments as for V' above. There is also another construction (communicated by Yu. Prokhorov) of V and $G \subset \mathrm{Aut}(V)$ such that compactification $V \supset U_0$ is G-equivariant. Indeed, take the standard compactification of $U_0 := \mathbb{C}^4$ by $\mathbb{P}^4$, with the divisor $B \subset \mathbb{P}^4$ at infinity, and extend the G-action to $\mathbb{P}^4$ in the usual way. Then there is a G-invariant smooth quadric $S \subset B = \mathbb{P}^3$. Let $\sigma : Y \longrightarrow \mathbb{P}^4$ be the blow up of S with the exceptional divisor $E := \sigma^{-1}(S)$. It is easy to see that the linear system $|2L - E|$, $L := \sigma^*(B)$, determines a birational contraction $\tilde\sigma : Y \longrightarrow \tilde Y$, mapping the proper transform $\sigma_*^{-1}(B) \sim L - E$ of the divisor B to a point. Moreover, since the normal bundle of $\sigma_*^{-1}(B) \simeq \mathbb{P}^3$ on Y is $\mathcal{O}_{\mathbb{P}^3}(-1)$, one immediately gets that $\tilde\sigma$ is the blow up of a smooth point on $\tilde Y$. Furthermore, $\tilde Y$ is a (smooth) Fano 4-fold, with $\mathrm{Pic}(\tilde Y) = \mathbb{Z} \cdot \tilde\sigma_*(L)$ and such that $\tilde\sigma^*(K_{\tilde Y}) = K_Y - 3\sigma_*^{-1}(B) = -4L$, i.e., the Fano index of $\tilde Y$ is 4. Hence, by Iskovskikh and Prokhorov [3, Theorem 3.1.14], $\tilde Y$ is a smooth quadric in $\mathbb{P}^5$. Finally, the construction of $\tilde Y$ implies that both σ and $\tilde\sigma$ are G-equivariant. Hence $\tilde Y$ $(= V)$ is a G-equivariant compactification of U_0. However, we could not obtain similar ("Italian") construction for V', since the way we have built V' is not actually birational. Yet we need V' to have, for instance, such properties as Lemma 3.3 below (which does not hold for the smooth quadric V).

Lemma 3.3. *The space* $H^0(V', \mathcal{O}_{V'}(D_0))$ *contains three linearly independent* G*-invariant elements.*

Proof. Note that D_0 and D_∞ are G-invariant. Moreover, since D_0 and D_∞ are hyperplane sections of $V' \subset \mathbb{P}^5$ which pass through the vertex $O \in V'$,[3] there is also a smooth G-invariant hyperplane section H of V'. Indeed, consider the linear projection $V' \dashrightarrow Q$ from O, with $Q \subset \mathbb{P}^4$ being a smooth quadric (cf. Lemma 2.15). Let also $f : V'' \longrightarrow V'$ be the blow up of O. Then we get $V'' = \mathbb{P}(\mathcal{E})$ for some $\mathbb{C}^2$-vector bundle $\mathcal{E}$ over Q such that the natural projection $V'' \longrightarrow Q$ is G-equivariant.

Further, since both $\mathbb{O}, \mathbb{I} \subset G$ are simple and commute with $\mathbb{C}^*$, the class of $\mathcal{E}$ in $H^1(Q, GL_2(\mathcal{O}_Q))$ is G-invariant. Hence the G-action on V'' extends to the one on $\mathcal{E}$. Now, $\mathcal{E}$ admits two G-invariant sections, the 0-section and the one corresponding to the exceptional divisor of f. This implies that the G-action on the fibers of the projection $V'' \longrightarrow Q$ coincides with the $\mathbb{C}^*$-action. The existence of the above H is now evident.

Finally, D_0, D_∞ and H are (obviously) linearly independent in $H^0(V', \mathcal{O}_{V'}(D_0))$. □

Lemma 3.4. *The* 3*-fold* V'/G *is rational.*

Proof. By Lemma 3.3, we may assume the equation of $V' \subset \mathbb{P}^5 = \mathrm{Proj}\big(\mathbb{C}[x_0, \ldots, x_5]\big)$ to be $x_0x_1 + x_2x_3 + x_4^2 = 0$, with $\mathbb{C}^* \subset G$ acting diagonally and $\mathbb{O} \times \mathbb{I} \subset G$ fixing x_0, x_1, x_5. Let $V' \dashrightarrow \mathbb{P}^2$ be the restriction to V' of the linear projection from the G-invariant plane $\Pi := (x_2 = x_3 = x_4 = 0)$. Note that $V' \cap \Pi$ is a pair of distinct lines (with trivial $\mathbb{O} \times \mathbb{I}$-action). Then, blowing up V' at $V' \cap \Pi$, we get a normal 4-fold $V'' \approx_G V'$ together with a G-equivariant morphism $V'' \longrightarrow \mathbb{P}^2$ which has at least three G-invariant sections and generic fiber $\approx$ [a quadratic cone]. In particular, we get

$$V' \approx_G [\text{quadratic cone with trivial } (\mathbb{O} \times \mathbb{I})\text{-action}] \times \mathbb{P}^2,$$

which implies that V'/G is rational. □

Lemma 3.5. *The* 3*-fold* V/G *is rational.*

Proof. We have

$$\mathbb{C}^4/G = \mathbb{C}^4/(\mathbb{O} \times \mathbb{I} \times \mathbb{C}^*) \simeq \mathbb{C}^4/(\mathbb{O} \times \mathbb{I} \times \mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z}) = U_0/G$$

for the (non-canonical) isomorphism $\mathbb{C}^* \simeq \mathbb{C}^*/(\mathbb{Z}/2\mathbb{Z})$. Now the statement follows from Lemma 3.4 because $\mathbb{C}(V'/G) = \mathbb{C}(U_0/G)$. □

Lemma 3.5 proves Theorem 1.1.

[3] Indeed, we have $D_0 \cap U_0 = (X_1X_4 - X_2X_3 = 0)$, hence $O \in D_0$, and similarly for D_∞ on U_1.

Acknowledgments

I would like to thank I. Cheltsov, S. Galkin and Yu. G. Prokhorov for valuable remarks and exceptional patience during the preparation of this paper. Also the referee's remarks have allowed to improve the exposition.

References

[1] M. Artin, *Algebraic spaces*, Yale University Press, New Haven, CT, 1971.

[2] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in Math., **156**, Springer Verlag, Berlin, 1970.

[3] V.A. Iskovskikh and Yu.G. Prokhorov, *Algebraic geometry V: Fano varieties*, in 'Encyclopaedia of mathematical sciences' (A.N. Parshin, I.R. Shafarevich, eds.), **47**, Springer Verlag, Berlin, 1999.

[4] Yu. G. Prokhorov, *Fields of invariants of finite linear groups*, in 'Cohomological and geometric approaches to rationality problems', 245–273, Progr. Math., **282**, Birkhäuser Boston, Boston, MA, 2010.

Courant Institute, New York University, 251 Mercer street, New York, NY 10012, USA

E-mail address: karzhema@cims.nyu.edu