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EULERIAN GRADED $\mathscr{D}$-MODULES

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Abstract. Let $R = K[x_1, \ldots, x_n]$ with K a field of arbitrary characteristic and $\mathscr{D}$ be the ring of differential operators over R. Inspired by Euler formula for homogeneous polynomials, we introduce a class of graded $\mathscr{D}$-modules, called *Eulerian* graded $\mathscr{D}$-modules. It is proved that a vast class of $\mathscr{D}$-modules, including all local cohomology modules $H^{i_1}_{J_1} \ldots H^{i_s}_{J_s}(R)$ where $J_1, \ldots, J_s$ are homogeneous ideals of R, are Eulerian. As an application of our theory of Eulerian graded $\mathscr{D}$-modules, we prove that all socle elements of each local cohomology module $H^{i_0}_{\mathfrak{m}} H^{i_1}_{J_1} \cdots H^{i_s}_{J_s}(R)$ must be in degree $-n$ in all characteristic. This answers a question raised in [12]. It is also proved that graded F-modules are Eulerian and hence the main result in [12] is recovered. An application of our theory of Eulerian graded $\mathscr{D}$-modules to the graded injective hull of R/P, where P is a homogeneous prime ideal of R, is discussed as well.

1. Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring in n indeterminates over a field K with the standard grading, i.e., $\deg(x_i) = 1$ for each x_i and $\deg(c) = 0$ for each nonzero $c \in K$. Let $\partial_i^{[j]}$ denote the jth order differential operator $\dfrac{1}{j!} \cdot \dfrac{\partial^j}{\partial x_i^j}$ with respect to x_i for each $0 \le i \le n$ and $j \ge 1$. By [4, Théorème 16.11.2], $\mathscr{D} = R\langle \partial_i^{[j]} | 1 \le i \le n, 1 \le j \rangle$ is the ring of K-linear differential operators of R (note if K has characteristic 0, $\mathscr{D}$ is the same as the Weyl algebra $R\langle \partial_1, \ldots, \partial_n \rangle$). The ring of K-linear differential operators $\mathscr{D}$ has a natural $\mathbb{Z}$-grading given by $\deg(x_i) = 1$, $\deg(\partial_i^{[j]}) = -j$, and $\deg(c) = 0$ for each x_i, $\partial_i^{[j]}$ and each nonzero $c \in K$.

The classical Euler formula for homogeneous polynomials says that

$$\sum_{i=1}^{n} x_i \partial_i f = \deg(f) f$$

for each homogeneous polynomial $f \in R$. Inspired by Euler formula, we introduce a class of $\mathscr{D}$-modules called Eulerian graded $\mathscr{D}$-modules: the graded $\mathscr{D}$-modules whose homogeneous elements satisfy a series of "higher order Euler formulas" (cf. Definition 2.1). One of our main results concerning Eulerian graded $\mathscr{D}$-modules is the following (proved in Sections 2 and 5):

Theorem 1.1. *Let $R = K[x_1, \ldots, x_n]$, $\mathfrak{m} = (x_1, \ldots, x_n)$, and $J_1, \ldots, J_s$ be homogeneous ideals of R. Then*

(1) *$R(\ell)$ is Eulerian if and only if $\ell = 0$.*
(2) *Let ${}^*\mathrm{E}$ be the graded injective hull of $R/\mathfrak{m}$. Then ${}^*\mathrm{E}(\ell)$ is Eulerian if and only if $\ell = n$.*

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(3) *Each local cohomology module* $H^{i_1}_{J_1}(\cdots(H^{i_s}_{J_s}(R)))$ *is Eulerian for all* $i_1,\ldots,i_s$.

As an application of our theory of Eulerian graded $\mathscr{D}$-modules, we have the following result on local cohomology (proved in Section 5):

Theorem 1.2. *Let notations be as in the previous theorem. Then all socle elements of each* $H^{i_0}_{\mathfrak{m}}(H^{i_1}_{J_1}(\cdots(H^{i_s}_{J_s}(R)))$ *must have degree* $-n$*, and consequently each* $H^{i_0}_{\mathfrak{m}}(H^{i_1}_{J_1}(\cdots(H^{i_s}_{J_s}(R)))$ *is isomorphic (as a graded* $\mathscr{D}$*-module) to a direct sum of copies of* ${}^*\mathrm{E}(n)$.

This result is characteristic-free, in particular it gives a positive answer to a question stated in [12] and recovers the main theorem in [12].

The paper is organized as follows. In Section 2, Eulerian graded $\mathscr{D}$-modules are defined over an arbitrary field and some basic properties of these modules are discussed. In Sections 3 and 4, we consider Eulerian graded $\mathscr{D}$-modules in characteristic 0 and characteristic p, respectively; in particular, we show in Section 4 that each graded F-module (introduced in [12]) is Eulerian. In Section 5, we apply our theory of Eulerian graded $\mathscr{D}$-modules to local cohomology modules; Theorem 1.2 is proved in this section. Finally, in Section 6, an application of our theory to graded injecitve hull of R/P, where P is a homogeneous prime ideal, is considered.

We finish our introduction by fixing our notation throughout the paper as follows. $R = K[x_1,\ldots,x_n]$ denotes the polynomial ring in n indeterminates over a field K. The jth order differential operator $\frac{1}{j!}\cdot\frac{\partial^j}{\partial x_i^j}$ with respect to x_i is denoted by $\partial_i^{[j]}$ and $\mathscr{D} = R\langle\partial_i^{[j]}|1\le i\le n, 1\le j\rangle$ denotes the ring of differential operators over R. It follows from [10, Corollary 2.2] that every element of $\mathscr{D}$ may be uniquely written as a linear combinations of monomials in x's and ∂'s. The natural $\mathbb{Z}$-grading on R and $\mathscr{D}$ is given by

$$\deg(x_i)=1,\ \deg(\partial_i^{[j]})=-j,\ \deg(c)=0$$

for each x_i, $\partial_i^{[j]}$ and nonzero $c\in K$ (it is evident that R is a graded $\mathscr{D}$-module).

A graded $\mathscr{D}$-module is a left $\mathscr{D}$-module with a $\mathbb{Z}$-grading that is compatible with the natural $\mathbb{Z}$-grading on $\mathscr{D}$. Given any graded $\mathscr{D}$-module M, the module $M(\ell)$ denotes M with degree shifted by ℓ, i.e., $M(\ell)_i = M_{\ell+i}$ for each i.

For each integer a and a nonnegative integer b, we will use $\binom{a}{b}$ to denote $\frac{a\cdot(a-1)\cdots(a-b+1)}{b!}$ (note that this number is still well-defined when $\mathrm{char}(K) = p > 0$).

The irrelevant maximal ideal $(x_1,\ldots,x_n)$ of R is denoted by $\mathfrak{m}$. The graded injective hull of $R/\mathfrak{m}$ is denoted by ${}^*\mathrm{E}$. It may be identified with the K-vector space with a basis $\{\frac{1}{x_1^{e_1}\cdots x_n^{e_n}}|e_1,\ldots,e_n\ge 1\}$ and it has a natural $\mathscr{D}$-module structure given by

$$\partial_i^{[j]}\cdot\frac{1}{x_1^{e_1}\cdots x_n^{e_n}} = \binom{-e_i}{j}\frac{1}{x_1^{e_1}\cdots x^{e_i+j}\cdots x_n^{e_n}}.$$

${}^*\mathrm{E}$ is graded to the effect that the element $\frac{1}{x_1\cdots x_n}$ has degree 0 (cf. [2, Example 13.3.9]).

2. Eulerian graded $\mathscr{D}$-modules

In this section, we introduce Eulerian graded $\mathscr{D}$-modules and discuss some of their basic properties. We begin with the following definition.

Definition 2.1. The rth Euler operator, denoted by E_r, is defined as

$$E_r := \sum_{i_1+i_2+\cdots+i_n=r, i_1\geq 0,\ldots,i_n\geq 0} x_1^{i_1}\cdots x_n^{i_n}\partial_1^{[i_1]}\cdots\partial_n^{[i_n]}.$$

In particular, E_1 is the usual Euler operator $\sum_{i=1}^n x_i\partial_i$.

A graded $\mathscr{D}$-module M is called *Eulerian*, if each homogeneous element $z \in M$ satisfies

$$E_r \cdot z = \binom{\deg(z)}{r} \cdot z \tag{1}$$

for every $r \geq 1$.

We start with an easy lemma.

Lemma 2.2. *For all positive integers s and t, we have $\partial_i^{[s]}\partial_i^{[t]} = \binom{s+t}{s}\partial_i^{[s+t]}$.*

Proof. It is easy to check that (in all characteristic)

$$\partial_i^{[s]}\partial_i^{[t]} = \frac{\partial_i^s}{s!}\frac{\partial_i^t}{t!} = \binom{s+t}{s}\frac{\partial_i^{s+t}}{(s+t)!} = \binom{s+t}{s}\partial_i^{[s+t]}.$$

□

The following lemma is a special case of [10, Proposition 2.1] (with $f = x_i^t$), which will be needed in the sequel.

Lemma 2.3. *For all positive integers s and t, we have*

$$\partial_i^{[s]}x_i^t = \sum_{j=0}^{\min\{s,t\}} \binom{t}{j} x_i^{t-j}\partial_i^{[s-j]}$$

for each i.

The following proposition indicates a connection among Euler operators.

Proposition 2.4. *For every $r \geq 1$, we have $E_1 \cdot E_r = (r+1)E_{r+1} + rE_r$.*

Proof. By Lemma 2.2 we know $\partial_i\partial_i^{[j]} = (j+1)\partial_i^{[j+1]}$. Now we have

$$\begin{aligned}
E_1 \cdot E_r &= \sum_j x_j\partial_j \cdot \sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\\
&= \sum_j \sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1}\cdots(x_j\partial_j x_j^{i_j})\cdots x_n^{i_n}\partial_1^{[i_1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\\
&= \sum_j \sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_i}\cdots(x_j^{i_j+1}\partial_j + i_jx_j^{i_j})\cdots x_n^{i_n}\partial_1^{[i_1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\\
&\quad \text{by lemma 2.3}\\
&= \sum_{i_1+i_2+\cdots+i_n=r} \sum_j x_1^{i_1}\cdots x_j^{i_j+1}\cdots x_n^{i_n}\partial_1^{[i_1]}\cdots(\partial_j\partial_j^{[i_j]})\cdots\partial_n^{[i_n]}\\
&\quad + \sum_{i_1+i_2+\cdots+i_n=r} \sum_j i_jx_1^{i_1}\cdots x_n^{i_n}\partial_1^{[i_1]}\cdots\partial_n^{[i_n]}\\
&= \sum_{i_1+i_2+\cdots+i_n=r} \sum_j (i_j+1)x_1^{i_1}\cdots x_j^{i_j+1}\cdots x_n^{i_n}\partial_1^{[i_1]}\cdots\partial_j^{[i_j+1]}\cdots\partial_n^{[i_n]} + rE_r\\
&= (r+1)E_{r+1} + rE_r.
\end{aligned}$$

□

Some remarks are in order.

Remark 2.5. (1) If M is an Eulerian graded $\mathscr{D}$-module, then $M(\ell)$ is Eulerian if and only if $\ell = 0$ (consequently, for any graded $\mathscr{D}$-module M, $M(\ell)$ is Eulerian graded for at most one ℓ). This follows from the following claim.

Claim 2.5.1. $\binom{a}{r} = \binom{b}{r}$ for all $r \in \mathbb{N}$ if and only if $a = b$.

Proof. The claim is trivial in characteristic 0 (by simply setting $r = 1$). Next, we consider the case in characteristic $p > 0$. Assume that $\binom{a}{r} = \binom{b}{r}$ for all $r \in \mathbb{N}$ and we wish to show that $a = b$. If one of a, b is 0, the conclusion is clear. Hence, there are three possibilities:

(i) $a, b > 0$;
(ii) $a, b < 0$;
(iii) $ab < 0$.

(i) If both a and b are positive, then by a theorem of Lucas [6], we have

$$\binom{a}{r} = \prod \binom{a_i}{r_i},$$

where $a = \sum a_ip^i$ and $r = \sum r_ip^i$ are the p-adic decompositions of a and r. In particular, if we apply $\binom{a}{r} = \binom{b}{r}$ to $r = p^i$, we get $a_i = b_i$ for every i, hence $a = b$.

(ii) When a and b are both negative, we look at the p-adic expansions of $-a-1$ and $-b-1$:

$$-a-1 = \sum_i a_ip^i, \quad -b-1 = \sum_i b-ip^i.$$

Setting $r = p^i$, we get

$$(-1)^{p^i} \cdot (a_i + 1) = (-1)^{p^i} \binom{-a + p^i - 1}{p^i} = \binom{a}{p^i} = \binom{b}{p^i}$$
$$= \binom{-b + p^i - 1}{p^i} = (-1)^{p^i} \cdot (b_i + 1).$$

Hence we have $a_i = b_i$ for every i. Therefore $-a-1 = -b-1$ whence $a = b$. (iii). If $a > 0$ and $b < 0$, we can pick $r = p^j \gg a - b$, then direct computation (or using the theorem of Lucas [6]) gives $\binom{a}{p^j} = 0$ while $\binom{b}{p^j} = (-1)^{p^j}\binom{-b+p^j-1}{p^j} = (-1)^{p^j}$, which is a contradiction. Likewise, it is also impossible to have $a < 0, b > 0$. This finishes the proof of our claim. □

(2) Definition 2.1 does not depend on the characteristic of K. However, we will see in Section 3 that, in characteristic 0, we only need to consider E_1, the usual Euler operator.

(3) R is Eulerian and a proof goes as follows. For each monomial $x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$, where j_n's are arbitrary integers (we allow negative integers), we have (for each $r \geq 1$)

$$E_r \cdot x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n} = \left(\sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\right)$$
$$\cdot (x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}) = \sum_{i_1+i_2+\cdots+i_n=r}\binom{j_1}{i_1}\cdots\binom{j_n}{i_n}\cdot x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$$
$$= \binom{j_1+\cdots+j_n}{r}\cdot x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}.$$

We explain the last equality: we use an induction argument. It is clear when $r = 1$, now suppose we have the equality for $r - 1$. Then

$$\sum_{i_1+i_2+\cdots+i_n=r}\binom{j_1}{i_1}\cdots\binom{j_n}{i_n}$$
$$= \sum_{i_n=0}^{r}\binom{j_n}{i_n}\sum_{i_1+i_2+\cdots+i_{n-1}=r-i_n}\binom{j_1}{i_1}\cdots\binom{j_{n-1}}{i_{n-1}}$$
$$= \sum_{i_n=0}^{r}\binom{j_n}{i_n}\binom{j_1+\cdots+j_{n-1}}{r-i_n}$$
$$= \binom{j_1+\cdots+j_n}{r},$$

where the second equality is by induction hypothesis for $r-1$, and the last equality is by the Chu–Vandermonde identity (see [3, page 44])

$$\binom{a}{r} = \sum_{k=0}^{r} \binom{j}{k}\binom{a-j}{r-k}.$$

Since $x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$ clearly has degree $j_1+\cdots+j_n$ in R and the E_r's clearly preserve addition, we can see from the computation above that $E_r \cdot z = \binom{\deg(z)}{r} \cdot z$ for every homogeneous $z \in R$. Therefore, R is Eulerian.

(4) $\mathscr{D}$ is *not* Eulerian. It is clear that the identity $1 \in \mathscr{D}$ is homogeneous with degree 0, but $E_1 \cdot 1 = (\sum_{i=1}^n x_i\partial_i)\cdot 1 = \sum_{i=1}^n x_i\partial_i \neq 0$.

One of the main results in this section is that, to check whether a graded $\mathscr{D}$-module is Eulerian, it suffices to check whether each element of any set of homogeneous generators satisfies (1).

Theorem 2.6. *Let M be a graded $\mathscr{D}$-module. Assume that $\{g_1, g_2, \dots\}$ is a set of homogeneous $\mathscr{D}$-generators of M. Then, M is Eulerian if and only if each g_j satisfies Euler formula (1) for every r.*

Proof. If M is Eulerian, then it is clear that each g_j satisfies Euler formula (1) for every r. Assume that each g_j satisfies Euler formula (1) for every r and we wish to prove that M is Eulerian. To this end, it suffices to show that, if a homogeneous element $z \in M$ satisfies Euler formula (1) for every r, then so does $x_1^{s_1}\cdots x_n^{s_n}\partial_1^{[j_1]}\cdots\partial_n^{[j_n]}\cdot z$. And it is clear that it suffices to consider $x_i^s\partial_i^{[j]}\cdot z$. Without loss of generality, we may assume that $i=1$. We will prove this in two steps; first we consider $\partial_1^{[j]}\cdot z$ and then $x_1^s \cdot z$ (once we finish our first step, we may replace $\partial^{[j]}\cdot z$ by z and then our second step will finish the proof).

First we will use induction on r to show that $\partial_1^{[j]}\cdot z$ satisfies Euler formula (1) for each r. When $r=1$, we compute

$$\begin{aligned}
E_1 \cdot (\partial_1^{[j]} z) &= \sum_{i=1}^n x_i\partial_i \cdot (\partial_1^{[j]} z)\\
&= x_1\partial_1^{[j]}\partial_1 z + \partial_1^{[j]}\sum_{i\ge 2} x_i\partial_i \cdot z\\
&= \partial_1^{[j]}x_1\partial_1 z - \partial_1^{[j-1]}\partial_1 z + \partial_1^{[j]}\sum_{i\ge 2} x_i\partial_i\cdot z\\
&= \partial_1^{[j]}\sum_{i=1}^n x_i\partial_i\cdot z - j\partial_1^{[j]} z\\
&= (\deg(z)-j)\cdot\partial_1^{[j]} z.
\end{aligned}$$

Now for general r, suppose we know that $E_{r-k}\cdot(\partial_1^{[j]}z) = \binom{\deg(z)-j}{r-k}\cdot\partial_1^{[j]}z$ for every $1\le k\le r-1$. Then we have

$$\Big(\sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\Big)\cdot(\partial_1^{[j]}z)$$

$$= \sum_{i_1+i_2+\cdots+i_n=r}(x_1^{i_1}\partial_1^{[j]})x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\cdot z$$

(1)

$$= \sum_{i_1+i_2+\cdots+i_n=r}(\partial_1^{[j]}x_1^{i_1})x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\cdot z$$

$$- \sum_{i_1+i_2+\cdots+i_n=r}\sum_{k=1}^{\min\{i_1,j\}}\binom{i_1}{k}x_1^{i_1-k}\partial_1^{[j-k]}x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\cdot z$$

$$= \binom{\deg(z)}{r}\cdot(\partial_1^{[j]}z) - \sum_{i_1+i_2+\cdots+i_n=r}\sum_{k=1}^{\min\{i_1,j\}} x_1^{i_1-k}x_2^{i_2}\cdots x_n^{i_n}(\binom{i_1}{k}\partial_1^{[i_1]})\partial_2^{[i_2]}$$

(2)

$$\cdots\partial_n^{[i_n]}\cdot(\partial_1^{[j-k]}z)$$

$$= \binom{\deg(z)}{r}\cdot(\partial_1^{[j]}z) - \sum_{i_1+i_2+\cdots+i_n=r}\sum_{k=1}^{\min\{i_1,j\}} x_1^{i_1-k}x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1-k]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}$$

$$\cdot(\partial_1^{[k]}\partial_1^{[j-k]}z)$$

$$= \binom{\deg(z)}{r}\cdot(\partial_1^{[j]}z) - \sum_{i_1+i_2+\cdots+i_n=r}\sum_{k=1}^{\min\{i_1,j\}} x_1^{i_1-k}x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1-k]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}$$

(3)

$$\cdot\binom{j}{k}(\partial_1^{[j]}z)$$

$$= \binom{\deg(z)}{r}\cdot(\partial_1^{[j]}z) - \sum_{k=1}^{j}\binom{j}{k}\sum_{i_1'+i_2+\cdots+i_n=r-k} x_1^{i_1'}x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1']}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}$$

(4)

$$\cdot(\partial_1^{[j]}z)$$

(5)

$$= \binom{\deg(z)}{r}\cdot(\partial_1^{[j]}z) - \sum_{k=1}^{j}\binom{j}{k}\binom{\deg(z)-j}{r-k}\cdot(\partial_1^{[j]}z)$$

(6)

$$= \binom{\deg(z)-j}{r}\cdot(\partial_1^{[j]}z)$$

$$= \binom{\deg(\partial_1^{[j]}z)}{r}\cdot(\partial_1^{[j]}z),$$

where (1) follows from Lemma 2.3:

$$x_1^{i_1}\partial_1^{[j]} = \partial_1^{[j]}x_1^{i_1} - \sum_{k=1}^{\min\{i_1,j\}} \binom{i_1}{k} x_1^{i_1-k}\partial_1^{[j-k]};$$

(2) and (3) follow from Lemma 2.2;
(4) is obtained by setting $i_1' = i_1 - k$;
(5) is true by induction on r;
(6) follows from the Chu–Vandermonde identity ([3, page 44])

$$\binom{a}{r} = \sum_{k=0}^{j} \binom{j}{k}\binom{a-j}{r-k}.$$

This completes our first step.

Next we consider $x_1 \cdot z$ and we have

$$\begin{aligned}
&\left(\sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\right)\cdot(x_1\cdot z)\\
&= \left(\sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}(\partial_1^{[i_1]}x_1)\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\right)\cdot z\\
&= \left(\sum_{i_1+i_2+\cdots+i_n=r, i_1\ge 1} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}(x_1\partial_1^{[i_1]}+\partial_1^{[i_1-1]})\partial_2^{[i_2]}\cdots\partial_n^{[i_n]} + x_1\right.\\
&\qquad \left.\sum_{i_2+\cdots+i_n=r} x_2^{i_2}\cdots x_n^{i_n}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\right)\cdot z\\
&= x_1\left(\sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]} + \sum_{i_1+i_2+\cdots+i_n=r, i_1\ge 1}\right.\\
&\qquad \left. x_1^{i_1-1}x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1-1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]}\right)\cdot z\\
&= x_1\left(\binom{\deg(z)}{r} + \binom{\deg(z)}{r-1}\right)z\\
&= \binom{\deg(z)+1}{r}(x_1\cdot z)\\
&= \binom{\deg(x_1\cdot z)}{r}(x_1\cdot z).
\end{aligned}$$

This finishes our second step in the case when $s = 1$.

Now we consider $x_1^s \cdot z$ when $s \ge 2$. By an easy induction we may assume that $x_1^{s-1}z$ satisfies (1), now we have

$$E_r(x_1^s z) = E_r(x_1x_1^{s-1}z) = \binom{\deg(x_1(x^{s-1}z))}{r}(x_1(x^{s-1}z)) = \binom{\deg(x_1^s z)}{r}(x_1^s z),$$

where the second equality is the case when $s = 1$ (because we assume that $x_1^{s-1}z$ satisfies (1)). This completes the proof of our theorem. $\square$

An immediate consequence of Theorem 2.6 on cyclic $\mathscr{D}$-modules is the following.

Proposition 2.7. *Let J be a homogeneous left ideal in $\mathscr{D}$. Then $\frac{\mathscr{D}}{J}$ is Eulerian if and only if* $E_r = \sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\partial_1^{[i_1]}\partial_2^{[i_2]}\cdots\partial_n^{[i_n]} \in J$ *for every r.*

Proof. According to Theorem 2.6, $\frac{\mathscr{D}}{J}$ is Eulerian if and only if $\overline{1} \in \frac{\mathscr{D}}{J}$ satisfies Euler's formula (1). Since $\overline{1}$ has degree 0, $\overline{1} \in \frac{\mathscr{D}}{J}$ satisfies Euler's formula (1) if and only if $E_r \cdot \overline{1} = \overline{E_r} = 0 \in \frac{\mathscr{D}}{J}$, which holds if and only if $E_r \in J$ for every r. $\square$

Proposition 2.8. *If a graded $\mathscr{D}$-module M is Eulerian, so are each graded submodule of M and each graded quotient of M.*

Proof. Let N be a graded submodule of M. Since each homogeneous element is also a homogeneous element in M, it is clear that N is also Eulerian. Given a $\mathscr{D}$-linear degree-preserving surjection $\psi : M \to M'$ and a homogeneous element $z' \in M'$, there is a homogeneous $z \in M$ with the same degree such that $\psi(z) = z'$ and hence we have (for every r)

$$E_r \cdot z' = E_r \cdot \psi(z) = \psi(E_r \cdot z) = \psi\left(\binom{\deg(z)}{r} \cdot z\right) = \binom{\deg(z')}{r} \cdot z'.$$

This proves that M' is also Eulerian. $\square$

We end this section with the following result which is one of the key ingredients for our application to local cohomology.

Theorem 2.9. (1) *The graded $\mathscr{D}$-module $R(\ell)$ is Eulerian if and only if $\ell = 0$.*
(2) *The graded $\mathscr{D}$-module ${}^*\mathrm{E}(\ell)$ is Eulerian if and only $\ell = n$.*

Proof. By Remark 2.5 (1), it suffices to show that $R(0)$ and ${}^*\mathrm{E}(n)$ are Eulerian graded. It is clear that $R = R(0)$ is Eulerian by Remark 2.5 (3). Since ${}^*\mathrm{E}(\ell)$ is spanned over K by $x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$ with each $j_t \leq -1$. By the computation in Remark 2.5 (3), it is clear that ${}^*\mathrm{E}(n)$ is Eulerian graded (because in ${}^*\mathrm{E}(n)$, the element $x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$ has degree $j_1 + \cdots + j_n$). $\square$

3. Eulerian graded $\mathscr{D}$-modules in characteristic 0

Throughout this section K will be a field of characteristic 0. In this section, we collect some properties of Eulerian graded $\mathscr{D}$-modules when $\mathrm{char}(K) = 0$. The main result is that, if a graded $\mathscr{D}$-module M is Eulerian, then so is M_f for each $f \in R$. This is one of the ingredients for our application to local cohomology in Section 5. First we observe that, in characteristic 0, if each homogeneous element z in a graded $\mathscr{D}$-module M satisfies (1) for $r = 1$ (instead of for all $r \geq 1$), then M is Eulerian.

Proposition 3.1. *Let M be a graded $\mathscr{D}$-module. If $E_1 \cdot z = \deg(z) \cdot z$ for every homogeneous element $z \in M$. Then M is Eulerian.*

Proof. We prove by induction that $E_r \cdot z = \binom{\deg(z)}{r} \cdot z$ for every $r \geq 1$. When $r = 1$ this is exactly $E_1 \cdot z = \deg(z) \cdot z$ which is given. Now suppose we know $E_r \cdot z = \binom{\deg(z)}{r} \cdot z$. By Proposition 2.4, we know that $E_{r+1} = \frac{1}{r+1}(E_1 \cdot E_r - rE_r)$. So we have

$$\begin{aligned}
E_{r+1} \cdot z &= \frac{1}{r+1}(E_1 \cdot E_r - rE_r) \cdot z \\
&= \frac{1}{r+1}\left(E_1 \cdot \binom{\deg(z)}{r} \cdot z - r \cdot \binom{\deg(z)}{r} \cdot z\right) \\
&= \frac{1}{r+1} \cdot \binom{\deg(z)}{r} \cdot (\deg(z) \cdot z - rz) \\
&= \frac{1}{r+1} \cdot \binom{\deg(z)}{r}(\deg(z) - r) \cdot z \\
&= \binom{\deg(z)}{r+1} \cdot z,
\end{aligned}$$

where the second equality uses the induction hypothesis. This finishes the proof. $\square$

Remark 3.2. As we have seen, Lemma 2.4 is quite useful when $\mathrm{char}(K) = 0$. Unfortunately, this is no longer the case once we are in characteristic p. For instance, when $r \equiv -1 (\mathrm{mod}\ p)$, we cannot link E_{r+1} and E_r via Lemma 2.4. This is one of the reasons that we treat characteristic 0 and characteristic p separately in two different sections.

Corollary 3.3. *Let J be a homogeneous left ideal in $\mathscr{D}$. Then $\frac{\mathscr{D}}{J}$ is Eulerian if and only if $\sum_{i=1}^{n} x_i \partial_i \in J$.*

Proof. This is clear from Proposition 3.1 and (the proof of) Proposition 2.7. $\square$

Remark 3.4. One particular case of Corollary 3.3 is when $J = \mathscr{D} \cdot E_1$, the left ideal generated by the Euler operator E_1. It was pointed out to the authors by Professor Mircea Mustaţă that, according to [1, VII, 9.2], the ring of differential operators $\mathscr{D}(\mathbb{P}^{n-1}_{\mathbb{C}})$ on $\mathbb{P}^{n-1}_{\mathbb{C}}$ can be identified with $\left(\mathscr{D}/\mathscr{D} \cdot E_1\right)^{\mathbb{C}^*}$.

Proposition 3.5. *If M is an Eulerian graded $\mathscr{D}$-module, so is $S^{-1}M$ for each homogeneous multiplicative system $S \subseteq R$. In particular, M_f is Eulerian for each homogeneous polynomial $f \in R$.*

Proof. By Proposition 3.1, it suffices to show for each homogeneous $f \in S$ and $z \in M$, we have $E_1 \cdot \frac{z}{f^t} = \deg(\frac{z}{f^t}) \cdot \frac{z}{f^t}$. Now we compute

$$\begin{aligned} E_1 \cdot \frac{z}{f^t} &= \sum_{i=1}^{n} x_i \partial_i \cdot \frac{z}{f^t} \\ &= \sum_{i=1}^{n} x_i \cdot \frac{f^t \cdot \partial_i(z) - \partial_i(f^t) \cdot z}{f^{2t}} \\ &= \frac{1}{f^{2t}} \left(f^t \sum_{i=1}^{n} x_i \partial_i(z) - \sum_{i=1}^{n} x_i \partial_i(f^t) \cdot z \right) \\ &= \frac{1}{f^t} \cdot \deg(z) \cdot z - \frac{1}{f^{2t}} \cdot \deg(f^t) \cdot f^t \cdot z \\ &= (\deg(z) - \deg(f^t)) \cdot \frac{z}{f^t} \\ &= \deg(\frac{z}{f^t}) \cdot \frac{z}{f^t}. \end{aligned}$$

This finishes the proof. □

Remark 3.6. (1) It turns out that Eulerian graded $\mathscr{D}$-modules are *not* stable under extension because of the following short exact sequence of graded $\mathscr{D}$-modules:

$$0 \to \frac{\mathscr{D}}{\langle \sum_{i=1}^{n} x_i \partial_i \rangle} \xrightarrow{\sum_{i=1}^{n} x_i \partial_i} \frac{\mathscr{D}}{\langle (\sum_{i=1}^{n} x_i \partial_i)^2 \rangle} \to \frac{\mathscr{D}}{\langle \sum_{i=1}^{n} x_i \partial_i \rangle} \to 0,$$

where the map $\frac{\mathscr{D}}{\langle \sum_{i=1}^{n} x_i \partial_i \rangle} \xrightarrow{\sum_{i=1}^{n} x_i \partial_i} \frac{\mathscr{D}}{\langle (\sum_{i=1}^{n} x_i \partial_i)^2 \rangle}$ is the multiplication by $\sum_{i=1}^{n} x_i \partial_i$, i.e., $\overline{a} \mapsto \overline{a \cdot (\sum_{i=1}^{n} x_i \partial_i)}$.

(2) Since $\dim(\frac{\mathscr{D}}{\langle \sum_{i=1}^{n} x_i \partial_i \rangle}) = 2n - 1$ and $\frac{\mathscr{D}}{\langle \sum_{i=1}^{n} x_i \partial_i \rangle}$ is Eulerian, finitely generated (even cyclic) Eulerian graded $\mathscr{D}$-modules may not be holonomic when $n \geq 2$.

(3) When $n = 1$, it is rather straightforward to check that each finitely generated Eulerian graded $\mathscr{D}$-module is holonomic.

(4) As we will see in Section 5, in characteristic 0, a vast class of graded $\mathscr{D}$-modules (namely local cohomology modules of R) are both Eulerian and holonomic.

4. Eulerian graded $\mathscr{D}$-module in characteristic $p > 0$

Throughout this section K will be a field of characteristic $p > 0$. In this section, we prove that being Eulerian is preserved under localization. The proof is quite different from that in characteristic 0. We also show that each graded F-module is always an Eulerian graded $\mathscr{D}$-module, which will enable us to recover the main result in [12] in Section 5.

Proposition 4.1. *If M is an Eulerian graded $\mathscr{D}$-module, so is $S^{-1}M$ for each homogeneous multiplicative system $S \subseteq R$. In particular, M_f is Eulerian for each homogeneous polynomial $f \in R$.*

Proof. First notice that, $\partial_i^{[j]}$ is R^{p^e}-linear if $p^e \geq j+1$. So we have

$$\partial_i^{[j]}(z) = \partial_i^{[j]}\left(f^{p^e} \cdot \frac{z}{f^{p^e}}\right) = f^{p^e} \cdot \partial_i^{[j]}\left(\frac{z}{f^{p^e}}\right).$$

This tells us that, if $p^e \geq r+1$ and $f \in S$, then $\partial_i^{[j]}(\frac{z}{f^{p^e}}) = \frac{1}{f^{p^e}}\partial_i^{[j]}(z)$ for every $j \leq r$ in $S^{-1}M$. In particular, we have

$$E_r \cdot \frac{z}{f^{p^e}} = \frac{1}{f^{p^e}} E_r \cdot z.$$

For any homogeneous $\frac{z}{f^t} \in S^{-1}M$, we can multiply both the numerator and denominator by a large power of f and write $\frac{z}{f^t} = \frac{f^{p^e-t}z}{f^{p^e}}$ for some $p^e \geq \max\{r+1, t\}$. So we have

$$\begin{aligned}
E_r \cdot \frac{z}{f^t} &= E_r \cdot \frac{f^{p^e-t}z}{f^{p^e}} = \frac{1}{f^{p^e}} E_r \cdot f^{p^e-t}z \\
&= \frac{1}{f^{p^e}} \binom{\deg(f^{p^e-t}) + \deg(z)}{r} \cdot f^{p^e-t}z \\
&= \binom{p^e \cdot \deg(f) - \deg(f^t) + \deg(z)}{r} \cdot \frac{f^{p^e-t}z}{f^{p^e}} \\
&= \binom{\deg(\frac{z}{f^t})}{r} \cdot \frac{z}{f^t},
\end{aligned}$$

where the last equality is because $p^e \geq r+1$ and we are in characteristic $p > 0$. This finishes the proof. $\square$

Recall the definition of a graded F-module as follows.

Definition 4.2 (cf. Definitions 2.1 and 2.2 in [12]). For each integer $e \geq 1$, let ${}^eR$ denote the R-bimodule that is the same as R as a left R-module and whose right R-module structure is given by $r' \cdot r = r^p r'$ for all $r' \in {}^eR$ and $r \in R$. An F-module is an R-module M equipped with an R-module isomorphism $\theta : M \to F(M) = {}^1R \otimes_R M$. An F-module (M, θ) is called a graded F-module if M is graded and θ is degree-preserving.

Remark 4.3. It is clear from the definition that, if (M, θ) is an F-module, the map

$$\alpha_e : M \xrightarrow{\theta} F(M) \xrightarrow{F(\theta)} F^2(M) \xrightarrow{F^2(\theta)} \cdots \to F^e(M)$$

induced by θ is also an isomorphism.

This induces a $\mathscr{D}$-module structure on M. To specify the induced $\mathscr{D}$-module structure, it suffices to specify how $\partial_1^{[i_1]} \cdots \partial_n^{[i_n]}$ acts on M. Choose e such that $p^e \geq (i_1 + \cdots + i_n) + 1$. Given each element z, we consider $\alpha_e(z)$ and we will write it as $\sum y_j \otimes z_j$ with $y_j \in {}^eR$ and $z_j \in M$. And we define

$$\partial_1^{[i_1]} \cdots \partial_n^{[i_n]} z := \alpha_e^{-1}\left(\sum \partial_1^{[i_1]} \cdots \partial_n^{[i_n]} y_j \otimes z_j\right).$$

See [5, Section 1] for more details.

When an F-module (M,θ) is graded, the induced map α_e is also degree-preserving. And hence M is naturally a graded $\mathscr{D}$-module. It turns out that each graded F-module is Eulerian as a graded $\mathscr{D}$-module.

Theorem 4.4. *If M is a graded F-module, then M is Eulerian graded as a $\mathscr{D}$-module.*

Proof. Pick any homogeneous element $z \in M$, we want to show $E_r \cdot z = \binom{\deg(z)}{r} \cdot z$ for each $r \geq 1$. Pick e such that $p^e \geq r+1$. Since M is a graded F-module, we have a degree-preserving isomorphism $M \xrightarrow{\alpha_e} F_R^e(M)$. Assume that $\alpha_e(z) = \sum_i y_i \otimes z_i$ where $y_i \in R$ and $z_i \in M$ are homogeneous and $\deg(z) = \deg(y_i \otimes z_i) = p^e \deg(z_i) + \deg(y_i)$ for each i. In particular, we have $\binom{\deg(y_i)}{r} = \binom{\deg(z)}{r}$ for every i (because we are in characteristic $p > 0$). So we know

$$\begin{aligned} E_r \cdot z &= \alpha_e^{-1}\left(\sum_i (E_r \cdot y_i) \otimes z_i\right) \\ &= \alpha_e^{-1}\left(\sum_i \binom{\deg(y_i)}{r} y_i \otimes z_i\right) \\ &= \alpha_e^{-1}\left(\binom{\deg(z)}{r} \sum_i y_i \otimes z_i\right) \\ &= \binom{\deg(z)}{r} \cdot z. \end{aligned}$$

This finishes the proof. $\square$

5. An application to local cohomology

Let R be an arbitrary commutative Noetherian ring and I be an ideal of R. We recall that if I is generated by $f_1, \ldots, f_l \in R$ and M is any R-module, we have the Čech complex:

$$0 \to M \to \bigoplus_j M_{f_j} \to \bigoplus_{j,k} M_{f_j f_k} \to \cdots \to M_{f_1 \cdots f_l} \to 0$$

whose ith cohomology module is $H_I^i(M)$. Here the map $M_{f_{j_1} \cdots f_{j_i}} \to M_{f_{k_1} \cdots f_{k_{i+1}}}$ induced by the corresponding differential is the natural localization (up to sign) if $\{j_1, \ldots, j_i\}$ is a subset of $\{k_1, \ldots, k_{i+1}\}$ and is 0 otherwise.

When R is graded and I is a homogeneous ideal (i.e., $f_1, \ldots, f_l$ are homogeneous elements in R) and M is a graded R-module, each differential in the Čech complex is degree-preserving because natural localization is so. It follows that each cohomology module $H_{J_1}^{i_1}(\cdots(H_{J_s}^{i_s}(R)))$ is a graded R-module.

When $R = K[x_1, \ldots, x_n]$ with K a field of characteristic $p > 0$ and $\mathfrak{m} = (x_1, \ldots, x_n)$, it is proven in [12] (using the theory of graded F-modules) that

Theorem 5.1 (Theorem 3.4 in [12]). *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K of characteristic $p > 0$ and $J_1, \dots, J_s$ be homogeneous ideals of R. Each local cohomology module $H^{i_0}_{\mathfrak{m}}(H^{i_1}_{J_1} \cdots (H^{i_s}_{J_s}(R)))$ is isomorphic to a direct sum of copies of ${}^*\mathrm{E}(n)$[1] (i.e., all socle elements of $H^{i_0}_{\mathfrak{m}}(H^{i_1}_{J_1} \cdots (H^{i_s}_{J_s}(R)))$ must have degree $-n$).*

It is a natural question (and is asked in [12]) whether the same result holds in characteristic 0. Using our theory of Eulerian graded $\mathscr{D}$-module, we can give a characteristic-free proof of the same result. In particular, we answer the question in characteristic 0 in the affirmative.

We begin with the following easy observation.

Proposition 5.2. *Let $J_1, \dots, J_s$ be homogeneous ideals of R, then each local cohomology module $H^{i_1}_{J_1}(\cdots (H^{i_s}_{J_s}(R)))$ is a graded $\mathscr{D}$-module.*

Proof. Since natural localization map is $\mathscr{D}$-linear (and so is each differential in the Čech complex), our proposition follows immediately from the Čech complex characterization of local cohomology. □

Theorem 5.3. *Let $J_1, \dots, J_s$ be homogeneous ideals of R, then each local cohomology module $H^{i_1}_{J_1}(\cdots (H^{i_s}_{J_s}(R)))$ (considered as a graded $\mathscr{D}$-module) is Eulerian.*

Proof. This follows immediately from Propositions 3.5, 4.1 and 2.8, and the Čech complex characterization of local cohomology. □

Proposition 5.4 (cf. Proposition 2.3 in [7] in characteristic 0 and Lemma (b) on page 208 in [9] in characteristic $p > 0$). *Let $R = K[x_1, \dots, x_n]$ and $\mathfrak{m} = (x_1, \dots, x_n)$. There is a degree-preserving isomorphism*

$$\mathscr{D}/\mathscr{D}\mathfrak{m} \to {}^*\mathrm{E}.$$

Proof. It is proven in Proposition 2.3 in [7] in characteristic 0 and Lemma (b) on page 208 in [9] in characteristic $p > 0$ that the map $\mathscr{D}/\mathscr{D}\mathfrak{m} \to {}^*\mathrm{E}$ given by

$$\partial_1^{[i_1]} \cdots \partial_n^{[i_n]} \mapsto (-1)^{i_1 + \cdots + i_n} x_1^{-i_1 - 1} \cdots x_n^{-i_n - 1} \tag{5.4.1}$$

is an isomorphism. The grading on $\mathscr{D}/\mathscr{D}\mathfrak{m}$ is induced by the one on $\mathscr{D}$ and hence $\deg(\partial_1^{[i_1]} \cdots \partial_n^{[i_n]}) = -(i_1 + \cdots + i_n)$. Since in ${}^*\mathrm{E}$, the socle element $x_1^{-1} \cdots x_n^{-1}$ has degree 0, it follows that $\deg(x_1^{-i_1-1} \cdots x_n^{-i_n-1}) = -(i_1 + \cdots + i_n)$. Therefore it follows that (5.4.1) defines a degree-preserving isomorphism $\mathscr{D}/\mathscr{D}\mathfrak{m} \to {}^*\mathrm{E}$. □

Proposition 5.5 (cf. Theorem 2.4(a) in [7] in characteristic 0 and Lemma (c) on page 208 in [9] in characteristic $p > 0$). *Let M be a graded $\mathscr{D}$-module. If $Supp_R(M) = \{\mathfrak{m}\}$, then as a graded $\mathscr{D}$-module $M \cong \bigoplus_j \frac{\mathscr{D}}{\mathscr{D}\mathfrak{m}}(n_j) \cong \bigoplus_j {}^*\mathrm{E}(n_j)$.*

Proof. M is a graded $\mathscr{D}$-module hence also graded as an R-module. We first claim that the socle of M can be generated by homogeneous elements and we reason as follows. Pick a generator g of the socle, we can write it as a sum of homogeneous elements $g = \sum_{i=1}^t g_i$ where each g_i has a different degree. For every $x_j \in \mathfrak{m}$, we have $\sum_{i=1}^t x_j \cdot g_i = x_j \cdot g = 0$ (since g is killed by $\mathfrak{m}$), hence $x_j \cdot g_i = 0$ for every i

[1] when $s = 1$, this is also proved in [11, page 615].

(because each $x_j \cdot g_i$ has a different degree). Therefore g_i is killed by every x_j, hence is killed by $\mathfrak{m}$, so g_i is in the socle for each i. This proves our claim. We also note that since the socle is killed by $\mathfrak{m}$, a minimal homogeneous set of generators is actually a homogeneous K-basis.

Let $\{e_j\}$ be a homogeneous K-basis of the socle of M with $\deg(e_j) = -n_j$. There is a degree-preserving homomorphism of $\mathscr{D}$-modules $\bigoplus_j \frac{\mathscr{D}}{\mathscr{D}\mathfrak{m}}(n_j) \to M$ which sends 1 of the jth copy to e_j. This map is injective because it induces an isomorphism on socles and $\bigoplus_j \frac{\mathscr{D}}{\mathscr{D}\mathfrak{m}}(n_j)$ is supported only at $\mathfrak{m}$ (as an R-module). By 5.4, $\bigoplus_j \frac{\mathscr{D}}{\mathscr{D}\mathfrak{m}}(n_j) \cong \bigoplus_j {}^*\mathrm{E}(n_j)$ is an injective R-module. So $M = \bigoplus_j \frac{\mathscr{D}}{\mathscr{D}\mathfrak{m}}(n_j) \bigoplus N$ where N is some graded R-module supported only at $\mathfrak{m}$. Since the map on the socles is an isomorphism, $N = 0$, so $M = \bigoplus_j \frac{\mathscr{D}}{\mathscr{D}\mathfrak{m}}(n_j) \cong \bigoplus_j {}^*\mathrm{E}(n_j)$. □

Theorem 5.6. *Let M be an Eulerian graded $\mathscr{D}$-module. If $Supp_R(M) = \{\mathfrak{m}\}$, then M is isomorphic (as a graded $\mathscr{D}$-module) to a direct sum of copies of ${}^*\mathrm{E}(n)$.*

Proof. Since M is supported only at $\mathfrak{m}$, we know it is isomorphic to $\bigoplus_j {}^*\mathrm{E}(n_j)$ as a graded $\mathscr{D}$-module by Proposition 5.5. By our assumption, M is Eulerian, so is $\bigoplus_j {}^*\mathrm{E}(n_j)$. It follows from Theorem 2.9 that $n_j = n$ for each j, i.e., M is isomorphic (as a graded $\mathscr{D}$-module) to a direct sum of copies of ${}^*\mathrm{E}(n)$. This finishes the proof. □

Corollary 5.7. *Let $J_1, \dots, J_s$ be homogeneous ideals of R, then $H^{i_0}_{\mathfrak{m}} H^{i_1}_{J_1} \cdots H^{i_s}_{J_s}(R)$ is isomorphic (as a graded $\mathscr{D}$-module) to a direct sum of copies of ${}^*\mathrm{E}(n)$ (or equivalently, all socle elements of each $H^{i_0}_{\mathfrak{m}} H^{i_1}_{J_1} \cdots H^{i_s}_{J_s}(R)$ must have degree $-n$).*

Proof. This follows immediately from Theorems 5.3 and 5.6. □

Remark 5.8. It is proven in [7] (resp, [8]) that every $H^{i_1}_{J_1}(\cdots(H^{i_s}_{J_s}(R)))$ is holonomic (resp, F-finite) as a $\mathscr{D}$-module (resp, F-module) in characteristic 0 (resp, characteristic $p > 0$). Therefore in any case we know $H^{i_1}_{J_1}(\cdots(H^{i_s}_{J_s}(R)))$ has finite Bass numbers (cf. Theorem 3.4(d) in [7] and Theorem 2.11 in [8]). It follows from this and Corollary 5.7 that $H^{i_0}_{\mathfrak{m}} H^{i_1}_{J_1} \cdots H^{i_s}_{J_s}(R) \cong {}^*\mathrm{E}(n)^c$ for some integer $c < \infty$.

6. Remarks on the graded injective hull of R/P when P is a homogeneous prime ideal

We have seen in Theorem 2.9 that ${}^*\mathrm{E}(\ell) = {}^*\mathrm{E}(R/\mathfrak{m})(\ell)$ is Eulerian graded if and only if $\ell = n$. In this section, we wish to extend this result to ${}^*\mathrm{E}(R/P)$ where P is a non-maximal homogeneous prime ideal (here ${}^*\mathrm{E}(R/P)$ denotes the graded injective hull of R/P, see cf. [2, Chapter 13.2]). To this end, we will discuss in detail the graded structures of ${}^*\mathrm{E}(R/P)$ as an R-module and as a $\mathscr{D}$-module. The underlying idea is that, there does *not* exist a canonical choice of grading on ${}^*\mathrm{E}(R/P)$ when it is considered as a graded R-module; however, there is a canonical grading when it is considered as a graded $\mathscr{D}$-module.

Remark 6.1. [${}^*\mathrm{E}(R/P)$ as a graded R-module] Since $P \neq \mathfrak{m}$, there is at least one x_i that is not contained in P. Hence the multiplication by x_i induces an automorphism on ${}^*\mathrm{E}(R/P)$, and consequently we have a degree-preserving isomorphism

$${}^*\mathrm{E}(R/P)(-1) \xrightarrow{\cdot x_i} {}^*\mathrm{E}(R/P).$$

It follows immediately that

$$^*\mathrm{E}(R/P) \cong {}^*\mathrm{E}(R/P)(m)$$

for each integer m in the category of graded R-modules. In other words, we have

$$^*\mathrm{E}(R/P)(i) \cong {}^*\mathrm{E}(R/P)(j)$$

for all integers i and j.

In some sense, this tells us that $^*\mathrm{E}(R/P)$ does not have a canonical grading when considered merely as a graded R-module.

However, as we will see, $^*\mathrm{E}(R/P)$ is equipped with a natural Eulerian graded $\mathscr{D}$-module structure, and from this point of view there is indeed a unique natural grading on $^*\mathrm{E}(R/P)$. The $\mathscr{D}$-module structure on $^*\mathrm{E}(R/P)$ is obtained via considering $H_P^{\mathrm{ht}\ P}(R)_{(P)}$ where $(\cdot)_{(P)}$ denotes homogeneous localization with respect to P (i.e., inverting all homogeneous elements not in P), which has a natural grading as follows.

Remark 6.2 (Grading on $H_P^{\mathrm{ht}\ P}(R)_{(P)}$). Choose a set of homogeneous generators $f_1, \ldots, f_t$ of P and consider the Čech complex

$$C^\bullet(P):\ 0 \to R \to \bigoplus_i R_{f_i} \to \cdots \to \bigoplus_{i_1<i_2<\cdots<i_j} R_{f_{i_1}\cdots f_{i_j}} \to \cdots \to R_{f_1\cdots f_t} \to 0.$$

Then, since each module has a natural grading and each differential is degree-preserving, $H^h(C^\bullet(P))$ also has a natural grading ($h = \mathrm{ht}\ P$), hence so is $H^h(C^\bullet(P))_{(P)}$. We will identify $H_P^{\mathrm{ht}\ P}(R)_{(P)}$ with $H^h(C^\bullet(P))_{(P)}$ with its natural grading.

Proposition 6.3. *$^*\mathrm{E}(R/P) \cong H_P^{\mathrm{ht}\ P}(R)_{(P)}$ in the category of graded R-modules.*

Proof. We have a graded injective resolution of R (or a *-injective resolution of R, cf. [2, Chapter 13])

(5.4.1)

$$0 \to R \to {}^*\mathrm{E}(R)(d_0) \to \cdots \to \bigoplus_{\mathrm{ht}\ Q=s} {}^*\mathrm{E}(R/Q)(d_Q) \to \cdots \to {}^*\mathrm{E}(R/\mathfrak{m})(d_\mathfrak{m}) \to 0,$$

where each d_Q is an integer depending on Q. Notice that when $Q \neq \mathfrak{m}$, $^*\mathrm{E}(R/Q)(i) \cong {}^*\mathrm{E}(R/Q)(j)$ for all integers i and j by Remark 6.1. So the above resolution can be written as

$$0 \to R \to {}^*\mathrm{E}(R) \to \cdots \to \bigoplus_{\mathrm{ht}\ Q=s} {}^*\mathrm{E}(R/Q) \to \cdots \to {}^*\mathrm{E}(R/\mathfrak{m})(d_\mathfrak{m}) \to 0 \tag{5.4.2}$$

Let $h = \mathrm{ht}\ P$. Then $H_P^h(R)_{(P)}$ is the homogeneous localization of the hth homology of (5.4.2) when we apply $\Gamma_P(\cdot)$. But when we apply $\Gamma_P(\cdot)$, (5.4.2) becomes

(5.4.3)

$$0 \to 0 \to \cdots \to {}^*\mathrm{E}(R/P) \to \Gamma_P(\bigoplus_{\mathrm{ht}\ Q=h+1} {}^*\mathrm{E}(R/Q)) \to \cdots \to \Gamma_P({}^*\mathrm{E}(R/\mathfrak{m})(d_\mathfrak{m})) \to 0$$

and when we do homogeneous localization at P to (5.4.3), we get

$$0 \to \cdots \to 0 \to {}^*\mathrm{E}(R/P) \to 0 \to \cdots \to 0.$$

So the hth homology is exactly $^*\mathrm{E}(R/P)$ (and when $P = \mathfrak{m}$, we get $H_\mathfrak{m}^n(R) \cong {}^*\mathrm{E}(R/\mathfrak{m})(d_\mathfrak{m})$, so actually $d_\mathfrak{m} = n$). This finishes the proof. $\square$

Remark 6.4 ($\mathscr{D}$-module structure on ${}^*\mathrm{E}(R/P)$). Since $H^h_P(R)_{(P)}$ has a natural graded $\mathscr{D}$-module structure, it follows from Proposition 6.3 that ${}^*\mathrm{E}(R/P)$ also has a natural graded $\mathscr{D}$-module structure.

Since ${}^*\mathrm{E}(R/P)$ is a graded $\mathscr{D}$-module, it is natural to ask the following question.

Question 6.5. Let P be a homogeneous prime ideal in R. Is there a natural grading on ${}^*\mathrm{E}(R/P)$ making it Eulerian graded?

Remark 6.6. $H^{\mathrm{ht}\ P}_P(R)_{(P)}$ is always Eulerian graded by Theorem 5.3, Propositions 3.5, and 4.1. From Remark 2.5(1), we know that, in the category of graded $\mathscr{D}$-modules, ${}^*\mathrm{E}(R/P)(\ell)$ is Eulerian graded for exactly one ℓ, we will identify this "canonical" ℓ.

Contrary to the case when we consider ${}^*\mathrm{E}(R/P)$ as a graded R-module, we can see that in the category of graded $\mathscr{D}$-modules we have

$${}^*\mathrm{E}(R/P)(i) \cong {}^*\mathrm{E}(R/P)(j) \text{ if and only if } i = j.$$

(Otherwise we would have ${}^*\mathrm{E}(R/P)(\ell) \cong {}^*\mathrm{E}(R/P)(\ell + j - i)$ for every ℓ, and hence there would be more than one choice of ℓ such that ${}^*\mathrm{E}(R/P)(\ell)$ is Eulerian.)

We wish to find the natural grading on ${}^*\mathrm{E}(R/P)$ that makes it Eulerian, and we need the following lemma (which may be well-known to experts).

Lemma 6.7. *We have a canonical degree-preserving isomorphism ($h = \mathrm{ht}\ P$)*

$$Ext^h_R(R/P, R) \cong Hom_R(R/P, H^h_P(R)).$$

Proof. $H^h_P(R)$ is the hth homology of (5.4.2) when we apply $\Gamma_P(\cdot)$, which is the hth homology of (5.4.3), which is the kernel of ${}^*\mathrm{E}(R/P) \to \Gamma_P(\oplus_{\mathrm{ht}\ Q=h+1}\, {}^*\mathrm{E}(R/Q))$. Since $\mathrm{Hom}_R(R/P, \cdot)$ is left exact, we know that $\mathrm{Hom}_R(R/P, H^h_P(R))$ is isomorphic to the kernel of

$$\begin{aligned} {}^*\mathrm{E}(R/P) &\to \mathrm{Hom}_R(R/P, \Gamma_P(\oplus_{\mathrm{ht}\ Q=h+1}\, {}^*\mathrm{E}(R/Q))) \\ &\cong \mathrm{Hom}_R(R/P, (\oplus_{\mathrm{ht}\ Q=h+1}\, {}^*\mathrm{E}(R/Q))). \end{aligned}$$

But this is exactly the hth homology of (5.4.2) when we apply $\mathrm{Hom}_R(R/P, \cdot)$, which by definition is $\mathrm{Ext}^h_R(R/P, R)$. And we want to emphasize here that the isomorphism obtained does not depend on the grading on ${}^*\mathrm{E}(R/P)$ as long as ${}^*\mathrm{E}(R/P)$ is equipped with the same grading when we calculate $\mathrm{Ext}^h_R(R/P, R)$ and $\mathrm{Hom}_R(R/P, H^h_P(R))$ as above. $\square$

Definition 6.8. For a d-dimensional graded K-algebra S with irrelevant maximal ideal $\mathfrak{m}$, the *a-invariant* of S is defined to be

$$a(S) = \max\{t \in \mathbb{Z} | H^d_{\mathfrak{m}}(S)_t \neq 0\}.$$

Proposition 6.9. *We have* $\min\{t | (Ann_{H^{\mathrm{ht}\ P}_P(R)} P)_t \neq 0\} = -a(R/P) - n$. *Hence we have a degree-preserving inclusion*

$$R/P \hookrightarrow H^{\mathrm{ht}\ P}_P(R)(-a(R/P) - n) \hookrightarrow H^{\mathrm{ht}\ P}_P(R)_{(P)}(-a(R/P) - n).$$

Proof. Let $h = \mathrm{ht}\ P$ and let $s = \min\{t | (\mathrm{Ann}_{H^{\mathrm{ht}\ P}_P(R)} P)_t \neq 0\}$. By lemma 6.7, we know

$$\mathrm{Ext}^h_R(R/P, R) \cong \mathrm{Hom}_R(R/P, H^h_P(R)) \cong \mathrm{Ann}_{H^h_P(R)} P,$$

so we know

$$s = \min\{t | (\mathrm{Ext}^h_R(R/P, R)_t \neq 0\} = \min\{t | (\mathrm{Ext}^h_R(R/P, R(-n))_t \neq 0\} - n$$

by graded local duality

$$\min\{t | (\mathrm{Ext}^h_R(R/P, R(-n))_t \neq 0\} = -\max\{t \in \mathbb{Z} | H^{n-h}_{\mathfrak{m}}(R/P)_t \neq 0\} = -a(R/P).$$

Hence we get $s = -a(R/P) - n$. The second statement follows from the first one by sending $\bar{1}$ in R/P to any element in $\mathrm{Ann}_{H_P^{\mathrm{ht}\ P}(R)} P$ of degree $-a(R/P) - n$. □

Remark 6.10. From what we have discussed so far, we can see that $H_P^{\mathrm{ht}\ P}(R)_{(P)}$ is a graded injective module (or *-injective module) and there is a degree-preserving inclusion $R/P \hookrightarrow H_P^{\mathrm{ht}\ P}(R)_{(P)}(-a(R/P)-n)$, always sending $\bar{1}$ in R/P to the lowest degree element in $\mathrm{Ann}_{H_P^{\mathrm{ht}\ P}(R)} P$. Therefore we propose a "canonical" grading on ${}^*\mathrm{E}(R/P)$ to the effect that ${}^*\mathrm{E}(R/P)$ can be identified with $H_P^{\mathrm{ht}\ P}(R)_{(P)}(-a(R/P)-n)$ (where the grading on $H_P^{\mathrm{ht}\ P}(R)_{(P)}$ is obtained via Čech complex).

We end with the following proposition.

Proposition 6.11 (Compare with Theorem 13.2.10 and Lemma 13.3.3 in [2])**.** *Given the grading on* ${}^*\mathrm{E}(R/P)$ *as proposed in Remark 6.10, we have that*

(1) ${}^*\mathrm{E}(R/P)(\ell)$ *is Eulerian if and only if* $\ell = a(R/P) + n$*;*
(2) *the minimal graded injective resolution (or *-injective resolution) of* R *can be written as*

$$0 \to R \to {}^*\mathrm{E}(R) \to \cdots \to \bigoplus_{\mathrm{ht}\ P = j} {}^*\mathrm{E}(R/P)(a(R/P)+n) \to \cdots \to {}^*\mathrm{E}(R/\mathfrak{m})(n) \to 0.$$

Proof. (1). This is clear by our grading on ${}^*\mathrm{E}(R/P)$ and the fact that there is a unique grading on $H_P^{\mathrm{ht}\ P}(R)_{(P)}$ that makes it Eulerian.

(2). This follows immediately from the calculation of $H_P^{\mathrm{ht}\ P}(R)_{(P)}$ using the minimal *-injective resolution of R (note that $a(R) = -n$ and $a(R/\mathfrak{m}) = 0$). □

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