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ON THE U_p OPERATOR IN CHARACTERISTIC p

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Abstract. For a perfect field κ of characteristic $p > 0$, a positive integer N not divisible by p, and an arbitrary subgroup Γ of $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$, we prove (with mild additional hypotheses when $p \leq 3$) that the U-operator on the space $M_k(\mathscr{P}_\Gamma/\kappa)$ of (Katz) modular forms for Γ over κ induces a surjection $U : M_k(\mathscr{P}_\Gamma/\kappa) \twoheadrightarrow M_{k'}(\mathscr{P}_\Gamma/\kappa)$ for all $k \geq p+2$, where $k' = (k - k_0)/p + k_0$ with $2 \leq k_0 \leq p+1$ the unique integer congruent to k modulo p. When $\kappa = \mathbf{F}_p$, $p \geq 5$, $N \neq 2, 3$, and Γ is the subgroup of upper-triangular or upper-triangular unipotent matrices, this recovers a recent result of Dewar [3].

1. Introduction

Fix a prime p, an integer $N > 0$ with $p \nmid N$, and a subgroup Γ of $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$. Let $\widetilde{\Gamma}$ be the preimage in $\mathrm{SL}_2(\mathbf{Z})$ of $\Gamma_0 := \Gamma \cap \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$, and write $\widetilde{M}_k(\widetilde{\Gamma})$ for the space of weight k mod p modular forms for $\widetilde{\Gamma}$ (in the sense of Serre [8, Section 1.2]). When $N = 1$, a classical result of Serre [8, Section 2.2, Théorème 6] asserts that the U_p operator is a contraction: for $k \geq p+2$, the map $U_p : \widetilde{M}_k(\Gamma(1)) \to \widetilde{M}_k(\Gamma(1))$ has image contained in $\widetilde{M}_{k'}(\Gamma(1))$ for some $k' < k$ satisfying $pk' \leq k + p^2 - 1$. In fact, Serre's result may be generalized and significantly sharpened:

Theorem 1.1. *Let κ be a perfect field of characteristic p and denote by $M_k(\mathscr{P}_\Gamma/\kappa)$ the space of weight k Katz modular forms for Γ over κ (see Section 3). Let k_0 be the unique integer between 2 and $p+1$ congruent to k modulo p, and if $p \leq 3$, assume that $N > 4$ and that Γ_0 is a subgroup of the upper-triangular unipotent matrices. Then for $k \geq p+2$, the U-operator (see Section 3) acting on $M_k(\mathscr{P}_\Gamma/\kappa)$ induces a surjection $U : M_k(\mathscr{P}_\Gamma/\kappa) \twoheadrightarrow M_{k'}(\mathscr{P}_\Gamma/\kappa)$, for $k' := (k - k_0)/p + k_0$.*

When $\widetilde{\Gamma} = \Gamma_\star(N)$ for $\star = 0, 1$ and $\kappa = \mathbf{F}_p$, the endomorphism U coincides with the usual Atkin operator U_p (see Corollary 3.3). In particular, if $p \geq 5$, so (by Theorems 1.7.1, and 1.8.1–1.8.2 of [5]) $\widetilde{M}_k(\widetilde{\Gamma}) \simeq M_k(\mathscr{P}_\Gamma/\mathbf{F}_p)$ and $N \neq 2, 3$, Theorem 1.1 is due to Dewar [3]. The proofs of both Serre's original result and Dewar's refinement of it rely on a delicate analysis of the interplay between the operators U_p, V_p, and θ acting on mod p modular forms. In the present note, we take an algebro-geometric perspective, and show how Theorem 1.1 follows immediately from a (trivial extension of a) general theorem of Tango[1] [9] on the behavior of vector bundles under the Frobenius map. In this optic, the contractivity of U_p in characteristic p is simply an instance of the "Dwork Principle" of analytic continuation along Frobenius. In

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[1]Tango's paper, which appeared the year prior to Serre's [8], is perhaps not as well-known as it should be.

particular, we use neither the θ-operator, nor the notion of "filtration" of a mod p modular form. Moreover, our formulation of Theorem 1.1 and its proof totally avoid the use of q-expansions, so should be readily adaptable to the Shimura curve setting.

2. Tango's Theorem

Fix a perfect field κ of characteristic p, and write $\sigma : \kappa \to \kappa$ for the p-power Frobenius automorphism of κ. Let X be a smooth, proper, and geometrically connected curve over κ of genus g. Attached to X is its *Tango number*:

$$n(X) := \max \left\{ \sum_{x \in X(\overline{\kappa})} \left\lfloor \frac{\mathrm{ord}_x(df)}{p} \right\rfloor \ : f \in \overline{\kappa}(X) \setminus \overline{\kappa}(X)^p \right\}, \tag{2.1}$$

where $\overline{\kappa}(X)$ is the function field of $X_{\overline{\kappa}}$. As in Lemma 10 and Proposition 14 of [9], it is easy to see that $n(X)$ is a well-defined integer satisfying $-1 \leq n(X) \leq \lfloor (2g-2)/p \rfloor$, with the lower bound an equality if and only if $g = 0$.

Proposition 2.1 (Tango)**.** *Let $S \neq X$ be a reduced closed subscheme of X with ideal sheaf $\mathscr{I}_S \subseteq \mathscr{O}_X$, and let $\mathscr{L}$ be a line bundle on X. If $\deg \mathscr{L} > n(X)$ then the natural σ-linear map*

$$F^* : H^1(X, \mathscr{L}^{-1} \otimes \mathscr{I}_S) \longrightarrow H^1(X, \mathscr{L}^{-p} \otimes \mathscr{I}_S) \tag{2.2}$$

induced by pullback by the absolute Frobenius of X is injective, and the natural σ^{-1}-linear "trace map"

$$F_* : H^0(X, \Omega^1_{X/\kappa}(S) \otimes \mathscr{L}^p) \longrightarrow H^0(X, \Omega^1_{X/\kappa}(S) \otimes \mathscr{L}) \tag{2.3}$$

given by the Cartier operator ([1], [7, *Section* 10]) *is surjective.*

Proof. The formation of (2.2) and (2.3) is compatible, via σ- (respectively σ^{-1}-) linear extension, with any scalar extension $\kappa \to \kappa'$ to a perfect field κ'; we may therefore assume that κ is algebraically closed. When $g = 0$ we have $X \simeq \mathbf{P}^1_\kappa$ and the proposition is easily verified by direct calculation, so we may further assume that $g > 0$. As the two assertions are dual[2] by Serre duality [7, Section 10, Proposition 9], it suffices to prove the injectivity of (2.2). The case $S = \emptyset$ is Tango's Theorem [9, Theorem 15]. In general, as $\deg(\mathscr{L}) > 0$ and $\mathscr{O}_X/\mathscr{I}_S^j$ is a skyscraper sheaf for all $j > 0$, one finds a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X, \mathscr{O}_X/\mathscr{I}_S) & \longrightarrow & H^1(X, \mathscr{L}^{-1} \otimes \mathscr{I}_S) & \longrightarrow & H^1(X, \mathscr{L}^{-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} & & \downarrow{\scriptstyle F^*} & & \\
0 & \longrightarrow & H^0(X, \mathscr{O}_X/\mathscr{I}_S^p) & \longrightarrow & H^1(X, \mathscr{L}^{-p} \otimes \mathscr{I}_S^p) & \longrightarrow & H^1(X, \mathscr{L}^{-p}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H^0(X, \mathscr{O}_X/\mathscr{I}_S) & \longrightarrow & H^1(X, \mathscr{L}^{-p} \otimes \mathscr{I}_S) & \longrightarrow & H^1(X, \mathscr{L}^{-p}) & \longrightarrow & 0.
\end{array}$$

in which the lower vertical arrows are induced by the inclusion $\mathscr{I}_S^p \subseteq \mathscr{I}_S$. Using that $\kappa = \overline{\kappa}$ and identifying $H^0(X, \mathscr{O}_X/\mathscr{I}_S)$ with κ^S, the left vertical composite is easily

[2]Note that κ-linear duality interchanges σ-linear maps with σ^{-1}-linear ones.

seen to coincide with the map $\oplus_S \sigma : \kappa^S \to \kappa^S$ which is σ on each factor; it is therefore injective. As the right vertical composite map is injective by Tango's Theorem, an easy diagram chase finishes the proof. $\square$

3. Modular forms mod p as differentials on the Igusa curve

In order to apply Tango's Theorem to prove Theorem 1.1, we must recall Katz's geometric definition of mod p modular forms, and Serre's interpretation of them as certain meromorphic differentials on the Igusa curve.

Let us write[3] $R_\Gamma := (\mathbf{Z}[\zeta_N])^{\det(\Gamma)}$, and for any R_Γ-algebra A denote by $\mathscr{P}_\Gamma/A$ the moduli problem $([\Gamma(N)]/\Gamma)^{R_\Gamma\text{-can}} \otimes_{R_\Gamma} A$ on (Ell/A) (see Section 3.1, Section 7.1, 9.4.2, and 10.4.2 of [6]) and by $M_k(\mathscr{P}_\Gamma/A)$ the space of weight k Katz modular forms for $\mathscr{P}_\Gamma/A$ (e.g., [10, Section 6]) that are holomorphic at ∞ in the sense of [5, Section 1.2]. Equivalently, $M_k(\mathscr{P}_\Gamma/A)$ is the A-submodule of level N, weight k modular forms in the sense of [2, VII.3.6] that are invariant under the natural action of $\Gamma_0 := \Gamma \cap \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$. Viewing $\mathbf{C}$ as an R_Γ-algebra via $\zeta_N \mapsto \exp(2\pi i/N)$, we note that $M_k(\mathscr{P}_\Gamma/\mathbf{C})$ *is* the "classical" space of weight k modular forms for $\widetilde{\Gamma}$ over $\mathbf{C}$ defined via the trancendental theory [2, VII.4].

Now fix a ring homomorphism $R_\Gamma \to \kappa$ with κ a perfect field of characteristic p. From here until the end of this section we will assume that $\mathscr{P}_\Gamma/\kappa$ is representable and that -1 acts without fixed points on the space of cusp-labels for Γ (see [6, Section 10.6] and cf. [6, 10.13.7–8]). We will later explain how to relax these hypotheses to those of Theorem 1.1. We write Y_Γ (respectively X_Γ) for the associated (compactified) moduli scheme; by [6, 10.13.12], one knows that X_Γ is a proper, smooth, and geometrically connected curve over κ. Writing $\rho : \mathscr{E} \to Y_\Gamma$ for the universal elliptic curve, our hypothesis that -1 acts without fixed points on the cusp labels for Γ ensures that the line bundle $\omega_\Gamma := \rho_* \Omega^1_{\mathscr{E}/Y_\Gamma}$ on Y_Γ admits a canonical extension, again denoted ω_Γ, to a line bundle on X_Γ [6, 10.13.4, 10.13.7]. By definition, $M_k(\mathscr{P}_\Gamma/\kappa) = H^0(X_\Gamma, \omega_\Gamma^k)$.

Let I_Γ be the *Igusa curve of level p over X_Γ*; by definition, I_Γ is the compactified moduli scheme associated to the simultaneous problem $(\mathscr{P}_\Gamma/\kappa, [\mathrm{Ig}(p)])$ on (Ell/κ) [6, Section 12]. By [6, 12.7.2], the Igusa curve is proper, smooth, and geometrically connected, and the natural map $\pi : I_\Gamma \to X_\Gamma$ is finite étale and Galois with group $(\mathbf{Z}/p\mathbf{Z})^\times$ outside the supersingular points and totally ramified over every supersingular point. Define $\omega := \pi^*\omega_\Gamma$, and recall [6, 12.8.2–3] there is a canonical section $a \in H^0(I_\Gamma, \omega)$ which vanishes to order 1 at each supersingular point and on which $d \in (\mathbf{Z}/p\mathbf{Z})^\times$ acts (via its action on I_Γ) through χ^{-1}, for $\chi : (\mathbf{Z}/p\mathbf{Z})^\times = \mathbf{F}_p^\times \hookrightarrow \mathbf{F}_p$ the mod p Teichmüller character. The following is a straightforward generalization of a theorem of Serre; see [6, Section 12.8] and cf. Propositions 5.7–5.10 of [4].

Proposition 3.1. *Fix an integer $k \geq 2$. For any integer $k_0 \leq k$ with $2 \leq k_0 \leq p+1$, the map $f \mapsto \pi^* f/a^{k_0-2}$ induces a natural isomorphism of κ-vector spaces*

$$M_k(\mathscr{P}_\Gamma/\kappa) \simeq H^0(I_\Gamma, \Omega^1_{I_\Gamma/\kappa}(\mathrm{cusps} + \delta_{k_0} \cdot \mathrm{ss}) \otimes \omega^{k-k_0})(\chi^{k_0-2}), \tag{3.1}$$

where $\delta_{k_0} = 1$ when $k_0 = p+1$ and is zero otherwise; here, ss *and* cusps *are the reduced supersingular and cuspidal divisors, respectively.*

[3] Here, we follow the notation of [6, Section 9.4]: By definition $\mathbf{Z}[\zeta_N]$ is the finite free $\mathbf{Z}$-algebra $\mathbf{Z}[X]/\Phi_N(X)$, where Φ_N is the N-th cyclotomic polynomial and ζ_N corresponds to X, equipped with its natural Galois action of $(\mathbf{Z}/N\mathbf{Z})^\times$.

Proof. The proof is a straightforward adaptation of Propositions 5.7–5.10 of [4]; for the convenience of the reader, we sketch the argument. Thanks to [6, 10.13.11], the Kodaira–Spencer map [6, 10.13.10] provides an isomorphism $\omega_\Gamma^2 \simeq \Omega^1_{X_\Gamma/\kappa}(\text{cusps})$ of line bundles on X_Γ which, after pullback along π, gives an isomorphism

$$\omega^2 \simeq \Omega^1_{I_\Gamma/\kappa}(-(p-2)\text{ss} + \text{cusps}) \tag{3.2}$$

of line bundles on I_Γ as π is étale outside ss and totally (tamely) ramified at each supersingular point.

Since $a \in H^0(I_\Gamma, \omega)$ has simple zeroes along ss, via (3.2) any global section f of ω_Γ^k induces a global section $\pi^* f/a^{k_0-2}$ of $\Omega^1_{I_\Gamma/\kappa}(\text{cusps} + \delta_{k_0}\cdot\text{ss})\otimes\omega^{k-k_0}$ on which $(\mathbf{Z}/p\mathbf{Z})^\times$ acts through χ^{k_0-2}; thus the map (3.1) is well-defined. As $\pi : I_\Gamma \to X_\Gamma$ is a degree $p-1$ generically étale branched cover, the canonical trace mapping $\pi_*\mathscr{O}_{I_\Gamma} \to \mathscr{O}_{X_\Gamma}$ of locally free $\mathscr{O}_{X_\Gamma}$-modules induces a trace mapping $\pi_* : H^0(I_\Gamma, \omega^k) \to H^0(X_\Gamma, \omega_\Gamma^k)$ which satisfies $\pi_*\pi^* = \deg\pi = p-1$; it follows easily that (3.1) is injective. To prove surjectivity, observe that by (3.2), a global section of $\Omega^1_{I_\Gamma/\kappa}(\text{cusps} + \delta_{k_0}\cdot\text{ss})\otimes\omega^{k-k_0}$ gives a meromorphic section h of ω^{k-k_0+2} satisfying $\text{ord}_x(h) \geq -(p-1)$ at each supersingular point x, with equality possible only when $k_0 = p+1$. If h lies in the (k_0-2)-eigenspace of the action of $(\mathbf{Z}/p\mathbf{Z})^\times$, then $f := a^{k_0-2}h$ descends to a meromorphic section of ω_Γ^k over X_Γ satisfying

$$(p-1)\,\text{ord}_x(f) = \text{ord}_x(h) + k_0 - 2 \geq k_0 - p - 1$$

at each supersingular point $x \in X_\Gamma(\overline{\kappa})$, with equality possible only when $k_0 = p+1$. Since the left side is a multiple of $p-1$ and $k_0 \geq 2$, we must have $\text{ord}_x(f) \geq 0$ in all cases, and f is a global (holomorphic) section of ω_Γ^k over X_Γ with $\pi^* f/a^{k_0-2} = h$. ☐

Using Proposition 3.1, the Cartier operator F_* on meromorphic differentials induces, by "transport of structure", a σ^{-1}-linear map $U : M_k(\mathscr{P}_\Gamma/\kappa) \to M_k(\mathscr{P}_\Gamma/\kappa)$. If G is any group of automorphisms of X_Γ, then the action of G commutes with F_* (ultimately because the p-power map in characteristic p commutes with all ring homomorphisms), and we likewise obtain a σ^{-1}-linear endomorphism U of $M_k(\mathscr{P}_\Gamma/\kappa)^G$. This allows us to define U even when $\mathscr{P}_\Gamma/\kappa$ is not representable as follows. Choose a prime $\ell > 3N$, and let Γ' be the unique subgroup of $\text{GL}_2(\mathbf{Z}/N\ell\mathbf{Z})$ projecting to the trivial subgroup of $\text{GL}_2(\mathbf{Z}/\ell\mathbf{Z})$ and to Γ in $\text{GL}_2(\mathbf{Z}/N\mathbf{Z})$. Then for any perfect field κ' of characteristic p admitting a map from $R_{\Gamma'}$, the moduli problem $\mathscr{P}_{\Gamma'}/\kappa'$ is representable, there is a natural action of $G := \text{SL}_2(\mathbf{Z}/\ell\mathbf{Z})$ on $M_k(\mathscr{P}_{\Gamma'}/\kappa')$, and one has $M_k(\mathscr{P}_\Gamma/\kappa') = M_k(\mathscr{P}_{\Gamma'}/\kappa')^G$ (cf. [2, VII.3.3] and [5, Section 1.2]). Via the canonical base-change isomorphism $M_k(\mathscr{P}_\Gamma/\kappa)\otimes_\kappa\kappa' \simeq M_k(\mathscr{P}_\Gamma/\kappa')$, we obtain the desired endomorphism U of $M_k(\mathscr{P}_\Gamma/\kappa)$ by descent, and it is straightforward to check that it is independent of our initial choices of ℓ and κ'. By post-composition with the σ-linear isomorphism[4] $M_k(\mathscr{P}_\Gamma/\kappa) \simeq M_k(\mathscr{P}_\Gamma^{\sigma^{-1}}/\kappa)$ induced by the "exotic isomorphism" of moduli problems $\mathscr{P}_\Gamma/\kappa \simeq \mathscr{P}_\Gamma^{\sigma^{-1}}/\kappa$ [6, 12.10.1] we obtain a κ-linear map $U^\# : M_k(\mathscr{P}_\Gamma/\kappa) \to M_k(\mathscr{P}_\Gamma^{\sigma^{-1}}/\kappa)$. When $\mathscr{P}_\Gamma$ is defined over[5] $\mathbf{F}_p$ in

[4]Explicitly, this isomorphism sends $f \in M_k(\mathscr{P}_\Gamma/\kappa)$ to the modular form f^σ defined by $f^\sigma(E,\alpha) := f(E^\sigma, \alpha^\sigma)$.

[5]A sufficient condition for this to happen is that $\det(\Gamma)$ contain the residue class of $p \bmod N$.

the sense that R_Γ admits a (necessarily unique) surjection to $\mathbf{F}_p$, one has canonically $\mathscr{P}_\Gamma/\mathbf{F}_p = \mathscr{P}_\Gamma^{\sigma^{-1}}/\mathbf{F}_p$ as problems on $(\mathrm{Ell}/\mathbf{F}_p)$, and $U^\#$ is an endomorphism of $M_k(\mathscr{P}_\Gamma/\mathbf{F}_p)$. The maps U and $U^\#$ are natural generalizations of Atkin's U_p-operator:

Proposition 3.2. *Suppose that $\mathscr{P}_\Gamma/\kappa$ is representable and let c be any cusp of $X(\Gamma)$ defined over κ. Then $q^{1/e}$ is a uniformizing parameter at c for some divisor e of N, and for $f \in M_k(\mathscr{P}_\Gamma/\kappa)$, the formal expansions of Uf at c and of $U^\# f$ at $c^{\sigma^{-1}}$ are:*

$$Uf = \sum_{n\geq 0} \sigma^{-1}(a_{np})q^{n/e} \quad \textit{and} \quad U^\# f = \sum_{n\geq 0} a_{np}q^{n/e} \textit{ respectively, for } f = \sum_{n\geq 0} a_n q^{n/e}.$$

Proof. Using the well-known local description of the Cartier operator on meromorphic differentials (e.g., [7, Section 10, Proposition 8]), the result follows easily from the arguments of Propositions 2.8 and 5.7 of [4]; see also (the proof of) [4, Proposition 5.9]. □

Corollary 3.3. *Suppose that $\widetilde{\Gamma} = \Gamma_\star(N)$ for $\star = 0, 1$. Then $R_\Gamma = \mathbf{Z}$ and the resulting endomorphisms U and $U^\#$ of $M_k(\mathscr{P}_\Gamma/\mathbf{F}_p)$ coincide with the Atkin operator U_p, whether or not $\mathscr{P}_\Gamma/\mathbf{F}_p$ is representable.*

Proof. That $R_\Gamma = \mathbf{Z}$ is clear, as $\det(\Gamma) = (\mathbf{Z}/N\mathbf{Z})^\times$. By the discussion above, we may reduce to the representable case, and the result then follows from Proposition 3.2 and the q-expansion principle. □

4. Proof of Theorem 1.1

We now prove Theorem 1.1. Fix k and let k_0 and k' be as in the statement of Theorem 1.1. First suppose that $\mathscr{P}_\Gamma \otimes_{R_\Gamma} \kappa$ is representable and that -1 acts without fixed points on the cusp-labels of Γ. Using (3.2) and the fact that a has simple zeroes along ss we compute (cf. [6, 12.9.4])

$$\deg \omega = \frac{2g-2}{p} + \frac{1}{p}\deg(\text{cusps}) > \left\lfloor \frac{2g-2}{p} \right\rfloor \geq n(I_\Gamma),$$

where g is the genus of I_Γ. Applying Proposition 2.1 with $X = I_\Gamma$, $S = \text{cusps} + \delta_{k_0} \cdot \text{ss}$, and $\mathscr{L} = \omega$, we conclude from (2.3) and the relation $k - k_0 = p(k' - k_0)$ that the Cartier operator

$$F_* : H^0(I_\Gamma, \Omega^1_{I_\Gamma/\kappa}(S) \otimes \omega^{k-k_0}) \longrightarrow H^0(I_\Gamma, \Omega^1_{I_\Gamma/\kappa}(S) \otimes \omega^{k'-k_0})$$

is surjective whenever $k - k_0 \geq p$. Passing to χ^{k_0-2}-eigenspaces for $(\mathbf{Z}/p\mathbf{Z})^\times$ and appealing to Proposition 3.1 then completes the proof in this case.

Now when $p \leq 3$, the hypotheses $N > 4$ and $\widetilde{\Gamma} \subseteq \Gamma_1(N)$ of Theorem 1.1 ensure that $\mathscr{P}_\Gamma \otimes_R \kappa$ is representable (as it maps to the moduli problem $[\Gamma_1(N)]$, which is representable for $N \geq 4$ by [6, 10.9.6]) and that -1 acts without fixed points on the cusp-labels of Γ [6, 10.7.4]. If $p \geq 5$, we may choose a prime $\ell > 3N$ with $\ell \not\equiv 0, \pm 1 \bmod p$, so that $p \nmid |\mathrm{SL}_2(\mathbf{Z}/\ell\mathbf{Z})|$. Then for $N' := N\ell$ and Γ' the subgroup $1 \times \Gamma$ of $\mathrm{SL}_2(\mathbf{Z}/\ell\mathbf{Z}) \times \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z}/N\ell\mathbf{Z})$, we have (after passing to an appropriate extension κ' of κ) that $\mathscr{P}_{\Gamma'} \otimes_{R_{\Gamma'}} \kappa'$ is representable with -1 acting freely on the cusp-labels of Γ' [6, 10.7.1, 10.7.3]. We conclude that the U-operator induces a surjection of $\kappa'[\mathrm{SL}_2(\mathbf{Z}/\ell\mathbf{Z})]$-modules $M_k(\mathscr{P}_{\Gamma'}/\kappa') \twoheadrightarrow M_{k'}(\mathscr{P}_{\Gamma'}/\kappa')$. Our choice of ℓ

ensures that the ring $\kappa'[\mathrm{SL}_2(\mathbf{Z}/\ell\mathbf{Z})]$ is semisimple, so passing to $\mathrm{SL}_2(\mathbf{Z}/\ell\mathbf{Z})$-invariants is exact. As the space of $\mathrm{SL}_2(\mathbf{Z}/\ell\mathbf{Z})$-invariant weight k modular forms for Γ' coincides with $M_k(\mathscr{P}_\Gamma/\kappa')$ (cf. the definition of U in Section 3), passing to invariants and descending from κ' to κ then completes the proof of Theorem 1.1 in the general case.

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