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ON DISCRETELY SELF-SIMILAR SOLUTIONS OF THE EULER EQUATIONS

Dongho Chae and Tai-Peng Tsai

Abstract. This note gives several criteria which exclude the existence of discretely self-similar solutions of the three-dimensional incompressible Euler equations.

1. Introduction

Let $I = (-\infty, 0)$ or $I = (0, \infty)$ be a time interval. We are concerned with the Euler equations for the homogeneous incompressible fluid flows in $\mathbb{R}^3 \times I$,

$$\text{(E)} \begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p, \\ \operatorname{div} v = 0, \end{cases}$$

where $v = (v_1, v_2, v_3)$, $v_j = v_j(x,t)$, $j = 1, 2, 3$, is the velocity of the flow, and $p = p(x,t)$ is the scalar pressure. It is called *backward* or *forward* depending on whether $I = (-\infty, 0)$ or $I = (0, \infty)$. Thanks to the time reversal symmetry of the Euler equations there is a one-to-one correspondence between backward and forward solutions, and we may only consider the backward case $I = (-\infty, 0)$.

Recall that the system (E) has the scaling property that if (v, p) is a solution of the system (E), then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$v^{\lambda,\alpha}(x,t) = \lambda^{\alpha} v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t) \tag{1.1}$$

are also solutions of (E). One can also include space-time translation in (1.1), but we omit it for simplicity. We say that a solution (v, p) of (E) is *self-similar* (SS) with respect to the space-time origin $(0, 0)$ if there exists $\alpha \in (-1, \infty)$ such that, for all $\lambda > 0$,

$$v^{\lambda,\alpha}(x,t) = v(x,t), \quad p^{\lambda,\alpha}(x,t) = p(x,t), \quad (x,t) \in \mathbb{R}^3 \times I. \tag{1.2}$$

It follows that $v(x,t) = \frac{1}{|t|^a} V(\frac{x}{|t|^b})$ for $V(y) = v(y, \text{sign } t)$ and

$$a = \frac{\alpha}{\alpha + 1}, \quad b = \frac{1}{\alpha + 1}, \quad \alpha > -1. \tag{1.3}$$

The condition $\alpha > -1$ ensures that the solution concentrates at the origin as $t \to 0$. If a solution satisfies (1.2) for one single $\lambda > 1$, we say it is *discretely self-similar*

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(DSS) with *factor* λ. It does not need to satisfy (1.2) for every λ, and a self-similar solution is considered as a special case. They are analogous to limit cycles, see explanations below, in particular (1.7). Many fractals are time-independent DSS objects. For example, the Cantor set is DSS with factor 3.

The possibilities of self-similar singularities in the Euler equations are studied in [3, 5–10]. The existence of DSS solutions of (E) has not been studied, and is the main concern of this note.

Our analysis is based on the self-similar transform. The self-similar transform with respect to $(0,0)$ is the map $(v,p) \mapsto (V,P)$ given by

$$v(x,t) = \frac{1}{(-t)^a} V(y,s), \quad p(x,t) = \frac{1}{b(-t)^{2a}} P(y,s), \tag{1.4}$$

where $a \in \mathbb{R}$ and $b > 0$ are given by (1.3), and

$$y = (-t)^{-b} x, \quad s = -\log(-t). \tag{1.5}$$

Substituting (1.4)–(1.5) into (E), we obtain the following system for (V,P):

$$\begin{cases} \dfrac{\partial V}{\partial s} + \frac{\alpha}{\alpha+1} V + \frac{1}{\alpha+1} (y \cdot \nabla) V + (V \cdot \nabla) V = -\nabla P, \\ \operatorname{div} V = 0. \end{cases} \tag{1.6}$$

A solution (v,p) of (E) is self-similar if and only if (V,P) is independent of s. A solution (v,p) of (E) is DSS with factor $\lambda > 1$ if and only if

$$V(y,s) = V(y, s + S_0), \quad \forall (y,s) \in \mathbb{R}^{3+1} \tag{1.7}$$

where $S_0 = (\alpha + 1) \log \lambda > 0$. In other words, (V,P) is a time periodic solution of (1.6) with period S_0.

DSS solutions of partial differential equations have been considered in many other contexts such as the cosmology. See the review [11, Section 5] and references therein.

We now sketch the structure of the rest of the paper. In Section 1.1 we review related results for Navier–Stokes (NS) equations. In Section 2, we give non-existence criteria based on vorticity integrability, and we will state and prove Theorems 2.1 and 2.2. In Section 3, we give non-existence criteria based on velocity integrability, and we will state and prove Theorems 3.1 and 3.2.

1.1. Related results for NS equations. For comparison, we review related results for NS equations for which we add Δv to the right side of (E). For (NS), the backward and forward cases are very different. Introduce the similarity variables: we take parameter $a < 0$ for the backward case and $a > 0$ for the forward case, and let

$$v(x,t) = \frac{1}{\sqrt{2at}} V(y,s), \quad p(x,t) = \frac{1}{2at} P(y,s), \tag{1.8}$$

where

$$y = \frac{x}{\sqrt{2at}}, \quad s = \frac{1}{2a} \log(2at). \tag{1.9}$$

The corresponding time-dependent *Leray's equations* for (V, P) read

$$\frac{\partial V}{\partial s} - \Delta V - aV - ay \cdot \nabla V + (V \cdot \nabla)V = -\nabla P, \quad \operatorname{div} V = 0. \tag{1.10}$$

For the forward case $a > 0$, one can consider the Cauchy problem for (NS) with initial data $v_0(x)$ which is also SS or DSS, i.e., it satisfies (1.2) with no time dependence. For small data, the unique existence of small mild solutions by [13] and revised by [1,2] implies those with SS or DSS data are also SS or DSS. For large SS data, the corresponding SS solution has recently been constructed by Jia and Sverak [14], and extended to large DSS data by Tsai [19] if the DSS-factor λ is sufficiently close to 1 according to the size of the data.

For the backward case $a < 0$, the existence question of self-similar solutions was raised by Leray [15]. It was excluded if $V \in L^3(\mathbb{R}^3)$ by Nečas et al. [16], and if $V \in L^q(\mathbb{R}^3)$, $3 < q \le \infty$, by Tsai [18]. Further extensions were given in [4,5,7,9]. The existence of backward DSS solutions has not been addressed in the literature, except that if $V \in L^\infty(\mathbb{R}, L^3(\mathbb{R}^3))$, which is equivalent to $v \in L^\infty((-\infty, 0), L^3(\mathbb{R}^3))$, it must be zero by the result of [12]. Thus one is concerned, e.g., if V only has the bound

$$|V(y,s)| \le \frac{C_*}{1 + |y|}, \quad \forall (y,s) \in \mathbb{R}^{3+1}, \tag{1.11}$$

for some large constant C_*. A special case that $V(y,s) = R(s\vec{k})\tilde{V}(y)$, with $R(s\vec{k})$ being the rotation about a fixed axis $\vec{k}$ by angle $s|\vec{k}|$, was proposed by G. Perelman.[1] Then $\tilde{V}$ satisfies a time-independent system. As pointed out to one of us by R. Kohn, the examples of Scheffer [17] are DSS solutions with singular DSS forces. In view of the forward case, one may hope that the case with the DSS-factor λ sufficiently close to 1 might be easier to exclude.

2. Criteria based on vorticity

In this section, we give non-existence criteria based on vorticity integrability.

Theorem 2.1. *Let $V(y,s) \in C_x^2 C_t^1(\mathbb{R}^{3+1})$ be a time periodic solution of* (1.6) *with period $S_0 > 0$ that has bounded first derivatives and satisfies*

$$\lim_{|y| \to \infty} V(y,s) = 0, \quad \forall s \in [0, S_0), \tag{2.1}$$

and for some $r > 0$,

$$\Omega := \operatorname{curl} V \in \bigcap_{0<q<r} L^q(\mathbb{R}^3 \times [0, S_0]). \tag{2.2}$$

Then $V = 0$ on $\mathbb{R}^{3+1}$.

Proof. We first observe that from the calculus identity

$$V(y,s) = V(0,s) + \int_0^1 \partial_\tau V(\tau y, s)\, d\tau = V(0,s) + \int_0^1 y \cdot \nabla V(\tau y, s)\, d\tau,$$

[1] Private communication of G. Seregin.

we have $|V(y,s)| \le |V(0,s)| + |y| \|\nabla V(s)\|_{L^\infty}$, and hence

$$\sup_{(y,s)\in\mathbb{R}^{3+1}} \frac{|V(y,s)|}{1+|y|} \le C_1 := \max_s |V(0,s)| + \|\nabla V\|_{L^\infty(\mathbb{R}^{3+1})}. \tag{2.3}$$

Let us consider the radial cut-off function $\sigma \in C_0^\infty(\mathbb{R}^N)$ such that

$$\sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases} \tag{2.4}$$

and $0 \le \sigma(x) \le 1$ for $1 < |x| < 2$. Then, for each $R > 0$, we define $\sigma_R(y) := \sigma\left(\frac{|y|}{R}\right) \in C_0^\infty(\mathbb{R}^N)$. We operate curl on (1.6),

$$\frac{\partial}{\partial s}\Omega + \Omega + \frac{1}{\alpha+1}(y\cdot\nabla)\Omega + (V\cdot\nabla)\Omega = (\Omega\cdot\nabla)V. \tag{2.5}$$

We multiply (2.5) by $|\Omega|^{q-2}\Omega\sigma_R$, and integrate over $\mathbb{R}^3 \times (0, S_0)$. The first term vanishes by periodicity. After integration by parts we get

$$\begin{aligned} &\left(1 - \frac{3}{(\alpha+1)q}\right)\int_0^{S_0}\int |\Omega|^q \sigma_R\, dy\, ds - \int_0^{S_0}\int_{\mathbb{R}^3} \xi\cdot\nabla V\cdot\xi|\Omega(s)|^q\sigma_R\, dy\, ds \\ &\quad = I := \frac{1}{q}\int_0^{S_0}\int_{\mathbb{R}^3} |\Omega(s)|^q \left(\frac{y}{\alpha+1} + V\right)\cdot\nabla\sigma_R\, dy\, ds, \end{aligned} \tag{2.6}$$

where we set $\xi = \Omega/|\Omega|$. Since $|\nabla\sigma_R(y)| \le \frac{\|\sigma'\|_{L^\infty}}{R} 1_{R\le|y|\le 2R}$, by (2.3) we have

$$|I| \le C(1+C_1)\int_0^{S_0}\int_{\{R\le|y|\le 2R\}} |\Omega|^q\, dy\, ds. \tag{2.7}$$

Passing $R \to \infty$, we have $I \to 0$ and (2.6) gives

$$\left|1 - \frac{3}{(\alpha+1)q}\right| \cdot \int_0^{S_0} \|\Omega(s)\|_{L^q}^q ds \le C_1 \int_0^{S_0} \|\Omega(s)\|_{L^q}^q ds. \tag{2.8}$$

This is true for all $q \in (0, r)$ for some $r > 0$. Passing $q \downarrow 0$ in (2.8), we get $\int_0^{S_0} \|\Omega(s)\|_{L^q}^q ds = 0$. Therefore $\Omega = \operatorname{curl} V = 0$ on $\mathbb{R}^3 \times (0, S_0)$. This, together with $\operatorname{div} V = 0$, provides us with the fact that $V(\cdot,s) = \nabla h(\cdot,s)$ for all $s \in [0, S_0)$ for a scalar harmonic function $h(\cdot,s)$ on $\mathbb{R}^3$. Using (2.1) we have $V(\cdot,s) = 0$ by Liouville theorem for harmonic functions. □

Remark after the proof: The above proof works for more general system

$$\begin{cases} V_s + aV + b(y\cdot\nabla)V + (V\cdot\nabla)V = -\nabla P, \\ \operatorname{div} V = 0, \end{cases} \tag{2.9}$$

where a, b are arbitrary real constants with $b \neq 0$.

Theorem 2.2. *Let $V(y,s) \in C^2_x C^1_t(\mathbb{R}^{3+1})$ be a time periodic solution of* (1.6) *with period $S_0 > 0$ that has bounded first derivatives satisfying*

$$\lim_{|y|\to\infty} \sup_{0<s<S_0} |V(y,s)| + |\nabla V(y,s)| = 0, \tag{2.10}$$

and there exists $q \in (0, \frac{3}{1+\alpha})$ such that

$$\Omega \in L^q(\mathbb{R}^3 \times [0, S_0]). \tag{2.11}$$

Then, $V = 0$ on $\mathbb{R}^{3+1}$.

Proof. Writing (2.11) in terms of spherical coordinates,

$$\int_0^{S_0} \int_{\mathbb{R}^3} |\Omega|^q dy\, ds = \int_0^\infty \int_0^{S_0} \int_{|y|=r} |\Omega|^q \, dS_r \, ds \, dr < \infty,$$

one finds that there exists a sequence $R_j \uparrow \infty$ such that

$$R_j \int_0^{S_0} \int_{|y|=R_j} |\Omega|^q dS_{R_j} ds \to 0 \quad \text{as } j \to \infty. \tag{2.12}$$

We multiply (2.5) by $\Omega|\Omega|^{q-2}$ and rewrite it

$$\begin{aligned} &\frac{1}{q}\frac{\partial}{\partial s}|\Omega|^q + |\Omega|^q + \frac{1}{q(\alpha+1)}\mathrm{div}\,(y|\Omega|^q) - \frac{3}{q(\alpha+1)}|\Omega|^q \\ &\quad = \hat{\alpha}|\Omega|^q - \frac{1}{q}\mathrm{div}\,(V|\Omega|^q), \end{aligned} \tag{2.13}$$

where $\hat{\alpha} = \xi \cdot \nabla V \cdot \xi$ with $\xi = \Omega/|\Omega|$. Note that $|\hat{\alpha}| \le |\nabla V|$. Let us fix an $R > 0$ and integrate (2.13) over the domain $(y,s) \in \{R < |y| < R_j\} \times (0, S_0)$. Applying the divergence theorem, we have

$$\begin{aligned} &\left(\frac{3}{q(\alpha+1)} - 1\right) \int_0^{S_0} \int_{R<|y|<R_j} |\Omega|^q dy\, ds + \frac{R}{q(\alpha+1)} \int_0^{S_0} \int_{|y|=R} |\Omega|^q dS_R ds \\ &- \frac{R_j}{q(\alpha+1)} \int_{|y|=R_j} |\Omega|^q dS_{R_j} ds = - \int_0^{S_0} \int_{R<|y|<R_j} \hat{\alpha}|\Omega|^q dy\, ds \\ &- \frac{1}{q}\int_0^{S_0} \int_{|y|=R} V_r |\Omega|^q dS_R ds + \frac{1}{q}\int_0^{S_0} \int_{|y|=R_j} V_r |\Omega|^q dS_{R_j} ds, \end{aligned}$$

where $V_r = V \cdot \frac{y}{|y|}$ and we used the fact $\int_0^{S_0} (\|\Omega(s)\|^q_{L^q})_s ds = 0$, following from the periodicity hypothesis. Passing $j \to \infty$, one obtains

$$
\begin{aligned}
(2.14) \qquad &\left(\frac{3}{q(\alpha+1)} - 1\right) \int_0^{S_0} \int_{|y|>R} |\Omega|^q dy\, ds + \frac{R}{q(\alpha+1)} \int_0^{S_0} \int_{|y|=R} |\Omega|^q dS_R ds \\
&= - \int_0^{S_0} \int_{|y|>R} \hat{\alpha} |\Omega|^q dy\, ds - \frac{1}{q} \int_0^{S_0} \int_{|y|=R} V_r |\Omega|^q dS_R ds.
\end{aligned}
$$

Using (2.10) and choosing R sufficiently large, we have

$$
|\hat{\alpha}| \le \frac{1}{2} \left(\frac{3}{q(\alpha+1)} - 1\right), \quad |V_r| \le \frac{R}{2(\alpha+1)},
$$

on the right-hand side of (2.14). Consequently,

$$
\int_0^{S_0} \int_{|y|>R} |\Omega|^q dy\, ds = \int_0^{S_0} \int_{|y|=R} |\Omega|^q dS_R ds = 0,
$$

and hence, $\Omega = 0$ on $\{y \in \mathbb{R}^3 \mid |y| > R\} \times (0, S_0)$. Thus our vorticity Ω satisfies the condition (2.2) of Theorem 2.1. Applying Theorem 2.1 we obtain $V = 0$ on $\mathbb{R}^{3+1}$. □

3. Criteria based on velocity

In this section, we give non-existence criteria based on velocity integrability.

We will need to estimate the pressure $P(y, s)$, which satisfies

$$
(3.1) \qquad -\Delta_y P(\cdot, s) = \sum_{i,j} \partial_i \partial_j (V_i V_j(\cdot, s))
$$

by taking the divergence of (1.6). One solution of (3.1) is given by

$$
(3.2) \qquad \tilde{P}(y, s) = -\frac{|v(y,s)|^2}{3} + \sum_{i,j} p.v. \int_{\mathbb{R}^3} K_{ij}(y-z) V_i V_j(z, s) dz,
$$

where the kernel is

$$
K_{ij}(y) = \frac{3 y_i y_j - \delta_{i,j} |y|^2}{4\pi |y|^5}.
$$

In general, for each fixed t, the difference $P - \tilde{P}$ is a harmonic function in x and may not be constant. We will assume $P = \tilde{P}$.

Theorem 3.1. *Suppose that $(V, P) \in C^1_{loc}(\mathbb{R}^{3+1})$ is a time periodic solution of (1.6) with period $S_0 > 0$, that the pressure P is given by (3.2), and that V satisfies for some $3 \le r \le 9/2$ one of the the following conditions,*

(i) *$\alpha > 3/2$ and $V \in L^3(0, S_0; L^r(\mathbb{R}^3))$, or*
(ii) *$-1 < \alpha < 3/2$ and $V \in L^2(0, S_0; L^2(\mathbb{R}^3)) \cap L^3(0, S_0; L^r(\mathbb{R}^3))$.*

Then $V = 0$ on $\mathbb{R}^{3+1}$.

Proof. Since P is given by (3.2), by the Calderon–Zygmund inequality we have

$$\|P(s)\|_{L^q} \le C_q \|V(s)\|^2_{L^{2q}} \quad \forall q \in (1,\infty), \forall s \in \mathbb{R}. \tag{3.3}$$

The case (i): Let σ_R is the cut-off function introduced in the proof of Theorem 1.1. We multiply (1.6) by $V\sigma_R$, and integrate over $\mathbb{R}^3 \times (0, S_0)$, then from the time periodicity condition and by integration by part we obtain

$$\begin{aligned} &\frac{1}{\alpha+1}\left(\alpha - \frac{3}{2}\right)\int_0^{S_0}\int_{\mathbb{R}^3} |V|^2 \sigma_R dy\, ds - \frac{1}{2(\alpha+1)}\int_0^{S_0}\int_{\mathbb{R}^3} |V|^2 y \cdot \nabla\sigma_R dy\, ds \\ &\quad = \frac{1}{2}\int_0^{S_0}\int_{\mathbb{R}^3} |V|^2 V \cdot \nabla \sigma_R dy\, ds + \int_0^{S_0}\int_{\mathbb{R}^3} PV \cdot \nabla\sigma_R dy\, ds. \end{aligned} \tag{3.4}$$

Since $y \cdot \nabla\sigma_R \le 0$ for all $y \in \mathbb{R}^3$, and the first term of the left-hand side of (3.4) is monotone non-decreasing function of R, we find that

$$\begin{aligned} &\frac{1}{\alpha+1}\left(\alpha - \frac{3}{2}\right)\int_0^{S_0}\int_{\mathbb{R}^3} |V|^2 \sigma_{R_1} dy\, ds \\ &\quad \le \frac{1}{2}\int_0^{S_0}\int_{\mathbb{R}^3} |V|^2 V \cdot \nabla\sigma_{R_2} dy\, ds + \int_0^{S_0}\int_{\mathbb{R}^3} PV \cdot \nabla\sigma_{R_2} dy\, ds \\ &\quad := I_1 + I_2 \end{aligned} \tag{3.5}$$

for all $0 < R_1 < R_2 < \infty$. Passing $R_2 \to \infty$, one has

$$\begin{aligned} I_1 &\le \frac{\|\nabla\sigma\|_{L^\infty}}{2R_2}\int_0^{S_0}\int_{R_2<|y|<2R_2} |V|^3 dy\, ds \\ &\le \frac{\|\nabla\sigma\|_{L^\infty}}{2R_2}\int_0^{S_0}\left(\int_{R_2<|y|<2R_2} |V|^r dy\right)^{\frac{3}{r}}\left(\int_{R_2<|y|<2R_2} dy\right)^{1-\frac{3}{r}} ds \\ &\le CR_2^{2-\frac{9}{r}}\int_0^{S_0} \|V(s)\|^3_{L^r(R_2<|y|<2R_2)} ds \to 0, \end{aligned}$$

and

$$\begin{aligned} I_2 &\le \frac{\|\nabla\sigma\|_{L^\infty}}{2R_2}\int_0^{S_0}\int_{R_2<|y|<2R_2} |V||P| dy\, ds \\ &\le \frac{\|\nabla\sigma\|_{L^\infty}}{2R_2}\int_0^{S_0}\left(\int_{R_2<|y|<2R_2} |V|^r dy\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^3} |P|^{\frac{r}{2}} dy\right)^{\frac{2}{r}}\left(\int_{R_2<|y|<2R_2} dy\right)^{1-\frac{3}{r}} ds \\ &\le CR_2^{2-\frac{9}{r}}\int_0^{S_0} \|V\|_{L^r(R_2<|y|<R_2)} \|V\|^2_{L^r} ds \\ &\le CR_2^{2-\frac{9}{r}}\left(\int_0^{S_0} \|V\|^3_{L^r(R_2<|y|<R_2)} ds\right)^{\frac{1}{3}}\left(\int_0^{S_0} \|V\|^3_{L^r} ds\right)^{\frac{2}{3}} \to 0, \end{aligned}$$

where we used (3.3). Therefore, we have

$$\left(\alpha-\frac{3}{2}\right)\int_0^{S_0}\int_{\mathbb{R}^3}|V|^2\sigma_{R_1}dy\,ds=0$$

for all $R_1>0$. This shows that $V=0$ on $\mathbb{R}^3\times(0,S_0)$.
The case (ii): In this case, from (3.4), we have

$$\begin{aligned}(3.6)\quad \frac{1}{\alpha+1}\left|\alpha-\frac{3}{2}\right|\int_0^{S_0}\int_{\mathbb{R}^3}|V|^2\sigma_R dy\,ds &\le \frac{1}{2(\alpha+1)}\int_0^{S_0}\int_{\mathbb{R}^3}|V|^2|y\cdot\nabla\sigma_R|dy\,ds\\ &+\frac{1}{2}\int_0^{S_0}\int_{\mathbb{R}^3}|V|^2|V\cdot\nabla\sigma_R|dy\,ds+\int_0^{S_0}\int_{\mathbb{R}^3}|P||V\cdot\nabla\sigma_R|dy\,ds\\ &:=J_1+J_2+J_3.\end{aligned}$$

From the above computations we know that $|J_2|+|J_3|\to 0$ as $R\to\infty$. For J_1 we estimate easily

$$|J_1|\le\frac{C}{R}\int_0^{S_0}\int_{R<|y|<2R}|V|^2|y|dy\,ds\le C\int_0^{S_0}\|V(s)\|^2_{L^2(R<|y|<2R)}ds\to 0$$

as $R\to\infty$. Hence $\int_0^{S_0}\int_{\mathbb{R}^3}|V|^2dy\,ds=0$, and $V=0$ on $\mathbb{R}^{3+1}$. □

The next result, Theorem 3.2, is an extension of Chae–Shvydkoy [10, Theorem 3.2] to the case of DSS solutions. An important role is played by the following lemma, which extends the local energy inequality in [10].

Lemma 3.1. *Suppose $(V,P)\in C^1_{loc}(\mathbb{R}^{3+1})$ is a time periodic solution of (1.6) with period $S_0>0$. Let $\lambda=e^{bS_0}>1$. For $-\infty<s_1<s_2<\infty$, let $l_j=e^{bs_j}$ and*

$$(3.7)\qquad I_j=l_j^{2\alpha-3}\int_0^{S_0}\int_{\mathbb{R}^3}|V(y,s_j+\tau)|^2\sigma(e^{-b(s_j+\tau)}y)dy\,d\tau,\quad (j=1,2),$$

where $\sigma(x)\ge 0$ is a radial function with $\sigma(x)=1$ for $|x|<1/2$ and $\sigma(x)=0$ for $|x|\ge 1$. We have

$$(3.8)\quad l_j^{2\alpha-3}\int_0^{S_0}\int_{|y|\le\frac{1}{2}l_j}|V(y,\tau)|^2dy\,d\tau\le I_j\le l_j^{2\alpha-3}\int_0^{S_0}\int_{|y|\le\lambda l_j}|V(y,\tau)|^2dy\,d\tau$$

for $j=1,2$, and for some constant $C=C(S_0)$

$$(3.9)\qquad |I_1-I_2|\le C\int_0^{S_0}\int_{\frac{1}{2}l_1\le|y|\le\lambda l_2}|y|^{2\alpha-4}(|V|^3+|PV|)(y,s)\,dy\,ds.$$

Note that λ is the factor for discrete self-similarity; see (1.7).

Proof. Let $t_j = -e^{-bs_j}$, $j = 1, 2$. Testing the Euler equation with σv in $\mathbb{R}^3 \times (t_1, t_2)$ we get

$$\int |v(x,t_2)|^2\sigma(x)dx - \int |v(x,t_1)|^2\sigma(x)dx = \int_{t_1}^{t_2}\int (|v|^2 + 2p)v \cdot \nabla\sigma(x)dx\,dt. \tag{3.10}$$

In self-similar variables (1.4)–(1.5) it becomes

$$\begin{aligned} e^{(2a-3b)s_2}\int |V(y,s_2)|^2\sigma(e^{-bs_2}y)dy - e^{(2a-3b)s_1}\int |V(y,s_1)|^2\sigma(e^{-bs_1}y)dy \\ = \int_{s_1}^{s_2} e^{(3a-3b-1)s}\int (|V|^2 + 2P)V \cdot \nabla\sigma(e^{-bs}y)dy\,ds. \end{aligned} \tag{3.11}$$

Assume now that v is DSS, so that $V(y, s)$ is periodic in s with period $S_0 > 0$.

Replacing s_j by $s_j + \tau$, dividing by $e^{(2a-3b)\tau}$, and integrating over $\tau \in [0, S_0]$, we get

$$I_1 - I_2 = I_3 \tag{3.12}$$

where I_1 and I_2 are given in (3.7), and

$$I_3 = \int_0^{S_0} e^{-(2a-3b)\tau}\int_{s_1+\tau}^{s_2+\tau} e^{(3a-3b-1)s}\int (|V|^2 + 2P)V \cdot \nabla\sigma(e^{-bs}y)dy\,ds\,d\tau. \tag{3.13}$$

The estimate (3.8) for I_1 and I_2 is because that $\sigma(e^{-b(s_j+\tau)}y)$ is supported in $\frac{1}{2}l_j \le |y| \le \lambda l_j$, and also using the periodicity.

For I_3, since $e^{-(2a-3b)\tau} \le C$ and $\sigma(e^{-bs}y)$ is supported in $\frac{1}{2}e^{bs} \le |y| \le e^{bs}$,

$$|I_3| \le C\int_E\int_0^{S_0}\int_{s_1+\tau}^{s_2+\tau} e^{(3a-3b-1)s}Q(y,s)|\nabla\sigma|(e^{-bs}y)ds\,d\tau dy, \tag{3.14}$$

where E denotes the spatial region $E = \{y : \frac{1}{2}l_1 \le |y| \le \lambda l_2\}$ and $Q = |V|^3 + |P||V|$. If we denote by $f(y, s)$ the integrand, the inner integral

$$\int_0^{S_0}\int_{s_1+\tau}^{s_2+\tau} f(y,s)ds\,d\tau \le S_0\int_{s_1}^{s_2+S_0} f(y,s)ds \le S_0\int_{J_y} f(y,s)ds, \tag{3.15}$$

where we have used $\sigma(e^{-bs}y)$ is supported in the time interval

$$J_y = \{s : |y| \le e^{bs} \le 2|y|\} = \left\{s : \frac{\ln|y|}{b} \le s \le \frac{\ln|y|}{b} + \frac{\ln 2}{b}\right\}. \tag{3.16}$$

In J_y we have $e^{(3a-3b-1)s} \le C|y|^{\frac{3a-3b-1}{b}} = C|y|^{2\alpha-4}$. Thus

$$|I_3| \le C\int_E\int_{J_y} |y|^{2\alpha-4}Q(y,s)ds\,dy. \tag{3.17}$$

Let k be the positive integer so that $(k-1)S_0 < \frac{\ln 2}{b} \le kS_0$. Using the periodicity of $Q(y, s)$ in s,

$$|I_3| \le Ck\int_E\int_0^{S_0} |y|^{2\alpha-4}Q(y,s)ds\,dy. \tag{3.18}$$

This shows (3.9). ☐

Theorem 3.2. *Suppose* $(V, P) \in C^1_{loc}(\mathbb{R}^{3+1})$ *is a time periodic solution of* (1.6) *in* $\mathbb{R}^{3+1}$ *with period* $S_0 > 0$, $V \in L^p(\mathbb{R}^3 \times ([0, S_0])$ *for some* $3 \le p \le \infty$, *and* P *is given by* (3.2). *If* $-1 < \alpha \le 3/p$ *or* $3/2 < \alpha < \infty$, *then* $V = 0$ *on* $\mathbb{R}^{3+1}$.

Proof. The proof of [10, Theorem 3.2] goes through with the help of Lemma 3.1. One adds the temporal integral $\int_0^{S_0} ds$ in front of every spatial integral in its proof. □

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Department of Mathematics, Chung-Ang University, Dongjak-gu Heukseok-ro 84, Seoul 156-756, Republic of Korea
E-mail address: dchae@cau.ac.kr

Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, BC, Canada V6T 1Z2
E-mail address: ttsai@math.ubc.ca